INT. J. GEOGRAPHICAL INFORMATION SCIENCE, 2000, VOL. 14, NO. 2, 153–172

Research Article

Fuzzy set theory and thematic maps: accuracy assessment and area estimation

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(*Received 12 August 1997; accepted 29 May 1999*)

Abstract. Traditionally, the classes in thematic maps have been treated as crisp sets, using classical set theory. In this formulation, map classes are assumed to be mutually exclusive and exhaustive. This approach limits the ability of thematic maps to represent the continuum of variation found in most landscapes. Substitution of fuzzy sets allows more flexibility for treatment of map classes in the areas of accuracy assessment and area estimation. Accuracy assessment methods based on fuzzy sets allow consideration of the magnitude of errors and assessment of the frequency of ambiguity in map classes. An example of an accuracy assessment from a vegetation map of the Plumas National Forest illustrates the implementation of these methods. Area estimation based on fuzzy sets and using accuracy assessment data allows estimation of the area of classes as a function of levels of class membership. The fuzzy area estimation methods are an extension of previous methods presented by Card (1982). One interesting result is that the sum of the areas of the classes in a map need not be unity. This approach allows a wider range of queries within a GIS.

1. Introduction

Historically, thematic maps have been a common (arguably the *most common*) method for displaying spatially organized data. Currently, their use continues to be common and pervades geographical information systems both in the form of inputs to a GIS and as a common result of analysis within a GIS. Their role and importance is unquestioned. However, thematic maps are an inherently limited way to represent landscapes. The limitation of thematic maps results from their use of categories to characterize the continuum of variation found in landscapes. Traditionally, the use of categories in thematic maps has followed classical set theory, in which each location in the landscape is assumed to belong to a single map category. Additionally, the map categories are assumed to be mutually exclusive and exhaustive. Categories, or sets, that meet these conditions are often termed *crisp sets*.

International Journal of Geographical Information Science
ISSN 1365-8816 print/ISSN 1362-3087 online © 2000 Taylor & Francis Ltd
http://www.tandf.co.uk/journals/tf/13658816.html

The use of thematic maps based on crisp sets results in several implications with regard to two issues of importance to GIS: map accuracy and area estimation. For map accuracy assessment, if each map location is a member of a single class, then all other map categories are equally and completely wrong. Experience in making thematic maps indicates this assumption is not always appropriate, as many locations occur near the definitional boundaries between map categories. Such a situation frequently arises in thematic maps of landcover, vegetation or soil classes, where the map classes represent a continuum of variation in the landscape. Thus, while there is usually a best map category for each site, there are clearly some categories that are more wrong than others.

Similarly, the use of crisp sets has profound implications for estimation of the areas of the map categories. As the categories are mutually exclusive and exhaustive, each location must be a member of exactly one map category. Thus the sum of the areas of all map categories must be unity. The purpose of this paper is to pursue methods of reducing the inherent weakness of thematic maps based on crisp sets in the related areas of accuracy assessment and area estimation. The intent is to use fuzzy set theory in place of classical set theory. More specifically, this paper includes: (1) an example of the use of fuzzy set theory in map accuracy assessment based on previously published methods (Gopal and Woodcock 1994); (2) development of methods for area estimation based on fuzzy sets; and (3) an example of the application of the fuzzy area estimation methods. The last two topics are new extensions to our previously published work.

The concept of a fuzzy set was introduced by Zadeh (1965) to describe imprecision that is characteristic of much of human reasoning. Fuzzy sets provide a quantitative approach for dealing with vagueness in complex systems. A primary difference between fuzzy set theory and classical set theory concerns membership functions. In classical set theory, each object or element is either a member of a set or it is not. Fuzzy set theory allows for grades of membership, providing considerable flexibility beyond that available using classical set theory. Fuzzy set theory has found use in many applications, ranging from pattern recognition, control engineering to modelling human decision making. Fuzzy sets are increasingly being used in GIS. A spatial decision support system based on fuzzy logic has been built for flood simulation and damage assessment (Leung *et al.* 1996). Fuzzy relational databases, like FRSIS (Kollias and Voliotis 1991), can handle imprecision in data representation and manipulation and allow for individualization of data. Fuzzy sets have been applied in land evaluation and suitability analysis (Burrough 1989, Banai 1993, Altman 1994).

In making thematic maps from remotely sensed data, three stages can be identified in the classification process (Foody 1999). The first stage is one of class definition, in which class descriptors are generated. In the second stage, class assignment, assuming crisp logic, each pixel is assigned to the class with which it has the greatest similarity. The third stage is one of accuracy assessment. Fuzzy sets have been applied in the second and third stages of classification of remotely sensed data in making thematic maps. Foody (1999) notes that the degree to which fuzziness is accommodated will be a function of the nature of data sets as well as practical constraints faced by the analyst.

In conventional classification, a pixel displays full and complete membership in a single class. This approach is suitable for mapping of classes that are discrete and mutually exclusive. But many land cover classes are continuous in nature. In addition, the pixel is rarely 'pure' and often contains mixtures of land-cover classes. The inability of 'hard' classification to represent data containing a significant proportion of mixed pixels has been a motivating factor for the development of alternative approaches, including fuzzy classification techniques. In fuzzy classification, a pixel can display any possible membership level, from full membership in one class to having varying membership amongst all classes.

Fuzzy classifications may be obtained in a number of ways: (*i*) fuzzy classifiers (Wang 1990, Foody 1992), (*ii*) softening the output of hard classification, such as from the maximum likelihood technique, (*iii*) neural networks (Carpenter *et al.* 1997). These three methods provide fuzzy membership values at each pixel for each class. Thus, one can easily generate fuzzy maps using these membership values which are useful in mapping continuous phenomena (Wang 1990, Foody 1992). Thus thematic maps are no longer constrained to only crisp sets.

