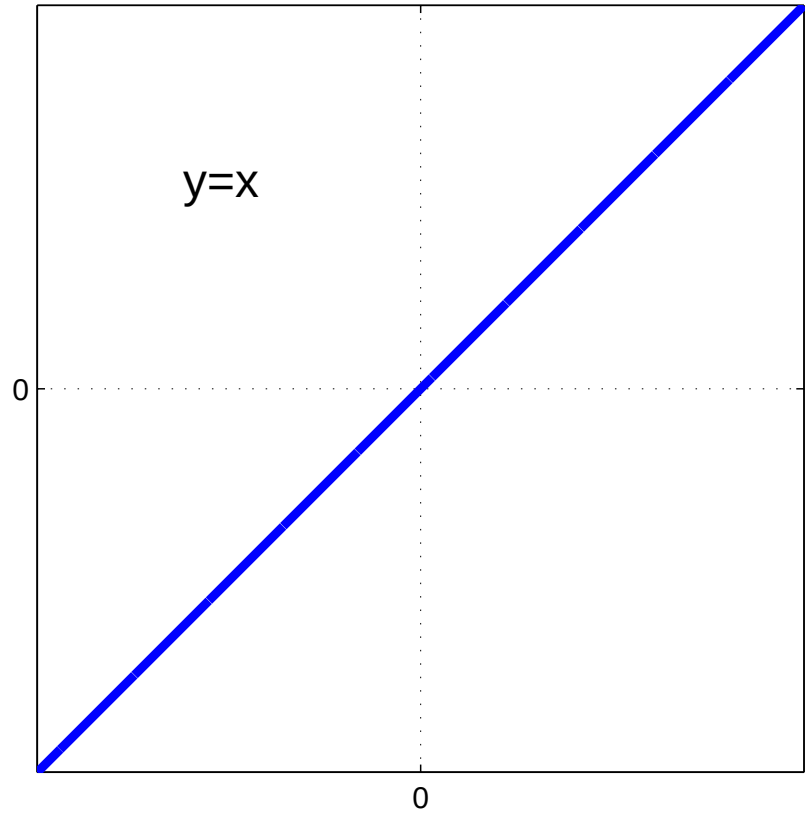
A Catologue of Beautiful Functions



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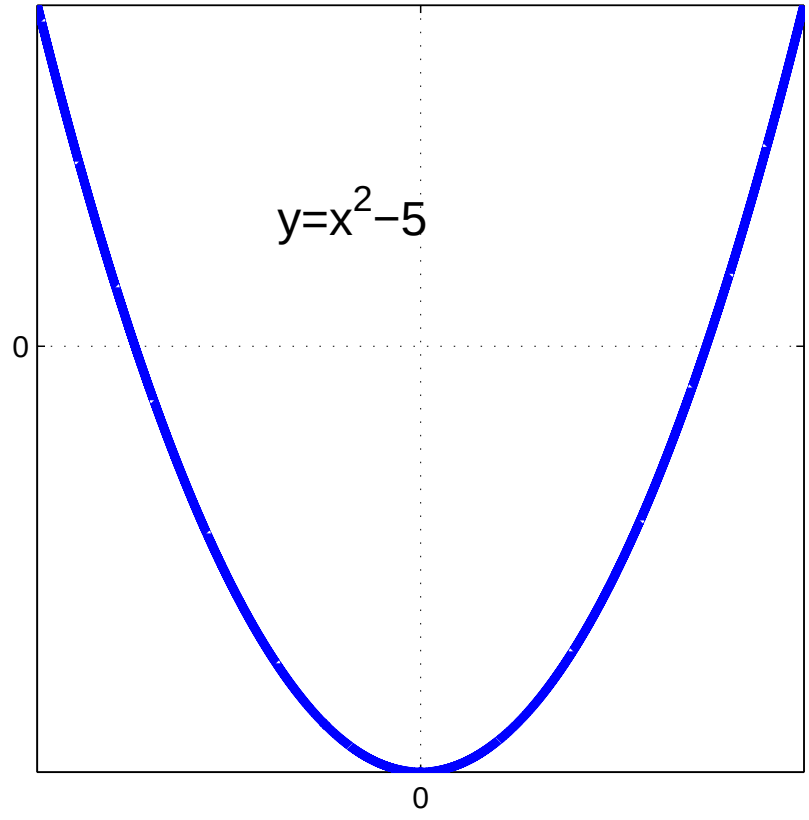

