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Review Paper

A Perspective on the Fundamentals of Fuzzy Sets and their Use in Geographic Information Systems

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Abstract
The development of fuzzy sets in geographic information systems (GIS) arose out of the need to handle uncertainty and the ability of soft computing technology to support fuzzy information processing. An overview of the fundamentals of fuzzy sets is used to illustrate its use in GIS. The use of some terms within both the GIS and fuzzy information processing community is clarified. Since one of the key problems when applying fuzzy sets to GIS problems is in the specification of grades of membership, the many methods used to specify memberships in fuzzy sets in GIS applications are presented. The *α-cut* is defined and shown to be of increasing importance in GIS. Non-compensatory and compensatory connectives are compared. Aggregation operators are reviewed and shown to be useful in a number of GIS studies. Fuzzy relations and fuzzy control systems are briefly discussed with reference to their use in GIS and in relation to the development of modern soft computing technology. Several features of fuzzy sets make that paradigm attractive for use in GIS. It is concluded that as GIS-related applications increase in their levels of complexity and sophistication fuzzy sets will play a major, cost effective role in their development.

1 Introduction

As early as the 1970's the modeling of fuzzy spatial events through fuzzy set theory (Zadeh 1965) has been proposed, primarily in application to areas of geographical analysis (Gale 1972, Leung 1979, Pipkin 1978). With the development of computer-based geographic information systems (GIS), the potential of fuzzy set theory in GIS began to be addressed. McBratney and Odeh (1997), in their perspective piece on fuzzy

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sets in soil science, note that the inadequacy of traditional Boolean logic for the design of spatial databases for GIS has been identified since the 1980's (Robinson and Strahler 1984, Robinson 1988). Furthermore, in his essay on the principles of logic in a GIS, Robinove (1989) discusses the potential value of fuzzy set theory. Thus, the relevance of fuzzy sets to GIS was recognized by a relatively small group of researchers many years ago. However, most GIS textbooks remain virtually devoid of any significant presentation of the basics of fuzzy set theory and its application in GIS. Those that do refer to fuzzy set theory, in general, do so in a limited fashion. Indeed, the only widely used textbook that devotes substantial attention to the topic of fuzzy sets is Burrough and McDonnell (1998) which is a text that builds on an earlier edition of Burrough (1986).

Several works address the issue of uncertainty and imprecision in GIS which led some like Altman (1994) to experiment with the use of fuzzy sets. Fisher (1996) suggested that an early work (Robinson and Frank 1985) which set out the differences between fuzzy and probability models of uncertainty within the context of GIS has not received the attention it deserves. Burrough (1996) discussed uncertainty in a GIS with a particular focus on indeterminate boundaries. More recently, Fisher (2000) traced the concept of vagueness back to its philosophical roots in the sorites paradox. He went on to show that many geographical objects and relations typically represented in a GIS are sorites susceptible and that, in spite of considerable research into accuracy (e.g. Goodchild and Gopal 1989, Unwin 1995) and quality (e.g. Guptil and Morrison 1995) in geographic databases, the notion that the objects being stored in a GIS are not well defined still seems to be ignored by many system developers and academic researchers. Importantly, it is recognized that fuzzy set theory can address many of the problems of vagueness and that one of its strengths is that fuzzy set approaches can be implemented (Fisher 2000). Cobb et al. (2000) presented many of the kinds of uncertainty that are found in spatial information systems, especially those like GIS. They consider the significance of modeling uncertainty as well as the application of fuzzy models in GIS. Their example of conflation and uncertainty using fuzzy sets and logic illustrates their promise in effectively addressing complex GIS tasks.

The cost of developing and implementing fuzzy computational tools may have been, in the past, an obstacle to incorporating fuzzy sets in GIS. To some extent, it remains an issue since it is still relatively rare for a commercially available GIS software system to support fuzzy information processing. A notable exception is the IDRISI GIS software package (Eastman 1999). Other tools, outside GIS software packages, have been developed. In fact, it has been argued that the increase in the frequency of fuzzy sets being used in GIS research is a reflection of the influence developments in soft computing technology are having on the field of GIS (Robinson 2002a). One of the earliest examples of how low-cost soft computing technology affected the use of fuzzy sets in GIS was the publication of the fuzzy c-means algorithm (Bezdek et al. 1984). Shortly after Bezdek et al.'s (1984) paper appeared, applications in both GIS and remote sensing (Fisher and Pathirana 1994, Robinson and Thongs 1986) began to appear and this trend continues to the present (e.g. Brown 1998b, Burrough et al. 2000). Further developments in soft computing technology have led to computational tools like the Fuzzy Logic Toolbox for MATLAB and the Fuzzy Systems Toolbox (Hall and Hathaway 1996), neural net packages such as the NeuDesk package and Fuzzy ARTMAP (Carpenter et al. 1997). Consequently, these are beginning to appear with some regularity in application to GIS problems. The MATLAB Fuzzy Systems Toolbox has been used to model mobile spatial objects (Palanciogla and Beard 2001). It was also used to model hierarchical pattern matching of land use maps to arrive at more sensible maps of land use change (Power et al. 2001) as well as for analyzing landscape pattern and soil variability (Ahn et al. 1999, Bartel 2000). It is unlikely any of the applications would have been attempted without the toolbox. Similarly, the NeuralDesk package and Fuzzy ARTMAP have been used in remote sensing applications that lead to development of the spatial databases underpinning GIS (Carpenter et al. 1997, Foody and Boyd 1999).

Although it is clear that the use of fuzzy sets and logic in GIS has increased in the past decade, there has been little perspective given to its development, particularly in relation to broader trends in the use of fuzzy logic technology. Yen's (1999) perspective on the historical development of fuzzy logic technology included a review of milestones in the development of fuzzy logic technology. One can draw several parallels between the development of fuzzy logic development in the artificial intelligence community and its development in the GIS community. He describes researchers in the USA and Japan encountering an antifuzzy atmosphere in the early days and persisting until relatively recently. In a similar sense, there was, and perhaps continues to be, an antifuzzy atmosphere in the GIS and related fields (e.g. remote sensing). Only recently has that atmosphere seemed to change as evidenced by the increased frequency of papers using fuzzy sets appearing in mainstream GIS journals. It is hoped that this brief perspective on fuzzy sets and GIS will provide the reader with an appreciation of the depth and diversity of tools available in fuzzy sets theory that can be applied to GIS problems.

This paper provides an overview of the basic concepts in fuzzy sets to illustrate how they have been addressed in GIS, and to clarify the use of some terms within both the GIS field and the fuzzy information processing community. First, the fundamental ideas underpinning fuzzy sets are presented. One of the key problems when applying fuzzy sets to GIS problems is in the specification of grades of membership. Thus, secondly, a substantial portion of this paper is devoted to presenting the diverse methods used to specify memberships in fuzzy sets in GIS applications. Discussion of membership functions is followed by a presentation of basic concepts and operations of fuzzy sets with special reference to GIS applications in which they may have played in important role. The last two major topics presented are fuzzy relations and fuzzy control systems. In both cases reference is made to GIS problems where fuzzy relations or fuzzy control systems have helped solve GIS problems. Finally, a brief set of concluding comments is offered that address the major features of fuzzy sets and examples of where those features have been evident in solving GIS problems.

2 What is a Fuzzy Set?

To understand fuzzy sets it is useful to first consider some of the fundamentals of classical set theory. In particular, consider the two fundamental laws of Boolean algebra – the law of excluded middle and law of contradiction. In logic, the proposition "every proposition is either true or false" excludes any third, or middle, possibility, which gave this principle the name of *the law of excluded middle*. From this we derive that a prediction may be only true or false. Thus, in classical, Boolean or crisp, set theory, membership of an element x in a set $\boldsymbol{A}$, is defined by a characteristic function which assigns a value of either 1 or 0 to each individual in the universal set $\boldsymbol{X}$. This is how it discriminates between members and nonmembers of the crisp set A (Yen 1999). For example, in geographical applications the percent slope is often calculated for locations in a study area. Let $\mathbf{X}$ be the universal set defining percent slope with values ranging from 0 to 100 and let A be the set 'gentle' slope. One might define a characteristic function for the set *gentle* as:

$$A(x) = \begin{cases} 1 & for \quad 5 < x \le 17 \\ 0 & for \quad otherwise \end{cases} \tag{1}$$

Furthermore, consider that the complement of $A(x)$, *not* $A(x)$, can be denoted as $\bar{A}(x)$. Thus, if $A \cup \bar{A} = X$ then the *law of excluded middle* is satisfied: in other words, the union of *gentle* and *not gentle* is the universal set X. The *law of contradiction* is satisfied by $A \cap \bar{A} = \emptyset$. In other words, no element of X can be both a member of A and *not* a member of A, i.e. $\bar{A}$. Therefore, no location can have both level and gentle, or gentle and steep, slope.

Let us consider the question of specifically at which value of percent slope (X) does a location (x) go from being gentle to not gentle? Equation (1) would suggest that any location with a percent slope of 17.01 would be classed as not gentle while one with 16.98 is gentle. Perhaps a slope of 17.01 is simply just not as gentle sloping as one of 16.98. Thus the location (x) with a slope of 17.01 might be considered both gentle and not gentle but to differing degrees. This is the fundamental proposition upon which fuzzy set theory is based. Figure 1 illustrates a comparison of a crisp and a fuzzy membership function defining gentle slope. It shows that now an element with a percent slope of 17.01 has a membership of less than 1 but greater than 0. Thus, the characteristic function of a crisp set, e.g. A, can be generalized, for example, by the function:

$$A(x) = \begin{cases} 0 & for \quad x < 3 \\ \dfrac{x-3}{11-3} & for \quad 3 \le x \le 11 \\ \dfrac{19-x}{19-11} & for \quad 11 \le x \le 19 \\ 0 & for \quad 19 < x \end{cases} \tag{2}$$

so that the values assigned to elements of X falling within a specified range, e.g. 3–19, indicate the grade of membership of elements (x) in that particular set (A). The larger values denote higher degrees of set membership (Figure 1). Such a function is referred to as a *membership function*. The set defined by such a membership function is a *fuzzy set*.