Conventional measures of accuracy assessment are designed for application with the results from 'hard' classification and are not suitable for fuzzy classification. There is a need to accommodate fuzziness in the ground truth data and accuracy assessment. The present research focuses on this issue. From the perspective of thematic maps and the topics of accuracy assessment and area estimation, the idea of varying degrees of membership in map classes for a single map polygon is central. Consider, for example, possible gradations between two simple classes in a map of *forest* and *residential*. At their extremes, the classes are easily differentiated. However, in rural areas housing density can vary considerably, with forest having houses interspersed. For high densities of houses, the class *residential* is unambiguously correct. Similarly, for a site with all trees and no houses the label of *forest* is correct and *residential* is wrong. However, for intermediate cases map polygons would have varying degrees of membership in both classes. Similar gradations and transitions between classes occur in many kinds of thematic maps. The use of fuzzy set theory provides the framework for acknowledging these varying levels of membership.

2. Map accuracy assessment

The accuracy assessment presented here uses the methods published by Gopal and Woodcock (1994). This example illustrates the utility of the methods and addresses issues involved in their implementation. The accuracy assessment data also play a critical role in the following section of fuzzy area estimation.

2.1. *Data and methods*

The vegetation map for the Plumas National Forest in California is used in both accuracy assessment and area estimation. This map was produced using Landsat TM imagery and digital terrain data for use in the management of National Forests for such purposes as timber inventory and wildlife habitat studies.

In the vegetation map are six main categories, or growth forms: *conifer* forest, *hardwood* forest, *brush*, *meadow*, *water*, and *barren/dry grass*. The *conifer*, *hardwood*, and *brush* growth forms were further divided into species associations loosely following a vegetation classification system called CALVEG (Matyas and Parker 1980). In addition, *conifer* polygons received labels for crown cover and tree size, which were derived from the inversion of a forest canopy reflectance model (Woodcock *et al.* 1994a). In this paper, only the most general level of the map is considered for accuracy assessment and area estimation. A complete set of accuracy assessment tables for species associations and conifer cover can be found in Woodcock *et al.* (1994b).

To assess the accuracy of the vegetation maps, locations assigned to each map category were visited. The sample sites were randomly selected within each of the six growth forms and transferred to air photos for field visits. More sites were allocated to the *conifer* growth form as it covers a large proportion of the area and had tree size, cover and species association labels that required assessment.

A larger number of sample sites were selected and transferred to air photos than could be visited in the 14 days available for fieldwork. The strategy was adopted to select daily routes that maximized the number of sites visited. This approach resulted in 160 sites being visited in the field. This strategy worked well except for the *hardwood* class. The reason is that hardwood trees frequently grow in the steep and inaccessible canyons and as a result *hardwood* sites were more frequently omitted than other map categories due to inaccessibility. Since the map categories of the sites were not known when selecting routes, this problem was not discovered until after the field data collection was completed. Thus, there are fewer sites than would be desirable in the *hardwood* class.

The accuracy assessment is based on comparing the map label assigned to each sample site with the evaluations given by the expert. At each site the expert assigns a rating for each possible map label. The rating system uses a linguistic scale, based on the premise that experts most often use linguistic constructs to describe map accuracy. The linguistic values and the descriptions used by the experts to evaluate a map class at a site are:

(5) *Absolutely right*: No doubt about the match. Perfect.
(4) *Good answer*: Would be happy to find this answer given on the map.
(3) *Reasonable or acceptable answer*: Maybe not the best possible answer but it is acceptable; this answer does not pose a problem to the user if it is seen on the map.
(2) *Understandable but wrong*: Not a good answer. There is something about the site that makes the answer understandable but there is clearly a better answer. This answer is a problem.
(1) *Absolutely wrong*: This answer is absolutely unacceptable and completely wrong.

The use of discrete levels of class membership is a practical solution to the difficult problem of determining levels of class memberships for accuracy assessment sites. One limitation of this approach is that it does not make full use of fuzzy sets, as fuzzy membership values are usually continuously varying between 0 and 1. One issue that merits further attention concerns methods for assigning levels of class membership to sites on a map. One possibility is to use quantitative field measurements which can be translated into levels of class membership. While we have made some efforts in this direction (Milliken and Woodcock 1996) there is a definite need for further work.

A few of the differences between traditional map accuracy assessment methods and the procedures used here are worth noting. First, each expert uses the above five linguistic values for assessing sample sites. These ratings or scores are used to estimate the membership value of each map class at each sample site. This allows the experts to acknowledge any existing heterogeneity in the vegetation cover in the site or ambiguity regarding map classes. The expert is not limited to a single match for a site or bound to the existence of a perfect match for each site. Second, a blind testing procedure is used, meaning the expert has no knowledge of the map data while assessing a site. The expert has to evaluate all map categories at each test site using the linguistic membership scale, so there is no need or temptation to provide the expert with the actual mapped category for each test site.

The analysis of the data for the sample sites using fuzzy functions results in a set of three tables. The nature of the tables have been described in detail by Gopal and Woodcock (1994), and are explained here briefly when used.

2.2. *Results*

Table 1 reports the results of two fuzzy measures called 'MAX' and 'RIGHT', which measure accuracy in terms of the frequency of matches and mismatches. MAX uses the highest rating given to a category for a given site to measure a match and provides a conservative estimate of accuracy. Thus, the matches in the MAX column indicate sites where the highest rating was given to the class assigned in the map. The second measure, RIGHT, accepts matches using any degree of right, which in the linguistic scale used here is any score greater than or equal to 3. The use of the MAX and RIGHT functions separates the traditional question of 'how accurate is the map?' into the following two more precise questions:

How frequently is the class assigned in the map the best choice for the site?
How frequently is the class assigned in the map acceptable?