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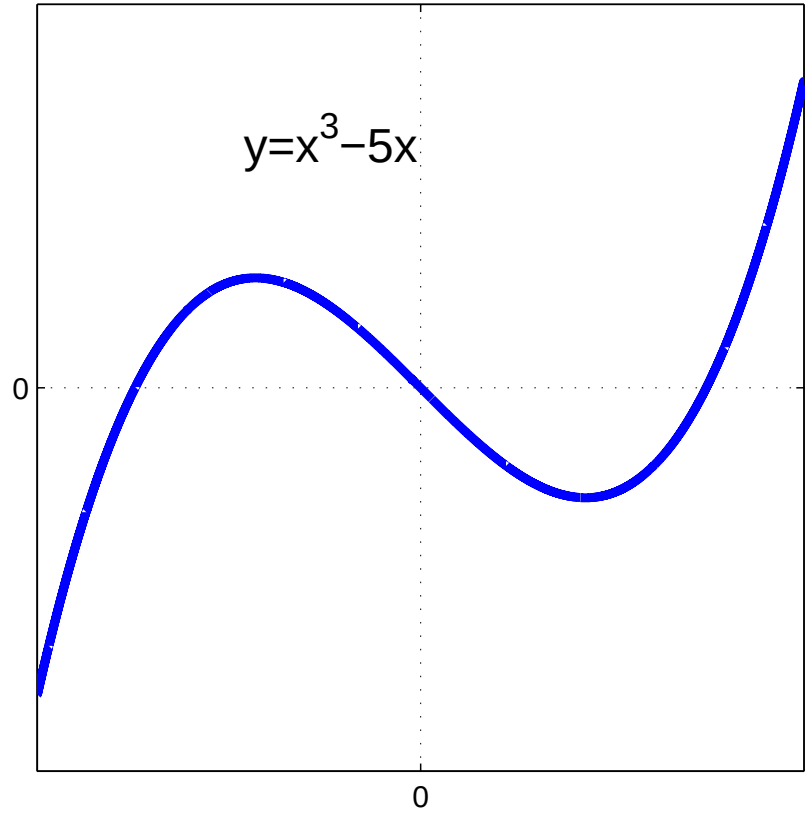