Fuzzy set theory violates both the law of excluded middle and contradiction by allowing an element to be a member in more than one set and to a varying degree. Since an element can belong to both sets A and $\bar{A}$, to varying degrees, it violates the *law of excluded middle*. To show that the *law of contradiction* is also violated for fuzzy sets, it need only be shown that $A \cap \bar{A} = \emptyset$ is violated for at least one $x \in X$. This is easy if the membership value of any x in $A(x) = 0.5$ then *not* $A(x)$, $\bar{A}(x) = 0.5$. Hence, $A(x) \cap \bar{A}(x) = 0.5$ and violates the *law of contradiction*. Since it is violated for any value $A(x) \in (0, 1)$ and is satisfied only for $A(x) \in \{0, 1\}$. The *law of contradiction* is satisfied only for crisp sets (Klir and Yuan 1995).

In the literature there are two commonly used notations that denote membership functions. In one notation the membership function of a fuzzy set $\mathbf{A}$ is denoted by μ_A. That is to say $\mu_A : X \to [0, 1]$ where the symbol, or name, of the fuzzy set (e.g. $\mathbf{A}$) is distinguished from the symbol of its membership function (i.e. μ_A). The other notation denotes the function simply as $\mathbf{A}$ and has the same form, namely $A : X \to [0, 1]$. In the

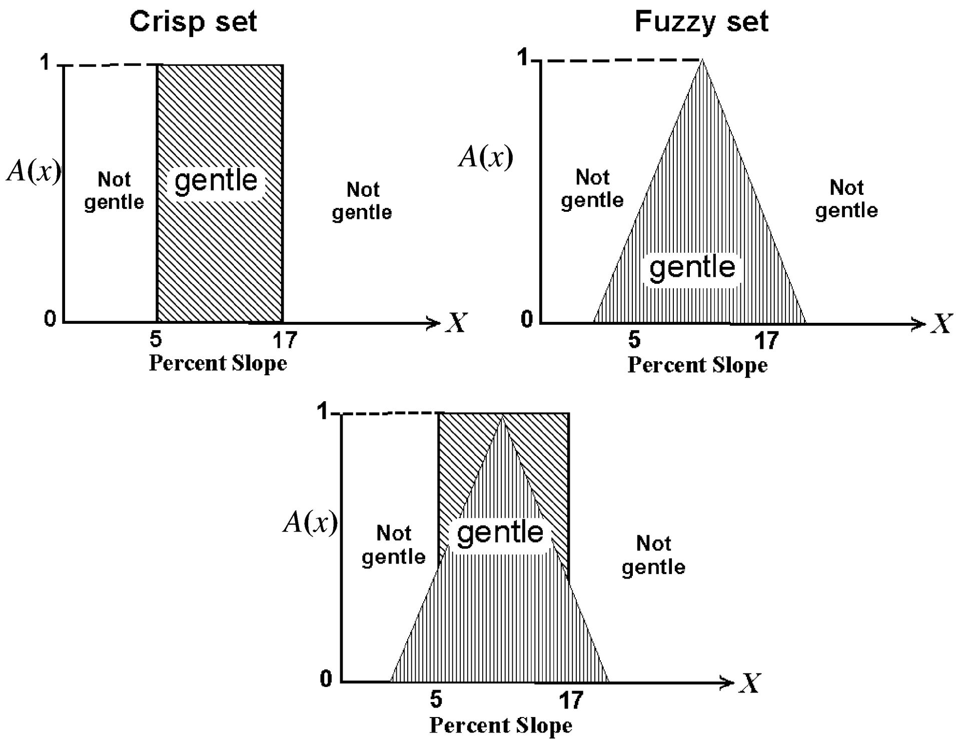


Figure 1 Characteristic function of gentle slope using classical set theory to define a crisp set versus using a characteristic membership function to define the same concept of gentle slope using fuzzy set theory

second form, the distinction between the fuzzy set and its membership function is not made. No ambiguity results from this double use of the same symbol since each fuzzy set is completely and uniquely defined by one particular membership function. Therefore, symbols of membership functions may also be used as labels of associated fuzzy sets (Klir and Yuan 1995).

In a more formal definition of a fuzzy set, let $\mathbf{X} = \{x\}$ be a finite set, or space, of points. These points could be elements, objects, or properties depending on the context of the application. Using the first notation, a fuzzy set $\mathbf{A}$ of $\mathbf{X}$ would then be defined by a function, μ_A, in the ordered pairs: $A = \{x, \mu_A(x)\}$ for each $x \in \mathbf{X}$. Where $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}$ then $\mathbf{A} = x_1, \mu_A(x_1) + x_2, \mu_A(x_2) + \cdots + x_n, \mu_A(x_n)$ such that + is defined in the set theoretic sense, not an arithmetic sum, so if $\mu_A = 0$ then $x, \mu_A(x)$ is omitted. In the literature one may see an equivalent formulation appear as $\mathbf{A} = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$ where the separator "/" can be interpreted as "with respect to" and is used to differentiate $\mu_A(x)$ and x. Regardless of the specific notation, it should be clear by now that specifying the membership function of a fuzzy set is just as important as specifying the characteristic function of a crisp set – perhaps more so. The basic idea, however simple or complex, is to assign a grade of membership to an element of X.

There have been many approaches taken to solve the problem of assigning membership values to elements. One, commonly used approach is to specify a standard membership function. Others specify membership functions that are meaningful only in the context of the problem. Both of these approaches would fit the Semantic Import model (Robinson 1988, McBratney and Odeh 1997) Some of these approaches have been combined with human-machine knowledge acquisition techniques to acquire memberships directly from experts, or users. A data-driven approach whereby fuzzy clusters are formed from data elements is also used (Bezdek et al. 1984). All of these approaches will be briefly considered with special reference to their application in GIS.

3 Fuzzy Membership Functions

The assignment of the membership function of a fuzzy set is subjective in nature. Thus, it should reflect the context in which the problem is viewed. Although assignment of the membership function of a fuzzy set is subjective, it should not be assigned arbitrarily. This still begs the question, as to what specific form the function will take. This section briefly presents the major approaches that may be used to assign grades of membership when applying fuzzy sets to GIS-related problems.

3.1 Standard Membership Functions

It is often convenient to express the membership function of a fuzzy subset in terms of a standard membership function whose parameters may be adjusted to fit a specific membership function in a fashion appropriate to the problem (Kandel 1986). This section provides a summary review of many of the standard functions with reference to their use in GIS. Note is taken of several studies that develop membership functions uniquely suited to their problem.

Most standard membership functions can be categorized as open form or closed form. Let us first consider open-form membership functions that are characterized as being non-decreasing and having values inside 0 and 1 only within a bounded interval (Bohlin et al. 2000).

3.1.1 Open form membership functions

The membership functions in Figure 2 are, as a group, sometimes referred to as 'linear' membership functions. Strictly speaking many of the so-called linear functions are actually open left shoulder, or open right shoulder, functions which are open forms of the closed form trapezoid function (e.g. Burrough and McDonnell 1998, Stefanakis et al. 1999, Zeng and Zhou 2001). Hence the labeling in Figure 2 of left and right trapezoidal functions. Thus, it is more precise to describe them as left and right trapezoid with linear reserved for cases where the values of X are finite and have a lower, or higher, bounding value that corresponds to the intercept of the line. For example, Wu (1998) defines a fuzzy set over the variable of percent land availability where X ranges from 0–100 percent with the function parameters $\alpha = 0$ and $\beta = 80$. Similarly, Rickel et al. (1998) defined the fuzzy set 'high precipitation' as a linear function of total precipitation.

Although these open form membership functions are simple to implement, they do not seem to be as popular in the GIS literature as other standard functions. A second major class of open form membership functions gives rise to S-shaped membership functions. The standard S-shape membership function is shown in Figure 3 as the right open shoulder S-shaped function S^+ (Novak 1992). The S^+ function was used by Charnpratheep et al. (1997) in their GIS-based study of landfill siting to define the membership function

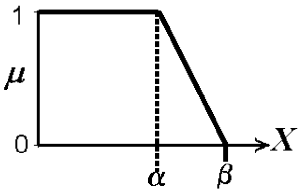
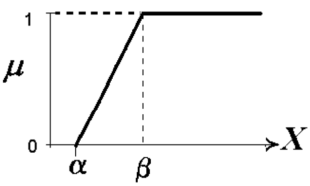
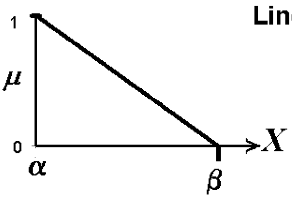
Description		References
Left Trapezoidal 	$\mu = max\left(min\left(1, \dfrac{\beta - x}{\beta - \alpha}\right), 0\right)$	Corne et al. (1999) De Bruin (2000) Robinson (2002) Wang (2000)
Right Trapezoidal 	$\mu = max\left(min\left(1, \dfrac{x - \alpha}{\beta - \alpha}\right), 0\right)$	Corne et al. (1999) Wu (1998) Zeng and Zhou (2001)
Linear 	$\mu = \begin{cases} \dfrac{\beta - x}{\beta - \alpha} & for \quad \alpha \le x \le \beta \\ 0 & for \quad otherwise \end{cases}$	Wu (1998) Rickel et al. (1998) Mackay and Robinson (2000)