The overall accuracy of the map is shown in two ways. First, the row labeled 'Total' in table 1 gives the number and percent of the accuracy assessment sites that are matches using the MAX function (83%) and the RIGHT function (94%). These numbers translate literally as the map label having been given the highest rating at 83% of the sites, and at least a RIGHT rating at 94% of the sites. Second, the bottom row of table 1 shows the weighted accuracy, where the accuracies of the individual classes are weighted by their areas. This measure of accuracy is the best overall assessment and shows 85% for the MAX function and 94% for the RIGHT function. Table 3 shows the *water* and *conifer* classes to be extremely accurate, with little room for improvement between the MAX and RIGHT results. The results for

Table 1. Results of the MAX and RIGHT functions. Notice the increase in accuracy associated with the use of the less stringent RIGHT function, particularly for the *brush* class.

		Expert evaluation			
		Matches using		Improvement	
Map label	Sites	MAX M	RIGHT R	(R–M)	Area weights
Water	23	23 (100.00%)	23 (100.00%)	0 (0.00%)	0.0118
Barren/grass	18	14 (77.78%)	17 (94.44%)	3 (16.67%)	0.0910
Meadows	20	16 (80.00%)	19 (95.00%)	3 (15.00%)	0.0047
Brush	28	15 (53.57%)	24 (85.71%)	9 (32.14%)	0.1923
Hardwoods	10	5 (50.00%)	6 (60.00%)	1 (10.00%)	0.0745
Conifers	61	60 (98.36%)	61 (100.00%)	1 (1.64%)	0.6305
Accuracy					
total	160	133 (83.12%)	150 (93.75%)	17 (10.62%)	1.0000
weighted		84.68%	94.22%	9.54%	

the *meadow* and *barren/grass* categories are also high but show definite improvement between the MAX and RIGHT functions. *Hardwood* and *brush* are the most troublesome categories. Both have low accuracies for the MAX function, and both improve using the RIGHT function, with *brush* accuracies becoming high. *Hardwood* accuracy is still undesirably low using the RIGHT function (60%), indicating it is the least reliable category in the map.

The DIFFERENCE function is designed to measure the magnitude of errors, and is calculated as the score for the class assigned in the map minus the highest score given to any other class. For the ideal case, where the mapped category is perfectly right (score = 5) and all other categories are absolutely wrong (score = 1), the DIFFERENCE function yields a 4. All sites that are matches using the MAX function have DIFFERENCE values greater than or equal to 0 and all mismatches are negative. A mismatch with a DIFFERENCE value of − 1 would correspond to a case where the map label received a score one less than the highest score given. Clearly, this kind of error is not as troublesome as those where a − 4 is found.

There are several interesting trends illustrated in the results from the DIFFERENCE function for the main classes (table 2). First, the many mismatches that exist in the *brush* class tend to be low in magnitude. Conversely, the errors in the *hardwood* class are often high in magnitude, adding to the conclusion that the *hardwood* class is the most unreliable in the map. Additionally, most of the mismatches for the *meadow* and *barren/grass* classes are small in magnitude, which is not surprising given the significant increase in accuracy of the RIGHT function compared to the MAX function.

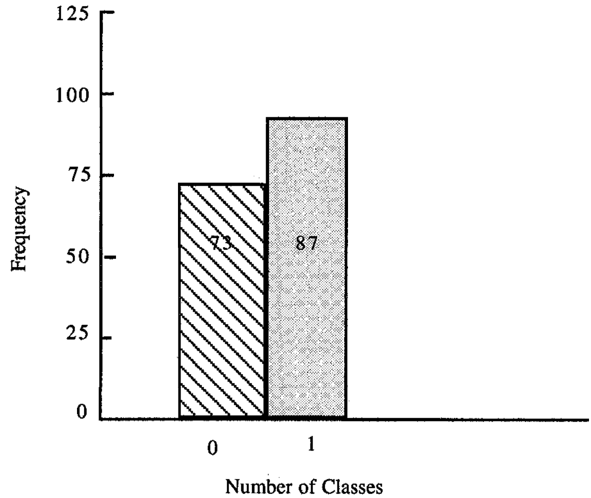
One important kind of information is the categorical nature of the errors, or between which categories confusion occurs. In a traditional analysis, this information is contained in the off-diagonal elements of an error or confusion matrix (Congalton 1991). Similar tables can be constructed using the CONFUSION and AMBIGUITY functions based on fuzzy sets. The CONFUSION function identifies classes with a rating greater than the mapped class and is identical to a traditional confusion matrix except that more than one map category can have a rating higher than the mapped class at a single site. The values from the CONFUSION function are the first values given in the columns headed with symbol ζ below the expert evaluations in table 3. The AMBIGUITY function identifies categories with the same rating as the mapped category. The values for the AMBIGUITY function are given in the same table in the column headed by the symbol η. The information on the equally

Table 2. Results of the DIFFERENCE function. Large errors have high negative scores. Notice the high frequency of low magnitude errors for the *brush* class.

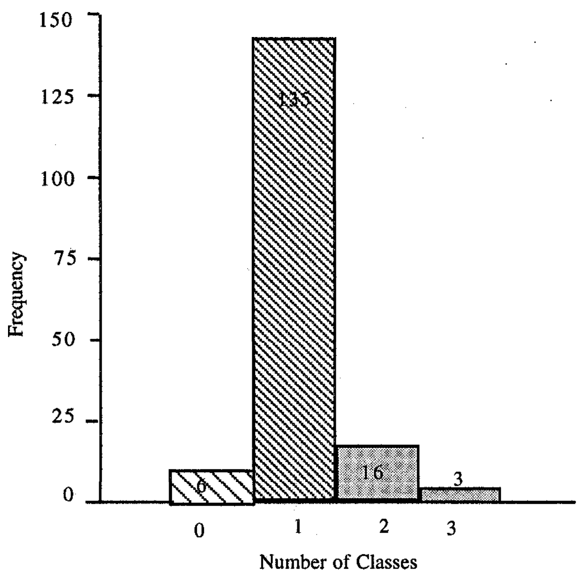
		Mismatches				Matches				
Map label	Sites	− 4	− 3	− 2	− 1	0	1	2	3	4
Water	23	0	0	0	0	0	0	0	0	23
Barren/grass	18	0	0	1	3	3	6	0	2	3
Meadows	20	1	0	0	3	9	6	1	0	0
Brush	28	0	1	3	9	3	5	4	3	0
Hardwoods	10	1	3	0	1	2	2	0	1	0
Conifers	61	0	0	0	1	1	0	10	18	31
Total	160	2	4	4	17	18	19	15	24	57

Table 3. Results of the CONFUSION and AMBIGUITY functions. This table is much like a traditional confusion matrix, except there are two entries for each cell, one for CONFUSION (ζ) and one for AMBIGUITY (η).