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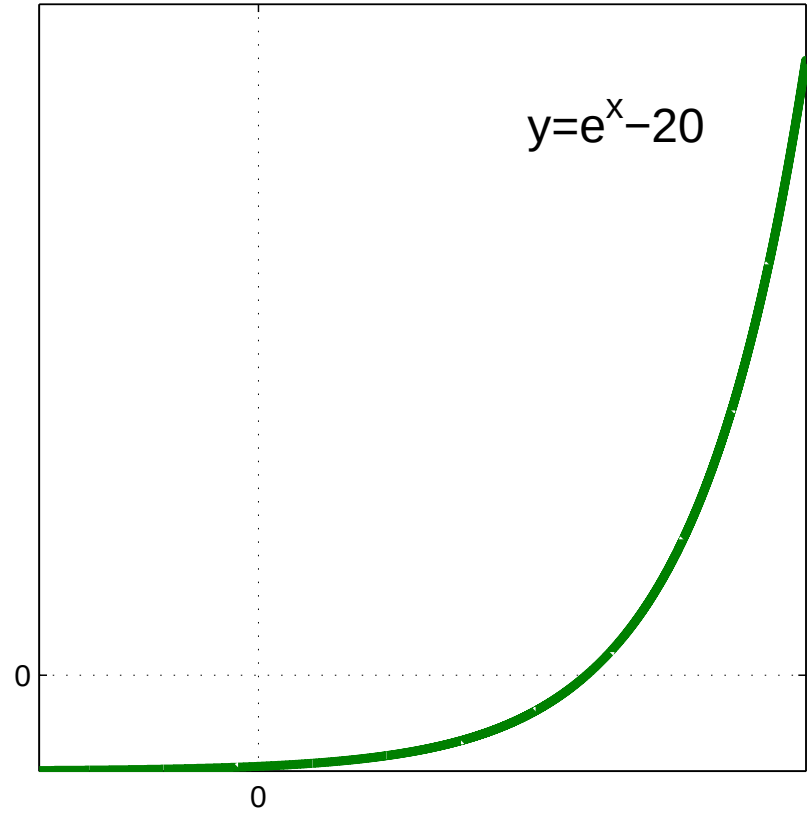

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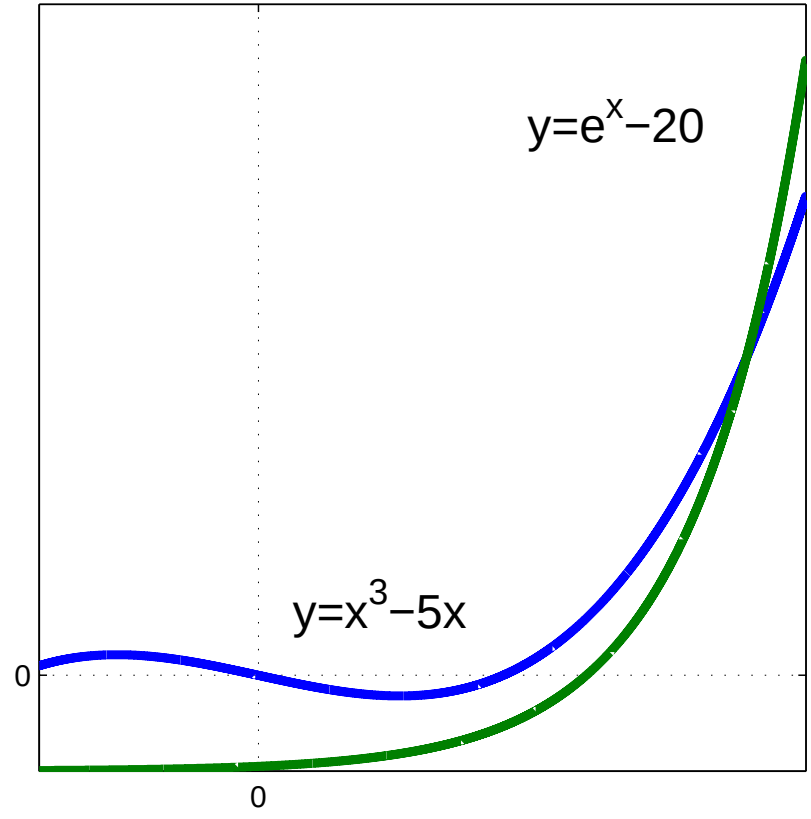