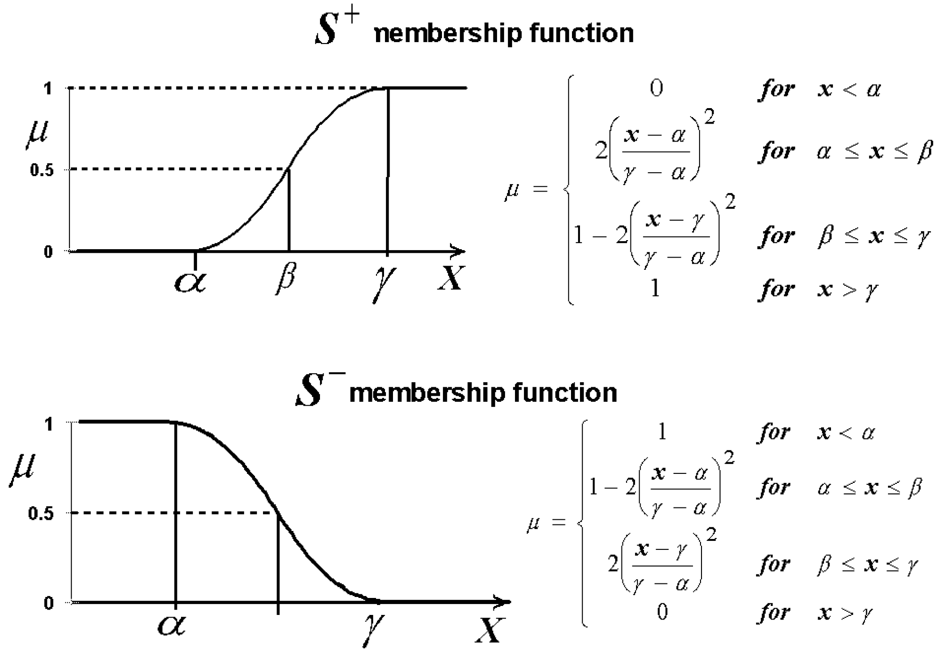
Figure 2 Open form trapezoidal and linear membership functions. References are studies that have used the corresponding membership function in a GIS problem domain

for *far*. The left open shoulder version of the S-shape function, S^-, was used in a GIS-based study to describe the *near_water* component in a fuzzy model of deer habitat (Rickel et al. 1998). Both are standard functions often seen in the fuzzy set literature (Figure 3).

Both Charnpratheep et al. (1997) and Rickel et al. (1998) used S-shaped curves to define grades of membership that described the nearness or farness of a location. To model μ as a function of distance from a point, Robinson (2000) and Leung and Yan (1997) used negative exponential functions of the general form of $\mu(x) = e^{-\beta x}$ where x in these two studies is distance between two points. Although an open form membership function, it does not result in an S-shaped function as is illustrated in Figure 4. In addition, its implementation is a bit more computationally intensive than the S-shaped functions in Figure 3.

The sigmoidal function, left or right open shoulder, also provides an S-shaped membership function (Figure 5). Power et al. (2001) used sigmoidal functions in combination with the closed form Gaussian membership function to fuzzify their variables for use in a fuzzy model of matching land use maps.

The generalized bell membership function produces another type of S-shaped function. It is commonly used in GIS related applications where the grade of membership is a function of distance from a location or object. Both Graniero and Robinson (2003) and DeGenst et al. (2001) have used bell curves where membership values are a function of distance from a location. DeGenst et al. (2001) use the form of the bell function



$$\mu = \begin{cases} 0 & \textbf{\textit{for}} \quad x < \alpha \\ 2\left(\dfrac{x-\alpha}{\gamma-\alpha}\right)^2 & \textbf{\textit{for}} \quad \alpha \le x \le \beta \\ 1-2\left(\dfrac{x-\gamma}{\gamma-\alpha}\right)^2 & \textbf{\textit{for}} \quad \beta \le x \le \gamma \\ 1 & \textbf{\textit{for}} \quad x > \gamma \end{cases}$$

$$\mu = \begin{cases} 1 & \textbf{\textit{for}} \quad x < \alpha \\ 1-2\left(\dfrac{x-\alpha}{\gamma-\alpha}\right)^2 & \textbf{\textit{for}} \quad \alpha \le x \le \beta \\ 2\left(\dfrac{x-\gamma}{\gamma-\alpha}\right)^2 & \textbf{\textit{for}} \quad \beta \le x \le \gamma \\ 0 & \textbf{\textit{for}} \quad x > \gamma \end{cases}$$

Figure 3 Left and right shoulder open S-shape membership functions

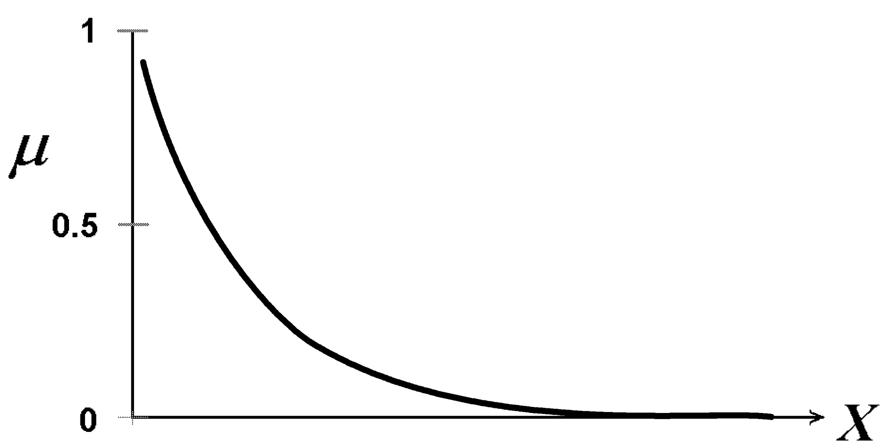
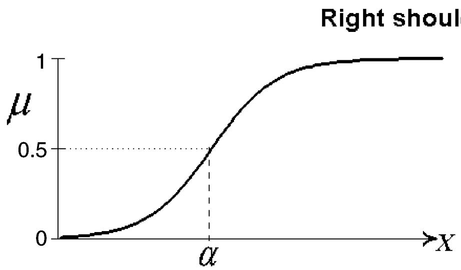


Figure 4 General form of the negative exponential membership function

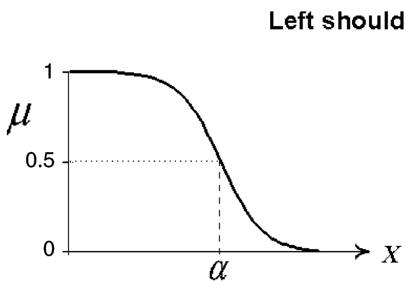
found in Burrough and McDonnell (1998) (see Figure 6b) while Graniero and Robinson (2003) use a more general form (Figure 6a).

3.1.2 Closed form membership functions

Several examples of closed form membership functions can be found in the GIS literature (e.g. Cheng et al. 2001, Cheng and Molenaar 1999, Power et al. 2001, Satur and Liur 1999, Stefanakis et al. 1999). A closed form membership function allows non-zero

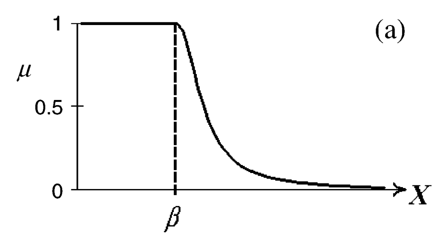


$$\mu = \frac{1}{1 + e^{-\beta(x-\alpha)}}$$

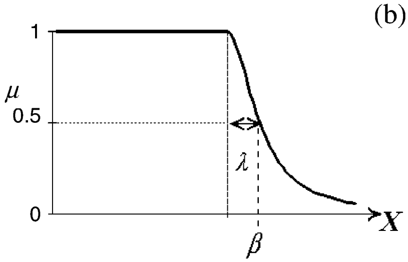


$$\mu = \frac{1}{1 + e^{\beta(x-\alpha)}}$$

Figure 5 Right and left shoulder sigmoidal membership functions where α is the crossover point and β controls the rate at which the $\mu(x) \rightarrow 1.0$



$$\mu = \begin{cases} \dfrac{1}{1 + \left(\dfrac{x-\beta}{\delta}\right)^{p}} & \textbf{for} \quad x > \beta \\ 1 & \textbf{for} \quad \textbf{otherwise} \end{cases}$$



$$\mu = \begin{cases} \dfrac{1}{1 + \left(\dfrac{x-\beta+\lambda}{\lambda}\right)^{p}} & \textbf{for} \quad x > (\beta - \lambda) \\ 1 & \textbf{for} \quad \textbf{otherwise} \end{cases}$$

Figure 6 Two versions of the generalized bell function used in GIS applications. In (a) the parameter δ determines how rapidly the function approaches 1.0. In (b), taken from Burrough and McDonnell (1998), β is at the crossover point ($\mu = 0.5$) and λ is the width of the transition zone where μ goes from 0.5 to 1.0. Thus, it controls the rate at which the function moves from 0 to 1, much like δ in (a) membership values only in a bounded interval. The function described earlier in Equation (2) describes a triangular membership function which is one of the simplest kinds of closed form membership functions. Other kinds of closed form membership functions include the trapezoidal, sigmoidal and generalized bell functions.

Although sometimes called a linear membership function (e.g. Burrough and McDonnell 1998, Stefanakis et al. 1999, Zeng and Zhou 2001), the triangular membership function $\Lambda : X \rightarrow [0, 1]$ is one of the most commonly used standard membership functions used in the fuzzy set literature. The membership function of Λ can be defined as:

$$\Lambda(x; \alpha, \beta, \gamma) = \mu_{\Lambda}(x) = \max(\min\left(\frac{x-\alpha}{\beta-\alpha}, \frac{\gamma-x}{\gamma-\beta}\right), 0) \tag{3}$$

where the parameters are as shown in Figure 7. Using water pressure in a glacial hydrologic modeling problem, Corne et al. (1999) used triangular membership functions, combined with a few left/right trapezoidal functions, defined over water pressure values to specify input/output variables for a rule-based fuzzy system. In a similar fashion, Stefanakis et al. (1999) used triangular membership functions to illustrate the difference in a fuzzy versus crisp approach to the classification of ground slope. Robinson (2002b) proposed using triangular membership functions to describe directions like north, southeast, west, etc. Satur and Liu (1999) used triangular functions extensively in their fuzzy cognitive mapping approach to GIS.