	Expert evaluation												No. of mismatches	
	Water		Barren		Meadows		Brush		Hardwood		Conifer			
Map label	ζ_{CC},	η_{CC},	ζ_{CC},	η_{CC},	ζ_{CC},	η_{CC},	ζ_{CC},	η_{CC},	ζ_{CC},	η_{CC},	ζ_{CC},	η_{CC},	ζ_{CC},	η_{CC},
Water	X	X	0	0	0	0	0	0	0	0	0	0	0	0
Barren	0	0	X	X	2	1	2	2	0	0	1	1	5	4
Meadows	0	0	1	4	X	X	2	3	2	3	1	0	6	10
Brush	0	0	1	2	1	0	X	X	2	0	10	1	14	3
Hardwoods	0	0	0	0	0	0	4	2	X	0	3	0	7	2
Conifers	0	0	0	1	0	0	0	0	1	0	X	X	1	1
Total	0	0	2	7	3	1	8	7	5	3	15	2	33	20



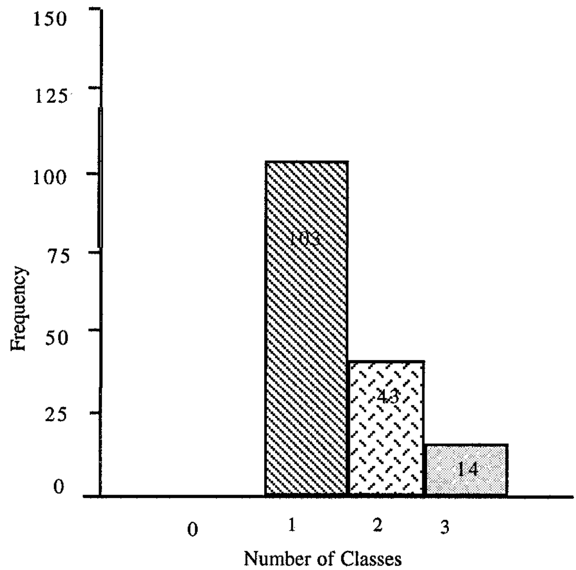
(*a*)



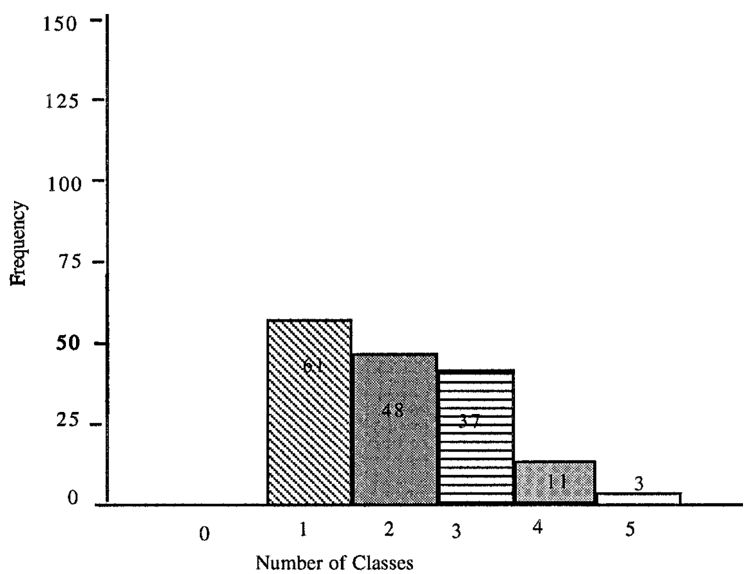
(*b*)

rated categories is particularly interesting for users of the maps when the matrix is not symmetric.

The most striking result in table 3 is that there are many *brush* sites (10) that received higher ratings for the *conifer* class than for the *brush* class. This result is undoubtedly related to the class definitions, which require polygons with greater than 10% cover of conifer trees to be assigned to the *conifer* class. Given that there may be extensive brush under such a sparse conifer canopy, it is often difficult to detect the conifer component of these sites using satellite remote sensing. A similar



(*c*)



(*d*)

Figure 1(*a–d*). These figures show the frequency of numbers of classes receiving different levels of class membership in the expert data. Figure 1(*a*) shows the frequency with which different numbers of classes received the highest level of class membership (5). Figure 1(*b*) is for class membership levels of greater than or equal to 4, 1(*c*) is for greater than or equal to 3, and 1(*d*) for greater than or equal to 2.

problem exists with the *hardwood* class, where three sites received higher ratings for the *conifer* class. There is also significant confusion between the *hardwood* and *brush* classes. Four sites labelled *hardwood* had higher ratings from the expert for the *brush* class. Additionally, the AMBIGUITY function shows that two sites where *hardwood* and *brush* received equal ratings were assigned to the *hardwood* class in the map.