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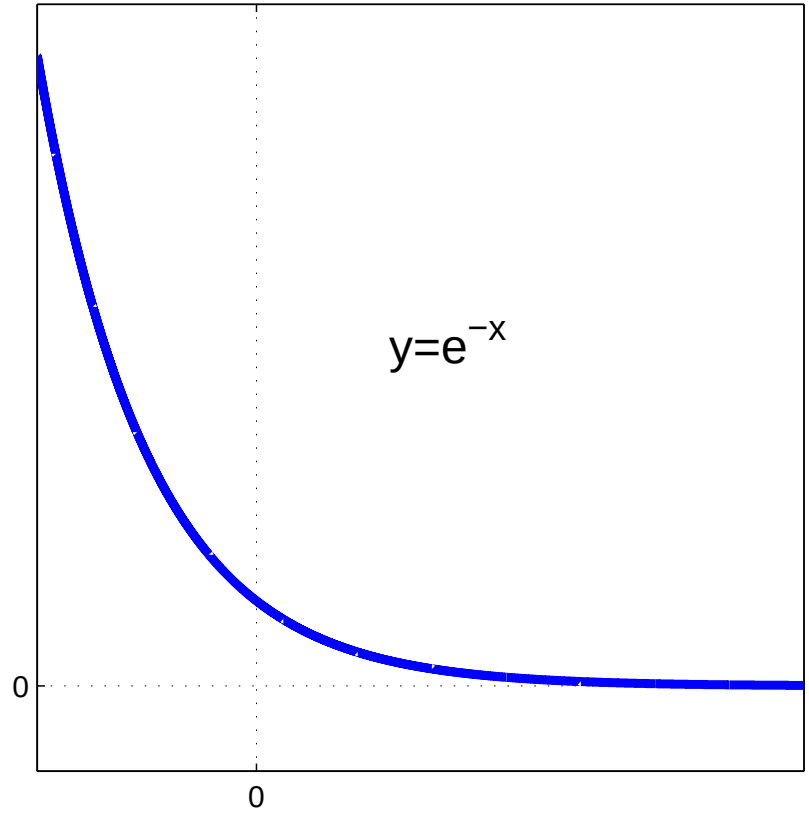
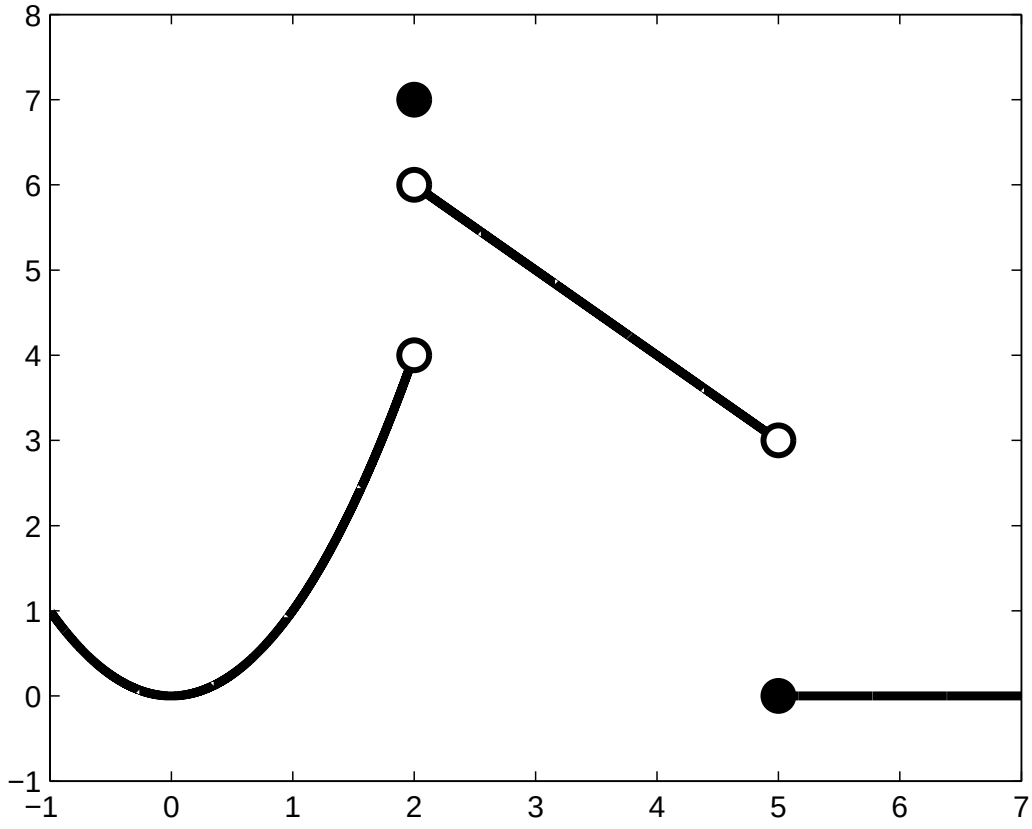