Trapezoidal and triangular membership functions are sometimes lumped together as linear membership functions (Stefanakis et al. 1999, Zeng and Zhou 2001). Similar to our definition of Λ, a closed form trapezoidal function $\Pi : X \rightarrow [0, 1]$ can be defined by:

$$\Pi(x; \alpha, \beta, \gamma, \delta) = \mu_{\Pi}(x) = \max(\min\left(1, \frac{x-\alpha}{\beta-\alpha}, \frac{\delta-x}{\delta-\gamma}\right), 0) \tag{4}$$

where the parameters are as shown in Figure 8.

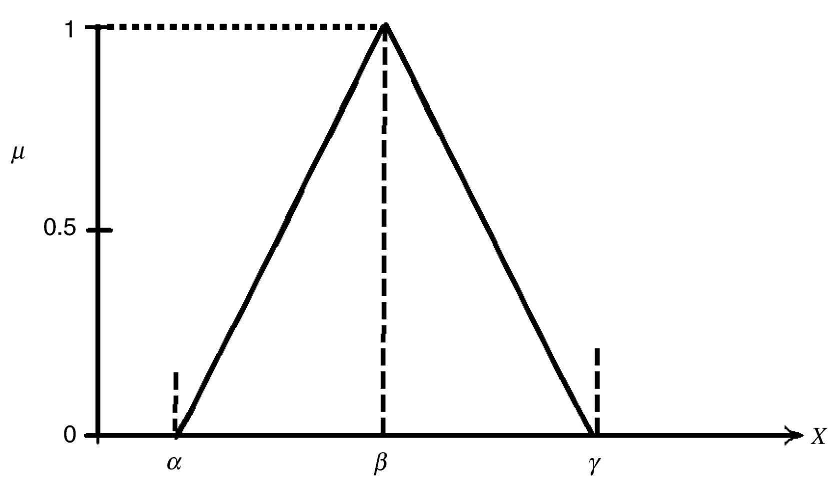


Figure 7 Triangular membership function that can be defined by

$$\mu(x) = \max(\min\left(\frac{x-\alpha}{\beta-\alpha}, \frac{\gamma-x}{\gamma-\beta}\right), 0)$$

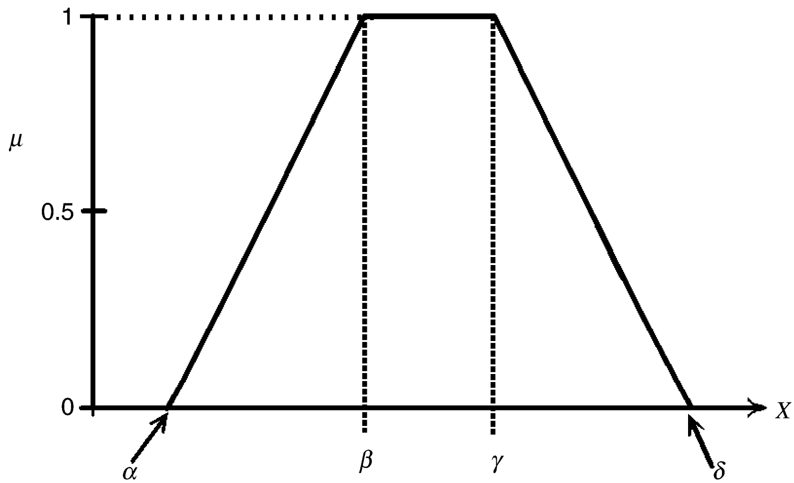


Figure 8 Closed form of the trapezoidal membership function defined as

$$\mu(x) = \max(\min\left(1, \frac{x-\alpha}{\beta-\alpha}, \frac{\delta-x}{\delta-\gamma}\right), 0)$$

Closed form trapezoidal membership functions seem to have been used sparingly in GIS applications. In their study of spatial query processing, Papadias et al. (1999) used trapezoidal functions to specify measures of similarity in distance and direction. Cheng et al. (2001) reported using trapezoidal functions to define fuzzy objects in a coastal landscape. Wong et al. (2001) formed trapezoidal membership functions in their development of a conservative fuzzy reasoning approach to spatial interpolation.

Despite the straightforward simplicity and elegance of the triangular and trapezoidal membership functions the Gaussian and sigmoidal functions seem to be more commonly used in GIS. DeGenst et al. (2001), Palanciogla and Beard (2001), and Power et al. (2001) use Gaussian membership functions in their GIS applications. It is common to define the Gaussian function $G : X \rightarrow [0, 1]$ as:

$$G(x; \alpha, \sigma, \beta) = \mu_G(x) = e^{-\left(\frac{x-\alpha}{\sigma}\right)^{2\beta}} \tag{5}$$

where α and σ can be related to the mean and variance of $\boldsymbol{X}$, respectively. The value of β is usually 1 and controls how flat G is at the upper limits. It should be noted that Palanciogla and Beard (2001) and Power et al. (2001) report using the MATLAB Fuzzy Toolbox where $G : X \rightarrow [0, 1]$ is defined by:

$$G(x; \alpha, \sigma, \beta) = \mu_G(x) = e^{-\left(\frac{x-\alpha}{2\sigma^2}\right)^{2}} \tag{6}$$

which is a slight variation on Equation (5). Not only does the Gaussian function produce a pleasing bell shaped curve (Figure 9), the parameters can easily be related to familiar concepts like mean and variance. This plus its availability in a generally available fuzzy software product like the MATLAB Fuzzy Toolbox (Hall and Hathaway 1996) have contributed to its being used in GIS (Robinson 2002a). On the other hand, it is easy to see that it is computationally more costly than the triangular or trapezoidal membership functions.

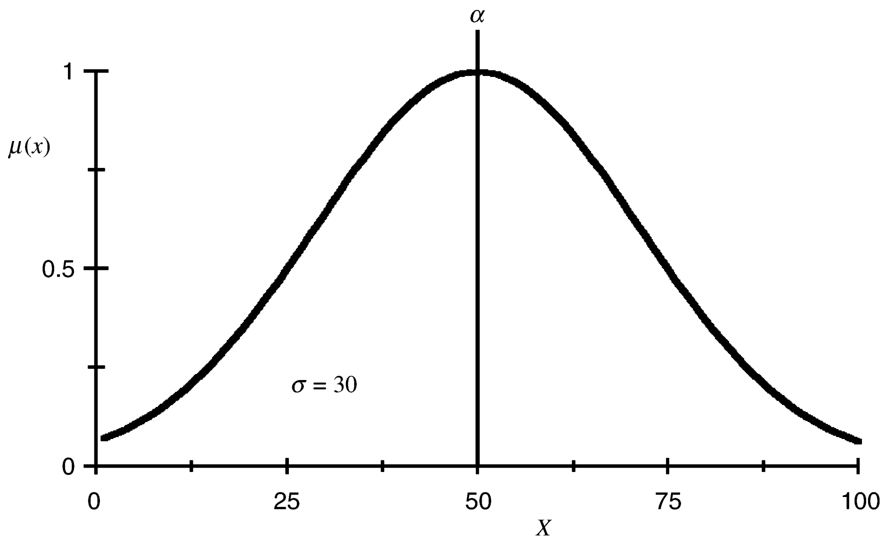


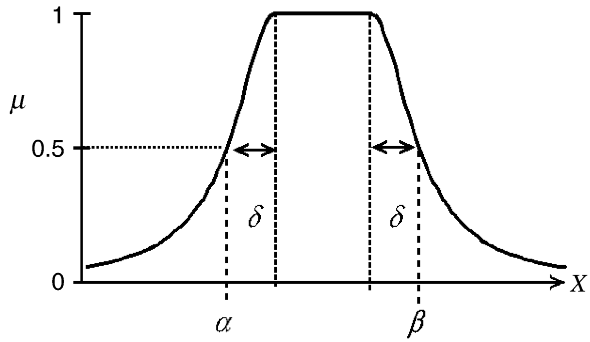
Figure 9 An example of a Gaussian membership function

Another class of bell-shaped closed form membership functions are occasionally referred to as sigmoid membership functions (e.g. Oberthur et al. 2000). For example, Macmillan et al. (2000) and Oberthur et al. (2000) use what they refer to as a symmetrical sigmoid membership function. The definition they present is essentially that described in Burrough and McDonnell (1998) which means it is a form of the generalized bell function that allows for parameterizing the rate at which the function makes the transition from 0 to 1 (see Figure 10).

Charnpratheep et al. (1997) and Rickel et al. (1998) take different approaches to forming bell-shaped functions. Basically, they combine left/right open shoulder S-shaped functions to form a bell shaped function. Thus, no matter which standard function, or combination thereof, is used there remains the necessity of having the function be applicable to the problem at hand. In other words, the parameters (e.g. α, β, δ, γ) must be specified so that the membership function makes sense in the context of the problem. Thus, rather than fit the problem to the membership function there have been a few cases where membership functions have been constructed that are unique to the problem at hand.