3. Area estimation based on fuzzy sets

One question frequently asked in relation to thematic maps is the areal extent of the various map categories. The area of a category as it appears in a map is one estimate of its true area. This estimate is accurate under either of the following two conditions: (1) the map is perfectly accurate, or (2) when the errors of omission match the errors of commission for each map category. Both these conditions are unlikely, and a variety of methods have been published to improve area estimates from maps using the data collected for a conventional accuracy assessment based on a set of samples from the map. The approach of Card (1982) is the most direct and easily implemented, and amounts to distributing the area associated with each accuracy assessment sample from a map category into its true category. Hay (1988) and Jupp (1989) also published methods which involve normalizing the confusion matrix and estimating conditional probabilities. All of these approaches are based on the use of crisp sets and result in the sum of the areas for the map categories being equal to unity. By substituting fuzzy sets for crisp sets, the conditions resulting in the sum of the areas being equal to unity do not frequently occur, and it is possible to consider the areas of map categories as a function of levels of class membership.

Critical to the problem of area estimation using fuzzy sets is an understanding of the distribution of levels of class membership. For the Plumas dataset, this distribution if illustrated in figure 1(*a–d*). Figure 1(*a*) shows the frequencies for different numbers of classes having received fuzzy set memberships of at least 5. This figure indicates that in 73 sites, no classes were assigned membership ratings of 5, and in the remaining 87 sites, only a single class received a rating of 5. This result is not surprising, as clearly all sites in a map are not perfect matches with one of the map classes. Similarly, it is hard to imagine a site that is a perfect match with more than one class, and figure 1(*a*) shows that no sites were given a rating of 5 for more than one class. Figure 1(*b*) shows the distribution of sites with respect to numbers of classes with ratings greater than or equal to 4. Interestingly, there were 6 sites which did not receive a rating of 4 or more for any map class. The overwhelming majority of sites received a rating of 4 or more for only a single map class. There were also three sites where 3 classes were deemed a good match. This trend of increasing numbers of classes being deemed a match as the level of class membership is decreased continues in figure 1(*c*), which uses a cut-off of class membership values greater or equal to 3. Figure 1(*d*) shows the frequency of class membership for the very weak level of membership of 2. As indicated, often there are sites where as many as three or four map classes meet this weak level of membership. Given that a 1 is the lowest level of class membership using the linguistic scale presented, all classes would meet this level of class membership for all sites, and thus plotting the distribution is meaningless.

3.1. *Methodology for fuzzy area estimation*

Background on Card's Method. The approach adopted here for estimating the area of map categories as a function of fuzzy membership levels is an extension of the methods used by Card (1982) for crisp sets. Card's method for estimating true map proportions ($\hat{p}_i$) requires a contingency table containing sample points selected either randomly or by random stratified sampling. Sampling points are assigned to cells in the contingency table depending on the map class (j) of the sample point, and the true class (i), as determined by some independent means. The values in the cells of the contingency table are counts, n_{ij}, of accuracy assessment samples and r refers to number of map classes. Also required are the map marginal proportions, π_j, which are readily available in most remote sensing and GIS applications (see figure 2(a)). Marginal proportions refer to relative areas of each map category. Row and column totals are calculated respectively as:

$$n_{i.} = \sum_{i=1}^{r} n_{ij} \tag{1}$$

$$n_{.j} = \sum_{j=1}^{r} n_{ij} \tag{2}$$

Estimates of the true map proportions ($\hat{p}_i$) are the sum of the cell probabilities for the row for each class, which are calculated as follows:

$$\hat{p}_i = \sum_{j=1}^{r} \hat{p}_{ij} = \sum_{j=1}^{r} \frac{\pi_j n_{ij}}{n_j} \tag{3}$$

True Class (i)	Map Class (j)			Total
1	n_{ij}			$n_{i.}$
2				
3				
sample total	$n_{.j}$			n
map prop	π_j			$\sum \pi_j = 1$

Figure 2(a). This figure shows a traditional contingency table for accuracy assessment data and area estimation, based on the formulation presented by Card (1982).

True Class (i)	Map Class (j)			True Prop
1	$\hat{p}_{ij}$			$\hat{p}_i$
2				
3				
total				$\sum\hat{p}_i = 1$

Figure 2(b). This figure shows cell probabilities as calculated from accuracy assessment data and map marginal totals, as presented by Card (1982).

Figure 2(b) is an example of the matrix of cell probabilities, $\hat{p}_{ij}$. Card's work is presented in terms of proportions of map classes, which can be easily converted to areas using the areas for the map classes and the total area of the map, both of which are usually readily available. Conceptually, Card's approach weighs each sample from a map class according to the number of samples from that class and the proportion of the map assigned to the class, or π_j/n_j. The distribution of the samples from map class j among the true classes i, as contained in the n_{ij}, determines how the area in map class j is redistributed.

Fuzzy Area Estimation. To use Card's method for estimating true class proportions together with our fuzzy set representation for class membership, a change is required in the way sample points are assigned to the contingency table. For this treatment, we have tried to continue and extend the notation used by Card and by Gopal and Woodcock (1994). The intent is to estimate the area of a class as a function of levels of fuzzy membership (μ), or $\hat{p}_{\mu i}$. This term corresponds to the estimate of the map proportion of class i for fuzzy membership values greater than or equal to μ.

Each sample is evaluated for inclusion in each column (j) of the map class (i) to which it belongs. A sample is counted toward the count $n_{\mu ij}$ if it meets the following conditions: the sample (x) is from map class i, and $\mu_j(x) \geqslant \mu$, where μ is a threshold of class membership. In this way, a new contingency table can be calculated for each value of μ. For use in fuzzy area estimation these contingency tables carry with them the original map marginal totals (π_j) and same column (or sample) totals (n_j), but the row totals ($n_{\mu i}$) are recalculated for each new contingency table (figure 3(a)). Thus the information on the original map proportions and numbers of samples within each map class are preserved in order to weight appropriately the area associated with each sample. This formulation results in several conditions which differ significantly from the formulation based on crisp sets. First, the row and column totals may not match, i.e.

$$\sum_{j=1}^{r} n_j \neq \sum_{i=1}^{r} n_{\mu i} \tag{4}$$

True Class (i)	Map Class (j)			Total
1	$n_{\mu ij}$			$\sum_{i=1}^{r} n_{\mu ij}$
2				
3				
fuzzy total	$\sum_{i=1}^{r} n_{\mu ij}$			$\sum_{i=1}^{r}\sum_{j=1}^{r} n_{\mu ij}$
sample total	$n_{.j}$			n
map prop	π_j			$\sum \pi_j = 1$

Figure 3(a). This figure shows how the traditional contingency table is modified for use in fuzzy area estimation.