An Odious Function



Differential Calculus

For any continuous function, $y = f(x)$,
the derivative is its instantaneous slope

Derivative is denoted $f'(x)$ or $\frac{df(x)}{dx}$ or $\frac{dy}{dx}$

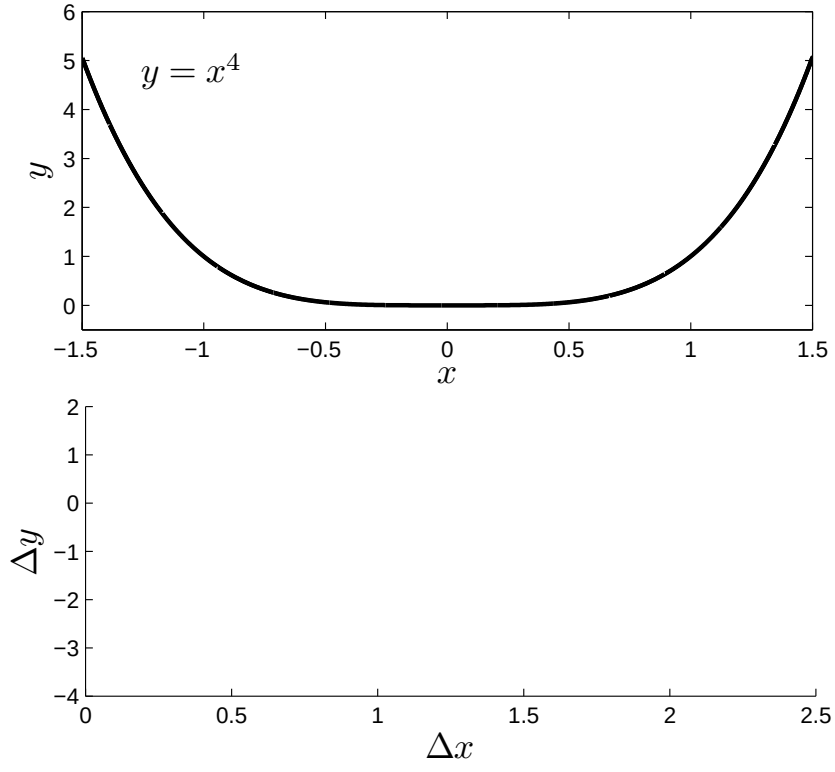
Slope = $\frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$

What do we mean by instanteous slope?