3.2 Problem Specific Membership Grades

Although standard functions are the most commonly used functions to assign membership values to elements, occasionally a problem is developed sufficiently so that a membership function particularly well-suited to a problem is formulated. That is the case where Wang and Hall (1996) developed a measure of membership in the boundary between two polygons using a function that was not dependent on standard functions such as those discussed above. Another example of developing a problem-specific membership function can be found in Brown (1998a) where using simple proportions he was able to assign the degree to which each tree species was a member of a forest type.



$$\mu = \begin{cases} \dfrac{1}{1+\left(\dfrac{x-\alpha-\delta}{\delta}\right)^{p}} & \textbf{\textit{for}}\ \ x < (\alpha-\delta) \\ 1 & \textbf{\textit{for}}\ \ (\alpha+\delta) \le x \le (\beta-\delta) \\ \dfrac{1}{1+\left(\dfrac{x-\beta+\delta}{\delta}\right)^{p}} & \textbf{\textit{for}}\ \ x > (\beta-\delta) \end{cases}$$

Figure 10 Fuzzy membership function that is a form of the generalized bell function, but is often referred to as a sigmoid, or symmetrical sigmoid, function

3.3 Specification of Standard Membership Functions

Yen (1999) noted that the specification and tuning of membership functions has been a source of much criticism leveled at the fuzzy logic approach in general and more specifically in GIS (Goodchild 2000). Often the parameterization of standard membership functions is summarily described as being chosen by experts (e.g. DeGenst et al. 2001, Satur and Liu 1999, Stefanakis et al. 1999). However, rigorous specification of membership functions from experts can be a difficult task (Zhu 1997a, 1999). In order to avoid the difficulty of determining membership values discrete levels of class membership rankings (1, 2, 3, 4, 5) were used by one research group to describe levels of accuracy for locations in their methodology for assessing the accuracy of thematic maps. Although this does indeed avoid the difficulty of membership function construction, the trade-off is that one loses the ability to make full use of fuzzy logic (Gopal and Woodcock 1994).

The challenging proposition of acquiring fuzzy memberships from domain experts has been addressed by automating the process to some extent in systems such as the Spatial Relations Acquisition Station (SRAS) (Robinson 2000), SOLIM (Zhu 1997a, 1999) and others (e.g. Foley et al. 1997). Each use a formal approach to acquiring fuzzy membership functions from experts. The SRAS uses intelligent human-machine question/answer sessions to specify the meaning of spatial relations such as 'near' or 'far.' The SOLIM effort is perhaps one of the few examples of an intensive knowledge acquisition effort that results in fuzzy representation of soil properties As part of an extensive knowledge acquisition methodology based on personal construct theory, membership curves are defined by the domain expert with the use of a graphical user interface (GUI) (Zhu 1997a, 1999).

As GIS applications become more complex, the use of fuzzy sets may require less reliance on human experts for defining membership functions. For example, where fuzzy logic were used for the purpose of controlling the selection of sub-models to be combined in a complex hydro-ecologic modeling system the membership functions were generated by the simulation model itself and used for self evaluation (Mackay and Robinson 2000).

3.4 Fuzzy Clustering

McBratney and Odeh (1997) and others (e.g. Altman 1994, Cheng et al. 2001, Stefanakis et al. 1999) have noted that another approach to grouping individuals into fuzzy sets (i.e. assigning membership values) is driven by the data at hand. This approach most often takes the form of the fuzzy c-means algorithm for clustering multivariate data into a finite number of fuzzy sets. The fuzzy c-means (FCM) algorithm is also known as fuzzy k-means (FKM) (e.g. Wilson and Burrough 1999, Burrough et al. 2000, 2001) or as the fuzzy ISODATA algorithm. FKM figured prominently in Wilson and Burrough's (1999) essay on 'new sneakers' for physical geography. Hence, it can be expected to continue to develop as an important method for the analysis of geographical phenomena.

FCM was originally developed by Dunn (1973) and later generalized by Bezdek (1974, 1981). To understand the basics of the FCM approach let us say that the set of all points to consider in our clustering problem is $X = \{x_1, x_2, \ldots, x_n\}$ ($\subset R^d$). Let us write $\mu_i : X \rightarrow [0, 1]$ for the i-th cluster where $i = 1, \ldots, c$ and μ_{ik} denotes the grade of membership of x_k in cluster μ_i. Let $U = \langle \mu_{ik} \rangle$ for the matrix of all membership values. The midpoint of μ_i is $\upsilon_i(R^d)$ and is computed as:

$$\upsilon_i = \frac{\sum_{k=1}^{n} (\mu_{ik})^m x_k}{\sum_{k=1}^{n} (\mu_{ik})^m} \tag{7}$$

where m, $1 \le m \le \infty$, is a parameter used as a weighting exponent which is application dependent. When $m = 1$ the fuzzy c-means coincides with the ISODATA algorithm and when $m \rightarrow \infty$ then all $\mu_{ik} \rightarrow 1/c$.

Membership of fuzzy clusters must fulfill the condition $\sum_i^c \mu_{ik} = 1$. A distance measure, $\| \ \|$ in R^d will affect the shape of the clusters. Euclidean distance is typically used, but other distance metrics can be used should the problem justify its use. The objective of the fuzzy c-means algorithm is to select μ_i so as to minimize the error function:

$$J = \sum_{i=1}^{c} \sum_{k=1}^{n} (\mu_{ik})^m \left\| x_k - \upsilon_i \right\|^2 \tag{8}$$

The computation proceeds by iteration. Specifically, the membership of an individual (x_k) with respect to a class mean (υ_i) in multivariate space is computed and is assigned to the nearest class i. The υ_i is recomputed. The procedure is repeated until the criterion of a stopping rule is met.

FCM did not receive much attention for GIS applications until the publication of the algorithm in Bezdek et al. (1984). Shortly after the publication of FCM, Robinson and Thongs (1986) demonstrated how it could be used to obtain a fuzzy classification of land cover from Landsat data. It was subsequently shown how the results of a fuzzy classification of land cover could be represented in a geographical database using concepts from the fuzzy database field (Robinson 1988). Since that early exploratory work, FCM has been used in a variety of problem domains. It has been used to develop classifications of land cover in suburban areas (Fisher and Pathirana 1994, Zhang and Foody 1998). In studies of vegetation, FCM has been used to model vegetation using remote sensing data as input (Foody 1996) and as a method to allocate sample plots to species clusters (Brown 1998b). FCM was also used in conjunction with geostatistical techniques by Ahn et al. (1999) to model soil variability from hyperspectral remote sensing data as a means of improving soils maps. Because FCM is a multivariate clustering algorithm and GIS databases have developed to the point where terrain as well as soil and even climate data maybe available, there has been exploration of the utility of incorporating terrain and other data with vegetation or remote sensing data to improve on landscape characterizations (Burrough et al. 2000, 2001). It has also been suggested that FCM could be used in lieu of a combined GIS knowledge acquisition methodology to populate similarity vectors that form the basis of a fuzzy soil inference method (Zhu 1997b).

Although FKM figures prominently in their paper on new approaches for physical geographers, it is but one approach to fuzzy clustering, albeit the one almost exclusively seen in GIS applications. There is now a rich variety of methods for fuzzy clustering (Höppner et al. 1999) that might prove interesting to explore in the context of specific GIS-related problems. Regardless of the specific method used, fuzzy classification techniques create overlapping but meaningful classes of attributes in data space (Wilson and Burrough 1999). Such output can then be represented, managed, queried, and combined with other data in a GIS environment capable of handling fuzzy geographical data.

4 Basic Concepts and Operations

The previous section discussed the main approaches used in GIS to specify the membership values of a fuzzy set. Specification of the membership values is only the beginning. Operations and measures are used to address a broad set of problems in GIS. This section presents many of the basic concepts of fuzzy sets and logic with an eye towards introducing the reader to some issues and measures that are not usually considered in the fuzzy sets and GIS literature. In addition, those concepts that have been used extensively in GIS are noted, providing the reader with a perspective on what has been used in the way of fuzzy sets in GIS. First, some basic concepts will be defined, followed by a discussion of operations on fuzzy sets.

4.1 Basic Concepts

Inclusion. $\boldsymbol{A}$ is said to be included in $\boldsymbol{B}$ if $\mu_A(x) \leq \mu_B(x)\ x \in \boldsymbol{X}$ which can be denoted in set notation as $\boldsymbol{A} \subset \boldsymbol{B}$.

Equality. $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal if $\mu_A(x) = \mu_B(x)\ x \in \boldsymbol{X}$. And is denoted as $\boldsymbol{A} = \boldsymbol{B}$.

Inequality. When at least one $x \in \boldsymbol{X}$ does not satisfy the equality condition $\mu_A(x) = \mu_B(x)$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ are not equal. Such inequality is normally denoted as $\boldsymbol{A} \neq \boldsymbol{B}$.

α-cut. One of the more important concepts in fuzzy sets is that of an ***α-cut***. Given the fuzzy set $\boldsymbol{A}$ defined on $\boldsymbol{X}$ and any number $\alpha \in [0, 1]$, the ***α-cut***, ${}^{\alpha}\boldsymbol{A}$, is the crisp set ${}^{\alpha}\boldsymbol{A} = \{x \mid \boldsymbol{A}(x) \geq \alpha\}$. A variant of an ***α-cut*** is the *strong* ***α-cut*** defined as

${}^{\alpha}\boldsymbol{A} = \{x \mid \boldsymbol{A}(x) > \alpha\}$. In other words, the ***α-cut*** of A is the crisp set ${}^{\alpha}\boldsymbol{A}$ that contains all the elements of ***X*** whose membership grades in ***A*** are greater than or equal to (or only greater than in case of the strong ***α-cut***) the specified value of α (Klir and Yuan, 1995). Figure 11 illustrates the decomposition of a fuzzy set (***A***) using the concept of the ***α-cut***. An important implication of this decomposition example is that a fuzzy set, such as A, can be expressed in term of the concept of ***α-cuts*** without resorting to the membership function. This important implication has been largely ignored in the GIS applications of fuzzy sets. One exception is the development of the fuzzy object-oriented system FOOSBALL that exploits ***α-cuts*** to implement a more efficient way of representing and querying uncertain spatial data (Morris and Petry 1998).