True Class (i)	Map Class (j)			True Prop
1	$\hat{p}_{\mu ij}$			$\hat{p}_{\mu i}$
2				
3				
total				$\sum \hat{p}_{\mu i}$

Figure 3(*b*). This figure shows cell probabilities calculated based on the contingency table presented in figure 3(*a*).

For high values of μ, some samples will not satisfy the conditions above for any class. For the Plumas data this is illustrated for $\mu = 5$ in figure 1(*a*) by the 73 samples for which a value greater than or equal to 5 was not observed. Alternatively, for low values of μ, one sample may be counted toward more than one class (j).

The calculation of the estimates of the true class proportions for any value of μ proceed in the same manner as Card:

$$\hat{p}_{\mu j} = \sum_{j=1}^{r} \hat{p}_{\mu ij} = \sum_{j=1}^{r} \frac{\pi_j n_{\mu ij}}{n_j} \tag{5}$$

Notice that to solve for $\hat{p}_{\mu i}$ as a function of μ, the only variables that change are the elements $n_{\mu ij}$ of the contingency table. Figure 3(*b*) shows a matrix of fuzzy cell probabilities.

3.2. *Application in the Plumas National Forest*

The results of application of the fuzzy area estimation methods for the Plumas National Forest are given in tables 4(*a*,*b*), 5(*a*,*b*), 6(*a*,*b*) and 7(*a*,*b*) for class membership levels of 5, 4, 3, and 2 respectively. Table 4(*a*) shows that outside of the *water*

Table 4(*a*). This table shows the fuzzy contingency table for a class membership level of 5 for the Plumas vegetation map. The row marked 'Total' is the total number of sample sites that received a 5, while the 'Sample Total' is all the sites visited. Note that all of the *water* sites received a rating of 5, while very few of the *brush*, *meadow*, or *hardwood* sites received a 5. See figure 3(*a*) for a conceptual example of this table.

	Map class(j)						
True class(i)	Water	Barren	Meadow	Brush	Hwd	Conifer	Total
Water	23	0	0	0	0	0	23
Barren/grass	0	5	0	0	0	0	5
Meadow	0	0	1	0	0	0	1
Brush	0	0	0	5	2	0	7
Hardwood	0	0	0	0	1	0	1
Conifer	0	0	1	0	0	49	50
Total	23	5	2	5	3	49	87
Sample total	23	18	20	28	10	61	160
Map prop.	0.01	0.09	0.01	0.19	0.07	0.63	1.00

Table 4(*b*). This table shows the cell probabilities for a class membership level of 5 for the Plumas vegetation map. Notice that the sum of the area estimates for all classes at this conservative class membership level is only approximately 60% of the total map area. See figure 3(*b*) for a conceptual example which illustrates how the values in this table were calculated.

	Map class(j)						
True class(i)	Water	Barren	Meadow	Brush	Hwd	Conifer	Total
Water	0.0118	0.0000	0.0000	0.0000	0.0000	0.0000	0.0118
Barren/grass	0.0000	0.0253	0.0000	0.0000	0.0000	0.0000	0.0253
Meadow	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000	0.0002
Brush	0.0000	0.0000	0.0000	0.0343	0.0149	0.0000	0.0492
Hardwood	0.0000	0.0000	0.0000	0.0000	0.0075	0.0000	0.0075
Conifer	0.0000	0.0000	0.0002	0.0000	0.0000	0.5065	0.5067
Total							0.6007

Table 5(*a*). This table shows the fuzzy contingency table for a class membership level of greater than or equal to 4 for the Plumas vegetation map.

	Map class(j)						
True class(i)	Water	Barren	Meadow	Brush	Hwd	Conifer	Total
Water	23	0	0	0	0	0	23
Barren/grass	0	12	5	3	0	0	20
Meadow	0	3	15	1	0	0	19
Brush	0	3	5	18	4	0	30
Hardwood	0	0	5	2	4	1	12
Conifer	0	0	1	9	2	60	72
Total	23	18	31	33	10	61	176
Sample total	23	18	20	28	10	61	160
Map prop.	0.01	0.09	0.01	0.19	0.07	0.63	1.00

Table 5(*b*). This table shows the cell probabilities for a class membership level of greater than or equal to 4 for the Plumas vegetation map. Notice that the sum of the area estimates for all classes has increased from that shown in table 4(*b*).

	Map class(j)						
True class(i)	Water	Barren	Meadow	Brush	Hwd	Conifer	Total
Water	0.0118	0.0000	0.0000	0.0000	0.0000	0.0000	0.0118
Barren/grass	0.0000	0.0607	0.0012	0.0206	0.0000	0.0000	0.0825
Meadow	0.0000	0.0152	0.0035	0.0069	0.0000	0.0000	0.0256
Brush	0.0000	0.0152	0.0012	0.1236	0.0298	0.0000	0.1698
Hardwood	0.0000	0.0000	0.0012	0.0137	0.0298	0.0103	0.0550
Conifer	0.0000	0.0000	0.0002	0.0618	0.0149	0.6202	0.6971
Total							1.0418

and *conifer* classes, very few sites received class membership levels of 5. As all of the samples from the *water* class were given a 5 rating for *water*, the area estimate at this level of class membership for *water* is all the area assigned to *water* in the map.