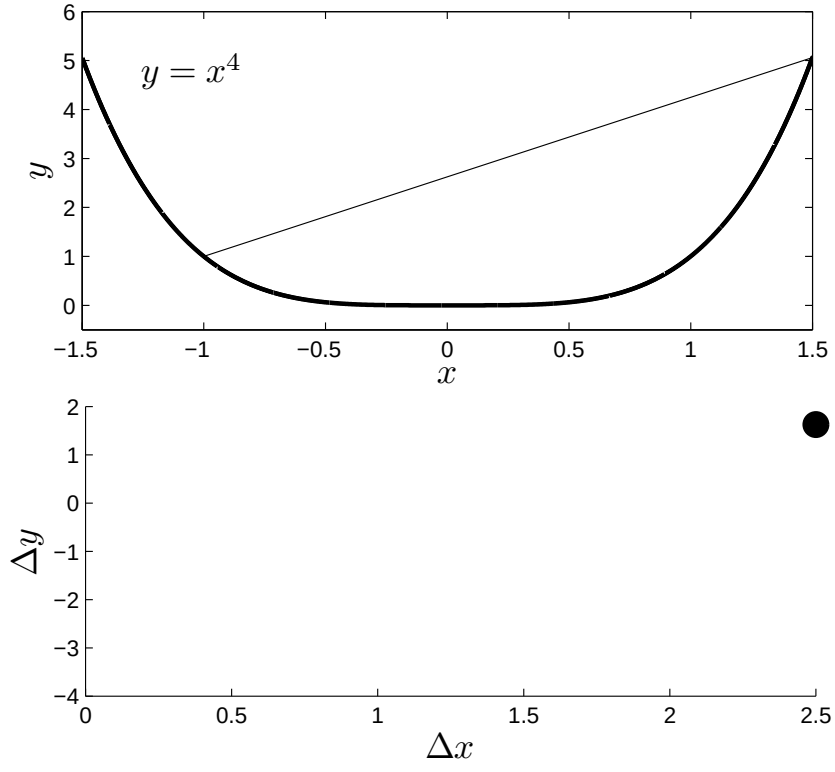
Make the “run” portion really really really small.

This concept should be intuitive

From Slope to Derivative



From Slope to Derivative



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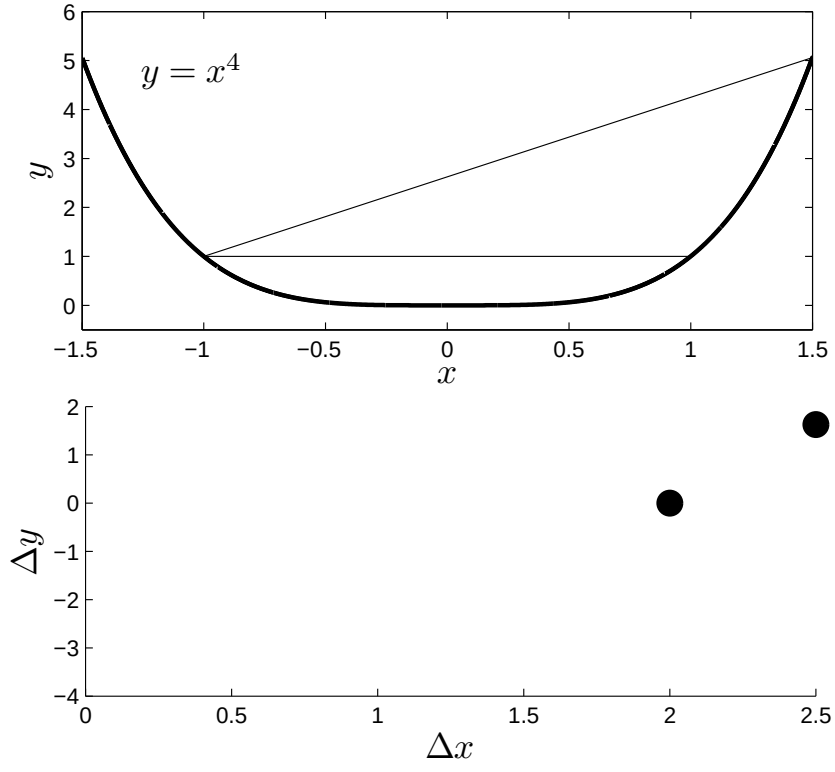