The *support* of a fuzzy set A within a universal set ***X*** is the crisp set that contains all the elements of ***X*** that have nonzero membership grades in ***A***. Note that this means the support of ***A*** is exactly the same as the strong ***α-cut*** of ***A*** for $\alpha = 0$. In the literature special symbols such as supp(***A***), or S(***A***) are sometimes used to denote the support of ***A***. Due to its clear relation to the ***α-cut***, Klir and Yuan (1995) used the natural symbol ${}^{0+}\boldsymbol{A}$. The 1-*cut*, ${}^{1}\boldsymbol{A}$, is called the *core* of ***A***.

The *height* of ***A***, $h(\boldsymbol{A})$, of fuzzy set ***A*** is the largest membership grade obtained by any element in that set. Formally it can be expressed as $h(\boldsymbol{A}) = \sup_{x \in X} A(x)$. The fuzzy set is called normal when $h(\boldsymbol{A}) = 1$ and subnormal when $h(\boldsymbol{A}) < 1$. It can also be viewed as the supermum of α for which ${}^{\alpha}A = \emptyset$ (Klir and Yuan 1995).

The ***α-cut*** has played in important role in a variety of applications of fuzzy sets in GIS. The ***α-cut*** has played in important role in Zhan's (1998) representation of a fuzzy region as well as in the development of an fuzzy query system based on an object-oriented model (Morris and Petry 1998). In complex GIS modeling applications, Robinson (2002b) has shown how they can be used to control the movement of mobile objects in spatially explicit ecological simulations and Mackay et al. (2003) use it in a system for automating the parameterization of a complex, GIS-based

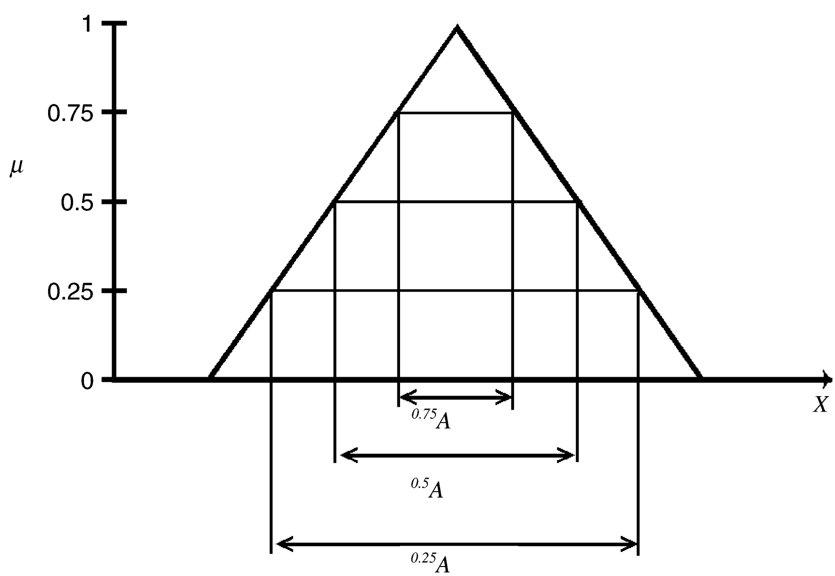


Figure 11 Decomposition of a fuzzy set using *α-cuts*

hydro-ecological model. Thus, the ***α-cut*** is becoming important in the application of fuzzy sets to GIS problems.

In crisp sets the number of members of a finite set A is called the cardinality of A and is denoted by $|A|$. The scalar cardinality of fuzzy set $\boldsymbol{A}$ defined on a finite universal set $\boldsymbol{X}$ is the summation of the membership grades of all the elements of $\boldsymbol{X}$ in $\boldsymbol{A}$. Thus, it is often denoted as:

$$|\boldsymbol{A}| = \sum_{x \in X} \mu_A(x) \tag{9}$$

The cardinality of the ***α-cut*** of A is denoted as $|{}^{\alpha}\boldsymbol{A}|$. Sometimes $|A|$ or $|{}^{\alpha}\boldsymbol{A}|$ are referred to as the *sigma count* of $\boldsymbol{A}$ (Klir and Yuan 1995). Although cardinality is not usually found in the GIS literature, it has begun to appear. In the work on querying probablistic land cover data, DeBruin (2000) used $|{}^{\alpha}\boldsymbol{A}|$ to compare results from crisp versus fuzzy query responses. Robinson (2002b) suggested the use of $|{}^{\alpha}\boldsymbol{A}|$ as the basis of one method for controlling the behavior of mobile animal objects in a spatially explicit ecological model. The $|{}^{\alpha}\boldsymbol{A}|$ plays a central role in defining the *U*-uncertainty used in Mackay et al. (2003) as part of an effort to automate the parameterization land surface models embedded in a GIS.

4.2 *Operations on Fuzzy Sets*

Originally proposed by Zadeh (1965), the following operations, or connectives, are widely used in GIS applications:

$$\begin{aligned} \neg a &= 1-a && \textit{complement} \\ a \wedge b &= \min\{a, b\} && \textit{intersection} \\ a \vee b &= \max\{a, b\} && \textit{union} \end{aligned}$$

Let us consider each of these connectives in more detail.
Complementation. The fuzzy sets $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be complementary if $\mu_B(x) = 1 - \mu_A(x)\ x \in \boldsymbol{X}$. This can be denoted as $\boldsymbol{B} = \bar{A}$ or $\bar{A} = \boldsymbol{B}$. The complement operator is generally regarded as corresponding to the set theoretic operator NOT.

Although rarely, if ever, seen in GIS application, involutive fuzzy complements are a class of complements that have played a role in other practical applications of fuzzy sets (Klir and Yuan 1995). In Sugeno class complements defined by $c_{\lambda}(A) = \dfrac{1-A}{1+\lambda A}$, where $\lambda \in (-1, \infty)$, a particular involutive fuzzy complement is obtained for each λ. Another involutive complement is the Yager class of complements defined as $c_w(A) = (1 - A^w)^{1/w}$ where $w \in (0, \infty)$. In similar fashion, for each w a particular involutive fuzzy complement is obtained. Note that when $w = 1$ the function becomes the classical fuzzy complement. There has been no consideration of either the Sugeno or Yager class complements in the GIS literature. Although use of complementation has not played as major a role in development of fuzzy sets in GIS applications to date, as the problems in GIS make more demands on intelligent solutions the use of complementation, especially involutive fuzzy complements may become more common.

Functions that qualify as fuzzy intersections and unions are referred to as t-norms and t-conorms, respectively (Klir and Yuan 1995). Jiang and Eastman (2000) discussed t-norms and t-conorms briefly in their paper on multi-criteria evaluation. Although the use of connectives is ubiquitous in the application of fuzzy sets to GIS problems, theirs is one of the few works in GIS to explicitly refer to t-norms/conorms. The most commonly used t-norms/conorms in the GIS literature are those originally proposed by Zadeh (1965), namely:

$$a \wedge b = \min(a, b) \quad \text{intersection, or logical AND} \qquad (10a)$$
$$a \vee b = \max(a, b) \quad \text{union, or logical OR} \qquad (10b)$$

The intuition for these connectives is obvious as is their equivalence to the crisp, or Boolean, logical connectives. However, it is clear that these connectives reflect worst and best case characterizations. Although equivalent to crisp logical connectives, their reflection of the worst/best case characterizations is also a disadvantage since it means that an outcome might remain unchanged even if we modify some value, e.g. min(0.7, 0.5) is the same as min(0.8, 0.5). Thus, they do not allow for compensation. That is, an element of the intersection of two sets cannot compensate a low degree of belonging to one of the intersected sets by a higher degree of belonging to another of them. If we desire any change in *a* or *b* to be effective we can use compensatory operators such as:

intersection		union	
algebraic product:	$a \cdot b$	algebraic sum:	$a + b - ab$
bounded difference:	$\max(0, a + b - 1)$	bounded sum:	$\min(1, a + b)$

In the connectives above, the outcome depends only on *a* and *b*, i.e. there are no additional parameters. Adding parameters introduces several interesting and potentially useful classes of connectives. Klir and Yuan (1995) provide a table with the definition of many such connectives. Among the most commonly cited, are the connectives proposed by Yager (1980):

$$a \wedge b = 1 - \min\{[(1 - a)^p + (1 + b)^p]^{1/p}, 1\}, \quad \textit{for } p > 0 \qquad (11a)$$
$$a \vee b = \min\{[a^p + b^p]^{1/p}, 1\}, \quad \textit{for } p > 0 \qquad (11b)$$

It is interesting to note that if $p \rightarrow \infty$ then Yager's connectives converge to Zadeh's connectives. If $p = 1$ then Yager's connective is the same as defined by Lukasiewczs and since drastic products and sums result when $p = 0$ (Figure 12e) it is common practice to set $p \geq 1$. Note how the Yager connective for intersection varies with p (Figure 12c–f) compared to the non-compensatory intersection (Figure 12b). Although one of the reasons for using fuzzy sets in GIS is to compensate for the degree to which locations may belong to more than one class of phenomena, it is relatively rare to see the use of compensatory connectives.