Table 6(*a*). This table shows the fuzzy contingency table for a class membership level of greater than or equal to 3 for the Plumas vegetation map.

True class(i)	Map class(j) Water	Barren	Meadow	Brush	Hwd	Conifer	Total
Water	23	0	0	0	0	0	23
Barren/grass	0	17	13	7	1	0	38
Meadow	0	4	19	1	1	0	26
Brush	0	9	7	26	5	2	49
Hardwood	0	0	5	2	6	1	14
Conifer	0	1	1	14	4	61	81
Total	23	31	45	51	17	64	231
Sample total	23	18	20	28	10	61	160
Map prop.	0.01	0.09	0.01	0.19	0.07	0.63	1.00

Table 6(*b*). This table shows the cell probabilities for a class membership level of greater than or equal to 3 for the Plumas vegetation map. Notice that the sum of the area estimates for all classes is now considerably larger than the total area of the map.

True class(i)	Map class(j) Water	Barren	Meadow	Brush	Hwd	Conifer	Total
Water	0.0118	0.0000	0.0000	0.0000	0.0000	0.0000	0.0118
Barren/grass	0.0000	0.0859	0.0031	0.0481	0.0075	0.0000	0.1446
Meadow	0.0000	0.0202	0.0045	0.0069	0.0075	0.0000	0.0391
Brush	0.0000	0.0455	0.0016	0.1786	0.0372	0.0207	0.2836
Hardwood	0.0000	0.0000	0.0012	0.0137	0.0447	0.0103	0.0699
Conifer	0.0000	0.0051	0.0002	0.0961	0.0298	0.6305	0.7617
Total							1.3107

Table 7(*a*). This table shows the fuzzy contingency table for a class membership level of greater than or equal to 2 for the Plumas vegetation map. For this minimal level of class membership, many sites are counted toward more than one class. The sum of the row totals greatly exceeds the number of sample sites.

True class(i)	Map class(j) Water	Barren	Meadow	Brush	Hwd	Conifer	Total
Water	23	0	0	0	0	0	23
Barren/grass	0	18	17	21	1	10	67
Meadow	0	7	19	2	3	2	33
Brush	0	13	10	28	6	21	78
Hardwood	0	1	9	5	7	9	31
Conifer	0	3	2	21	8	61	95
Total	23	42	57	77	25	103	327
Sample total	23	18	20	28	10	61	160
Map prop.	0.01	0.09	0.01	0.19	0.07	0.63	1.00

Table 7(*b*). This table shows the cell probabilities for a class membership level of greater than or equal to 2 for the Plumas vegetation map. Notice the higher estimates for map areas for most classes beyond their original area in the map.

	Map class(j)						
True class(i)	Water	Barren	Meadow	Brush	Hwd	Conifer	Total
Water	0.0118	0.0000	0.0000	0.0000	0.0000	0.0000	0.0118
Barren/grass	0.0000	0.0910	0.0040	0.1442	0.0075	0.1034	0.3501
Meadow	0.0000	0.0354	0.0045	0.0137	0.0223	0.0207	0.0966
Brush	0.0000	0.0657	0.0024	0.1923	0.0447	0.2171	0.5222
Hardwood	0.0000	0.0051	0.0021	0.0343	0.0521	0.0930	0.1866
Conifer	0.0000	0.0152	0.0005	0.1442	0.0596	0.6305	0.8500
Total							2.0173

The area estimate for the *conifer* class at level 5 includes area initially assigned to the *conifer*, *meadow* and *brush* classes, as indicated in tables 4(*a*) and (*b*). Notice that the total area in the map estimated to be 'perfect matches' is only 60% of the map area.

As the class membership level is relaxed to 4, several interesting patterns emerge (table 5(*a*, *b*)). Many of the samples from the *brush* class meet this level of class membership in other classes. This situation is particularly true for the *conifer* class, where 11 of the brush samples become members of the *conifer* class. Significant area is contributed to the *conifer* class from that orginally mapped as *brush*. Also note that the sum of the areas for the classes for this level of class membership exceeds the area of the map. This situation is the direct result of sample sites receiving class memberships of 4 or higher for more than one class, as illustrated in figure 1(*b*).

With the exception of water, as class membership levels are further relaxed the area estimates for the classes continue to increase, and the total area estimates grow well past unity (tables 6(*b*) and 7(*b*)). In particular, the *brush* class grows significantly at these more lenient levels of class membership. This result is due to the nature of the landscape in this National Forest, where chaparral vegetation occurs in varying degrees in many locations. Where conifer trees have only 10% cover, the area is best mapped as *conifer*. However, in many of these sparse conifer stands, brush is a common understorey, and so at lenient levels of class membership these sites become part of the *brush* class. Notice at the class membership level of 3 (a reasonable answer for the site, for which a better answer might exist), the area estimate for the *conifer* class is almost 76% (table 6(*b*)) of the area of the National Forest. This figure is in striking contrast to the 51% (table 4(*b*)) estimate of areas having the most conservative level of class membership (5, perfect match). This difference indicates that roughly one quarter of the area of the National Forest occurs with very sparse coverage of conifer trees mixed extensively with brush and hardwoods. Also note that this estimate differs significantly from the 63% of the area assigned to the *conifer* class in the map (table 4(*a*)).

Figure 4 summarizes the results of tables 4–7 by showing the ratio of the area estimate to the area in the map for each class as a function of class membership. The *water* class, which is easily defined using crisp sets, shows no change as a function of class membership. However, the other classes go through different rates of increase in areas estimates as levels of class membership are decreased.

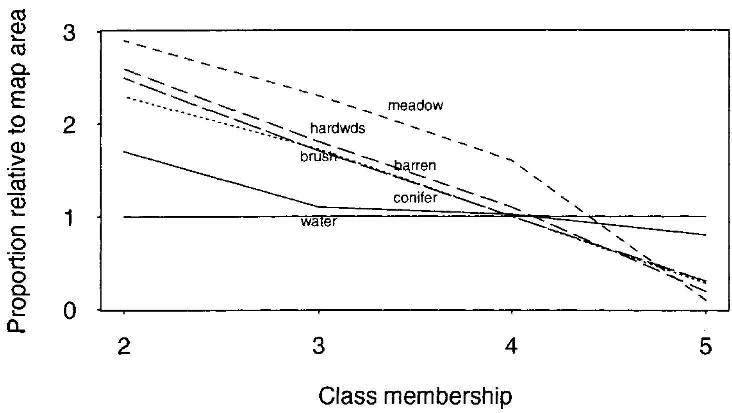


Figure 4. This figure shows the ratio of the estimated area to the area shown in the map for each class as a function of class membership levels. All the data for this graph is in tables 3(*b*)–6(*b*). Note how the area estimates for most classes increase significantly as levels of class membership decrease, with water being the notable exception.

Conventional Area Estimation. For comparison purposes, we provide a conventional confusion matrix using boolean logic or crisp sets for the Plumas data set (tables 8(*a*,*b*)). The accuracy of the map using conventional crisp sets is 80.625 (129/160). Note that in 21 (out of 160) sites, two or three classes got the same membership rating and one class was randomly picked as the winning class. The kappa value for this data set is 0.74. The total area estimate is 1.0 and the overall probability correct using Card's method is 0.85.

A comparison of the conventional contingency matrix using crisp sets (table 8) with those produced using the fuzzy sets (tables 4–7) reveal that the fuzzy area estimates vary with different membership values. The overall probability correct (sum of diagonals) using fuzzy area estimation methods for the Plumas National Forest for class membership levels of 5, 4, 3, and 2 are 0.58, 0.83, 0.94, 0.98 respectively. Card's method produces an overall probability correct of 0.85. Fuzzy sets are thus able to show what happens to area estimation when fuzzy class membership values are incorporated. There is very little difference in the two methods for *water* and

Table 8(*a*). For comparison purposes, this table shows a conventional confusion matrix for the data for the Plumas National Forest where the values in each cell are counts of samples.

	Map class(j)						
True class(i)	Water	Barren	Meadow	Brush	Hwd	Conifer	Total
Water	23	0	0	0	0	0	23
Barren/grass	0	13	2	2	0	1	18
Meadow	0	4	11	2	2	1	20
Brush	0	1	0	17	1	9	28
Hardwood	0	0	0	3	5	2	10
Conifer	0	0	0	0	1	60	61
Total	23	18	13	24	9	73	160
Sample total	23	18	20	28	10	61	160
Map prop.	0.01	0.09	0.01	0.19	0.07	0.63	1.00

Table 8(*b*). This table shows the cell probabilities as calculated using Card's methods for the Plumas data. Note that the map areas for the classes are constrained to sum to unity.

	Map class(j)						
True class(i)	Water	Barren	Meadow	Brush	Hwd	Conifer	Total
Water	0.0118	0.0000	0.0000	0.0000	0.0000	0.0000	0.0118
Barren/grass	0.0000	0.0657	0.0005	0.0137	0.0000	0.0103	0.0902
Meadow	0.0000	0.0202	0.0026	0.0137	0.0149	0.0103	0.0617
Brush	0.0000	0.0051	0.0000	0.1168	0.0075	0.0930	0.2224
Hardwood	0.0000	0.0000	0.0000	0.0206	0.0372	0.0207	0.0785
Conifer	0.0000	0.0000	0.0000	0.0000	0.0075	0.6202	0.6277
Total							1.0

conifers. But differences emerge for other classes. *Brush*, for example, would result in different estimates using the fuzzy set approach and may represent reality on the ground better than the conventional approach.

4. Discussion and conclusions

The use of fuzzy set representations for classes in thematic maps relaxes some of the restrictions imposed by classical set theory, and thus expands the kinds of information than can be provided through accuracy assessment. In addition to information on the frequency and categorical nature of errors in a thematic map, methods based on fuzzy sets can provide a measure of the magnitude of errors in a map, as well as assess ambiguity between map categories. The accuracy assessment presented for the Plumas National Forest illustrates the use of these methods and the kinds of information that can be provided.

Using crisp sets, the only query possible is the area of each class. Using fuzzy set theory it is possible to make queries regarding areas meeting certain criteria, or membership levels. Consider an example regarding wildlife habitat and vegetation maps. A species of bird might require the presence of conifer trees for nesting to make a site suitable habitat. Conifer trees would by definition occur in a class called *conifer forest*, but many conifer trees might also occur in areas better mapped as other classes, such as *hardwood forest*, or *shrub woodland*. Using the approaches presented here it would be possible to query regarding levels of membership in the *conifer forest* class which would be sufficiently low to include areas where conifer trees are only present, regardless of the fact that those areas had been mapped in other classes.

In some ways, the slightly different view of the area estimation problem presented in this paper is common to many problems involving GIS. Area estimation becomes a specific case of the more general question of determining which locations meet certain sets of conditions. This kind of problem is now common in GIS, where in a map overlay procedure the intersection or union of various map categories is determined and their area measured. A simple example might be the area of a particular vegetation type that occurs on a specific soil type. Given thematic maps of both soil and vegetation types, this can be easily accomplished in a GIS. Similarly, the area that is either the particular vegetation type or the specific soil type could be determined, with the area meeting the second set of conditions always being greater than or equal to the area satisfying the first set of conditions. For these kinds of questions of determining areas meeting various conditions, questions of their sums equaling unity are irrelevant. As the problem of area estimation is viewed from fuzzy set theory, this assumption of unity for the sums of the areas of map categories also becomes irrelevant.

Acknowledgments

The authors would like to thank the Region 5 Remote Sensing Laboratory of the US Forest Service for partial support for the research presented here.

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