4.3 Aggregation Operators

Aggregation operations on fuzzy sets are operations that combine several fuzzy sets in a desirable manner to produce a single fuzzy set (Klir and Yuan 1995). It is easily seen that fuzzy intersections and unions qualify as aggregation operations on fuzzy sets. Their property of associativity provides the mechanism for extending their definition to any number of arguments, not only the two used in the definitions above. Aggregation operators other than intersection and union have also been used in GIS applications. Charnpratheep et al. (1997) proposed the convex combination model (see Kandel 1986 for further details) because they argued it has an advantage over the fuzzy min-operator intersection with respect to the ability to integrate preferences into their landfill location screening process. In other words, a form of compensatory combination that was defined as:

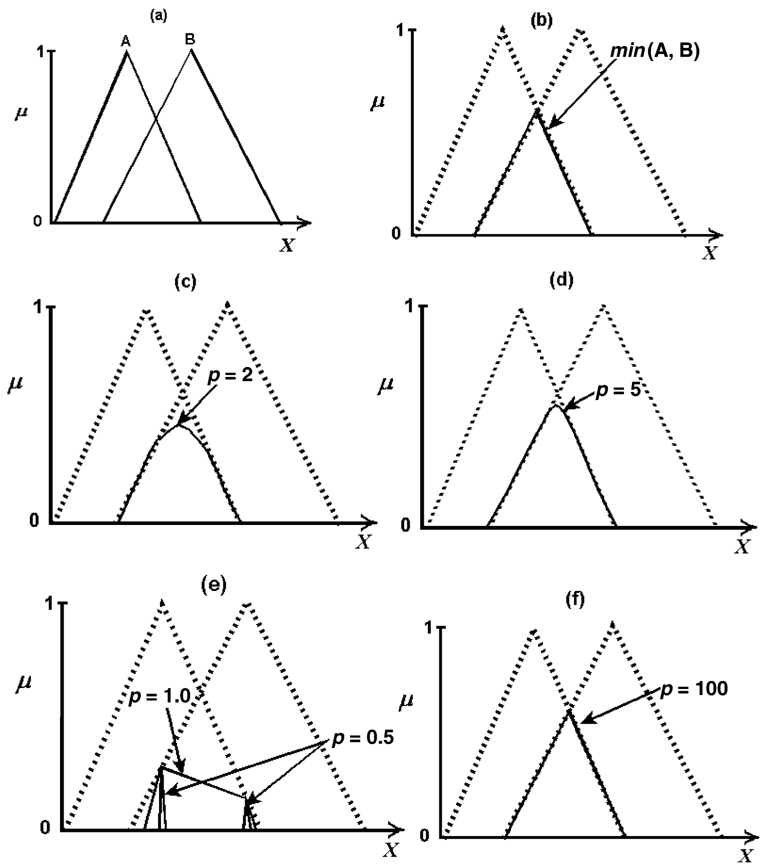


Figure 12 Intersection operator on fuzzy sets A and B (a), using the standard min (b), and the Yager connective with differing values of *p* (c–f)

$$F = \sum_{i=1}^{m} w_i \mu_i(x_{ij}) \tag{12}$$

where $\mu_i(x_{ij})$ is the membership value of the j^{th} grid cell in data layer i, and w_i is the weight assigned to layer i and where $\sum_{i=1}^{m} w_i = 1$ and $w_i > 0$. It can easily be seen that the weights are compensating for the relative 'importance' each layer should be given in the combination.

Several studies have used variations of the weighted aggregation operators. Using terminology from Burrough and McDonnell (1998) that lumps together many kinds of fuzzy relations into the term 'joint membership function' (JMF), Oberthur et al. (2000) used a weighted sum, or convex combination, in their study to combine sets of soil properties into indices of soil quality. Jiang and Eastman (2000) used the ordered weighted averaging operator that they modified from the standard formulation (see Klir and Yuan 1995 for a definition) to include weights that apply to the ranked criteria after the application of the usual criterion weights. Their study showed that different aggregation approaches can yield strikingly different results. Both Charnpratheep et al. (1997) and Jiang and Eastman

(2000) illustrate the subjectivity inherent in the formulation of a weighting scheme. Their studies also highlight the point that decision rules can be a source of uncertainty in GIS. As illustrated by the sensitivity analysis of a weighted membership function used in a fuzzy rule-based GIS for real estate decision making, small differences in subjective weights can lead to large variations in the results (Zeng and Zhou 2001).

5 Fuzzy Relations

The presence, or absence, of association, interaction or interconnectedness between elements of two or more sets is represented by a *crisp relation*. Rather than presence/absence (i.e. 1/0) of association, degrees of association can be represented by membership grades in a fuzzy relation in much the same way as degrees of set membership are represented in a fuzzy set. Thus, the classical notion of relation can be generalized into matter of degree as a fuzzy relation. For example, a fuzzy relation *Petite* between height and weight of a person describes the degree to which a person with a specific height and weight is considered petite (Yen 1999). In an ecological application, locations characterized as *midslope deciduous* forest may be a fuzzy relation that describes the degree to which a raster cell with a water flow accumulation value and a certain density of deciduous trees is considered *midslope deciduous*.

A fuzzy relation ***R*** between variables x (e.g. water flow accumulation value) and y (e.g. density of deciduous trees), whose domains are X and Y, respectively, is defined by a function that maps ordered pairs in $X \times Y$ to their degree in the relation, i.e. $\mu_R : X \times Y \to [0, 1]$. More generally, a fuzzy *n*-ary relation is characterized by a function $\mu_R : X_1 \times \cdots \times X_m \to [0, 1]$, where X_i are the universes of discourse and $X_1 \times \cdots \times X_m$ is the product space. Just as a classical relation can be viewed as a set, a fuzzy relation can be viewed as a fuzzy subset. From this perspective, the mapping, $\mu_R : X_1 \times \cdots \times X_m \to [0, 1]$, is equivalent to the membership function of a multidimensional fuzzy set. Note that since fuzzy relations are fuzzy sets, the operations of fuzzy sets (union, intersections, etc.) can be applied to them. For simplicity, let the range of accumulation flow values be $\boldsymbol{f} = \{2, 50, 200, 1024\}$. And let the range of deciduous tree density values, of interest to us be $\boldsymbol{d} = \{10, 25, 50, 90\}$. Thus we can express the fuzzy relation, *midslope deciduous* (***R***) in matrix form as:

$$R = \begin{array}{r} \\ 2 \\ 50 \\ 200 \\ 1024 \end{array} \begin{array}{c} \begin{array}{cccc} 10 & 25 & 50 & 90 \end{array} \\ \begin{bmatrix} 0.1 & 0.1 & 0.3 & 0.6 \\ 0.2 & 0.3 & 0.6 & 0.8 \\ 0.1 & 0.1 & 0.3 & 0.5 \\ 0.1 & 0.1 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

Each entry indicates the degree to which a location with the corresponding values of $\boldsymbol{f}$ and $\boldsymbol{d}$ is considered to be *midslope deciduous*. Having defined the fuzzy relation, we can answer two kinds of questions:

1. What is the degree to which a location with a specific density of deciduous forest and waterflow accumulation value is considered to be *midslope deciduous*?
2. What is the possibility that a midslope deciduous location has a specific pair of accumulation and density measures?

Despite their potential utility, there has been little explicit use of fuzzy relations described in the GIS literature. Although Zhan (1998) refers to it as a fuzzy subset, his definition of geographic fuzzy region is consistent with the general definition of a fuzzy relation. Specifically, a fuzzy geographical region R can be defined by the membership function $\mu_R : X \times Y \times \Phi \rightarrow [0, 1]$ where X is a set containing all possible x coordinates, Y is a set containing all possible y coordinates and Φ is a set containing all possible values of attribute ϕ. Each point(x, y) within the region is assigned a membership value for attribute $\phi : \mu_R(x, y, \phi), \phi \in \Phi$. Using *α-cuts* and fuzzy relations, he demonstrated a method for determining the eight binary (i.e. crisp) topological relations between fuzzy regions (Zhan 1998). It remains to be seen how future, more intelligent GIS applications will make use of fuzzy relations.

By adding restrictions to fuzzy relations we can obtain various types of fuzzy relations. Examples of these are similarity relations and fuzzy order relations (Terano et al. 1992). In GIS applications similarity relations have been shown to be useful while fuzzy order relations have not yet been explicitly employed. Furthermore, in the GIS literature there is reference to similarity vectors (Zhu 1997), or similarity measures (Papadias et al. 1999). Such vectors, or measures, should not be confused with similarity relations as they generally lack the specific characteristics that define fuzzy similarity relations.

One of the earliest examples of similarity relations used in GIS was in Robinson and Strahler (1984) where they used the landcover similarity relation (L):

$$L = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1.0 & 0.6 & 0.4 & 0.2 \\ 0.6 & 1.0 & 0.2 & 0.4 \\ 0.4 & 0.2 & 1.0 & 0.3 \\ 0.2 & 0.4 & 0.3 & 1.0 \end{bmatrix} \end{array} \quad \begin{array}{l} x_1 = \text{Wet Meadow} \\ x_2 = \text{Dry Meadow} \\ x_3 = \text{Riparian Hardwoods} \\ x_4 = \text{Sage Brush} \end{array}$$

to illustrate how the work in fuzzy databases (Buckles and Petry 1982, Petry 1996) could be applied to situations in GIS. The fuzzy relation $\boldsymbol{L}$ is a fuzzy relation that is restricted to meet the following conditions:

$$\begin{array}{ll} \textit{reflexivity}: & \mu_L(x, x) = 1 \\ \textit{symmetry}: & \mu_L(x, y) = \mu_L(y, x) \\ \textit{transitivity}: & \vee_y\{\mu_L(x, y) \wedge \mu_L(y, z)\} \le \mu_L(x, z) \end{array}$$

It is important to realize that similarity relations are often called fuzzy equivalence relations. In other words, they are useful when we want to ‘think’ about the equivalence of classes. Robinson (1988) showed how they could be used in the context of simple queries on a land-related database and it was further developed in Petry’s (1996) discussion of GIS applications in fuzzy databases. Fuzzy similarity relations figure prominently in the development of a rule-based approach to conflation that combines fuzzy sets with evidential reasoning (Cobb et al. 1998). This line of research was further developed to address significant problems in distributed spatial query systems (Cobb et al. 2000). Exploiting the equivalence relationships formalized by fuzzy similarity relations and *α-cuts*, Morris and Petry (1998) describe the FOOSBALL fuzzy object-oriented framework and query system for spatial databases.

6 Fuzzy Control Systems

The development of widely available computational tools like the Fuzzy Logic Toolbox for MATLAB and Fuzzy Systems Toolbox (Hall and Hathaway 1996) has led to the

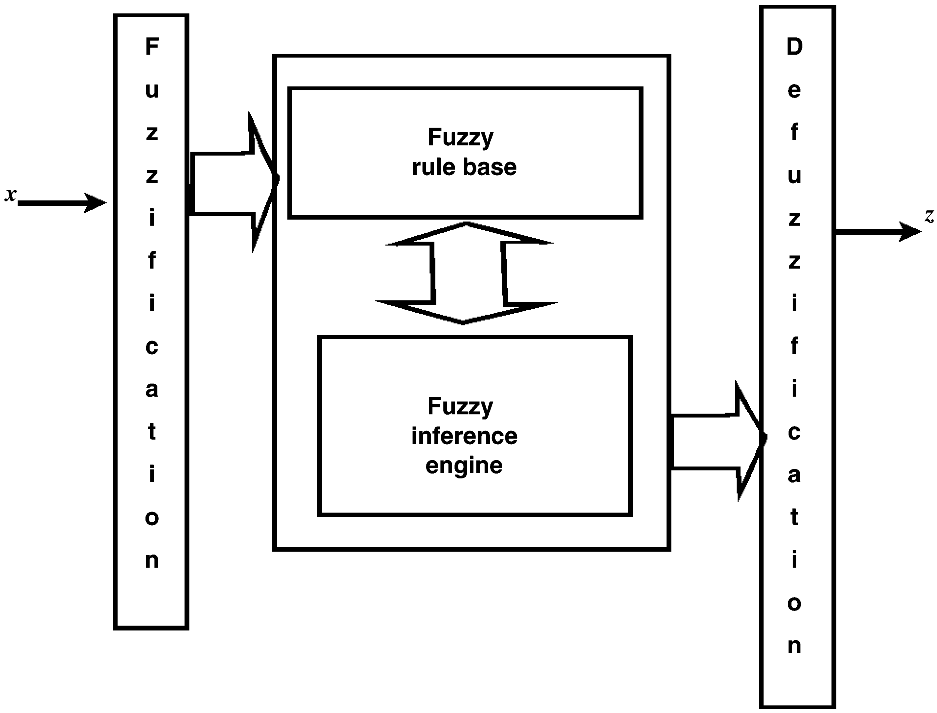


Figure 13 Components of a fuzzy controller (adapted from Klir and Yuan 1995, Bohlin et al. 2000)

application of fuzzy controller systems theory to GIS problems. Fuzzy control is conclusion based on rules. A fuzzy IF-THEN rule $\mathfrak{R}$ is symbolically expressed as:

$\mathfrak{R}$: IF (fuzzy criteria) THEN (fuzzy conclusion)

Such a rule can be seen as a causal relation between measurements and control values of the process. For example, Palanciogla and Beard (2001) had as one of their rules that:

$\mathfrak{R}_1$: IF (early morning, weekday, single lane, 25mph, low volume, bad weather) THEN (travel time per mile = VERY LOW)

Of course, each of the linguistic variables in fuzzy criteria had a membership function associated with measurements. For example, the time of day would be the measurement upon which a membership function would be defined to describe *early morning*.

As shown in Figure 13, a fuzzy controller consists respectively of fuzzification, fuzzy rule base, fuzzy inference mechanism, and defuzzification. The fuzzy rule base contains all the fuzzy rules needed for an inference mechanism to provide conclusions.

In a functional description, it can be described as a function $z = f(x)$, where x is a measurement (insignal) and z is the computed control value (outsignal). Generally $f(x)$ is given as:

$$z = defuzz[DISJ[\Phi\{CONJ_j[A_{ij}(x_j)], Z_i\}]] \tag{13}$$

where a contol value z is computed from measurements x_j. In the first step, measurements x_j are fuzzified to $A_{ij}(x_j)$ representing the truth value of "x_j *is* A_{ij}." A_{ij} is the linguistic variable in the *i-th* rule of the *j-th* measurement. Fuzzified values are then combined by a logical conjunction operator ($CONJ_i$) providing conjunction over all the values $A_{ij}(x_j)$. Thirdly, Φ is an implication operator that carries the conjunction value over to combine with the output membership function of the rule. Following that, a logical disjunction operator $DISJ_i$ computes the membership function on which finally a defuzzification (*defuzz*) is applied to provide the outsignal (z). There have been several inference methods proposed, but one of the most commonly used is that of Mamdani (see Mamdani and Assilian 1975). The Mamdani inference method is the one used by both Palanciogla and Beard (2001) and Power et al. (2001) in their application of fuzzy control systems to GIS problems. The Mamdani inference method produces fuzzy sets as inputs to the defuzzification process. There are several different methods for defuzzification (Klir and Yuan 1995, Bohlin et al. 2000). The most common method of defuzzification is the center of area method. It is the method used by Power et al. (2001) and Palanciogla and Beard (2001) in their GIS-related applications of fuzzy control systems.

While Palanciogla and Beard (2001) demonstrate fuzzy logic as a method for space-time reasoning about moving objects, they did not offer conclusions regarding its performance in comparison with a crisp (Boolean) approach. Power et al. (2001) do provide such an analysis, showing how the use of the MATLAB fuzzy inference system for hierarchical pattern matching of land use maps can be used to arrive at better maps of land use change. Their results clearly demonstrated the superiority of the fuzzy sets approach over the traditional crisp (Boolean) approach. They effectively showed that a fuzzy local polygon-by-polygon land use comparison is less affected by possible map registration problems than are the traditional methods. More importantly, they note that the fuzzy land use change possibility maps provide a better interpretation of the land use agreement characteristics of a dataset than did the crisp (Boolean) maps.

7 Conclusions

This paper has reviewed many of the major aspects of fuzzy set theory and its linkage to GIS. It may seem to the reader that an inordinate amount of space was allocated to the issue of membership functions. However, the real advantages of fuzzy logic can not be realized without membership functions upon which to operate. Thus, they are fundamental to all that followed. In fact, one of the main features of the fuzzy paradigm that makes it superior to the classical (nonfuzzy) paradigm is how it allows us to express irreducible observation and measurement uncertainties in their various manifestations and make these uncertainties intrinsic to empirical data (i.e. using grades of membership). When fuzzy data are processed their intrinsic uncertainties are processed as well, and the results obtained are more meaningful, than their counterparts obtained by processing the usual crisp data (Klir and Yuan 1995). This is a feature often noted in studies where locations, or land, are being evaluated. Examples include landfill suitability (Charnpratheep et al. 1997), land suitability (Oberthur et al. 2000), real estate properties (Zeng and Zhou 2001), spatial relations in habitat studies (DeGenst et al. 2001), or land use change (Power et al. 2001).

Another feature of fuzzy sets is its *greater expressive power*. For example, Corne et al. (1999) compared computational intelligence techniques, including fuzzy logic, in the modeling of subglacial water systems. They emphasized that even though their fuzzy logic model was not the most efficient technique, it did produce the most easily interpreted physical model. In addition, it is often noted that fuzzy sets theory has the capability to capture and deal with the meanings of sentences expressed in natural language. Work on acquiring linguistic terms as spatial relations (Robinson 2000) and developing natural language query abilities for GIS (Wang 2000) are among the examples of how this kind of expressive power can be used in a GIS setting.

It has been said that the fuzzy paradigm has the ability to capture human commonsense reasoning and decision making (Klir and Yuan 1995). Zhu's (1997a, 1999) work clearly demonstrates the utility of the paradigm for capturing and representing spatially explicit soils knowledge from human experts. Zhu and Mackay (2001) investigated the impacts of detailed and spatially continuous soils information on hydro-ecological model performance over watersheds of a medium spatial scale by comparing simulated hydro-ecological responses based on detailed soil information derived from a fuzzy logic-based inference approach with those based on soil information derived from a conventional soil map. Results from the fuzzy model tended to be superior in the lumped parameter model while the realism of results from the fuzzy model was less pronounced using the distributed model approach. It is possible to interpret their results as supporting the notion that the fuzzy paradigm is particularly useful the more complex the problem. The more complex the problem, especially those with spatial subtleties, the better fuzzy systems tend to perform (Robinson 2002a).

Perhaps one of the more important features of the fuzzy paradigm is its potential to offer greater resources for *managing complexity and controlling computational cost*. The general experience is that the more complex the problem involved, the greater the superiority of fuzzy methods (Klir and Yuan 1995, Yen 1999). This seems to hold in the domain of GIS as well. For example, we see fuzzy sets used to automatically reason about semantic error in complex geographic modeling situations (Mackay and Robinson 2000), automatically parameterize complex process models (Mackay et al. 2003), control spatial interpolation processes (Wong et al. 2001), manage complex queries in a heterogeneous distributed GIS environment (Cobb et al. 2000), and control the spacing of sample points along a transect (Graniero and Robinson 2003).

As GIS-related applications increase in their levels of complexity and sophistication it is reasonable to expect fuzzy sets to play a major role in their development. A review of fuzzy sets in GIS indicates that, in many cases, fuzzy sets have provided cost effective solutions to GIS problems (Robinson 2002a). Robinson (2002a) also speculated that the contribution of fuzzy sets to cost-effective solutions in GIS will increase as the complexity of GIS problems escalates. Furthermore, in their vision of the future of physical geography, Wilson and Burrough (1999) include fuzzy classification techniques as one of the methods that future physical geographers should be comfortable using. Thus, this overview of the basics of fuzzy sets provides a brief glimpse at what is now a rich, developing field that will certainly be useful in solving the GIS problems of the future. It is hoped that this brief perspective on fuzzy sets and GIS has provided the reader with an appreciation of the depth and diversity of tools available in fuzzy sets theory that can be applied to GIS problems.

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