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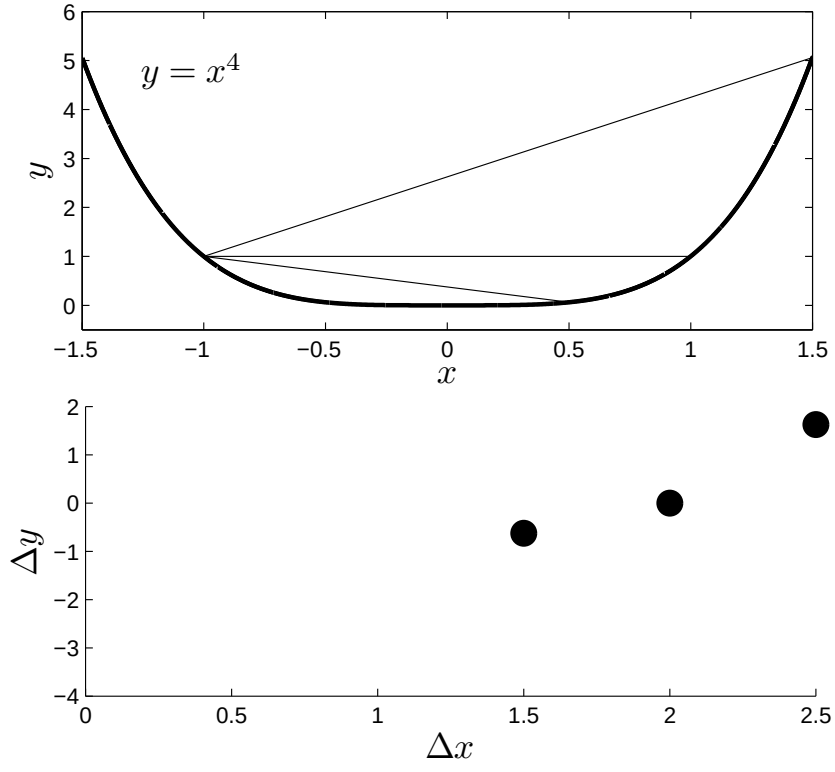

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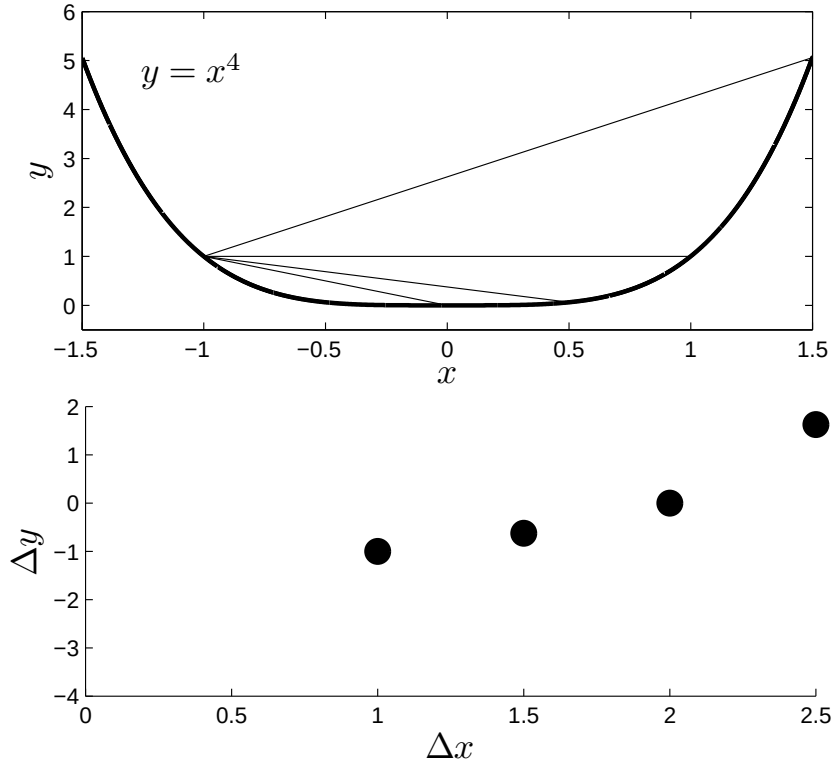

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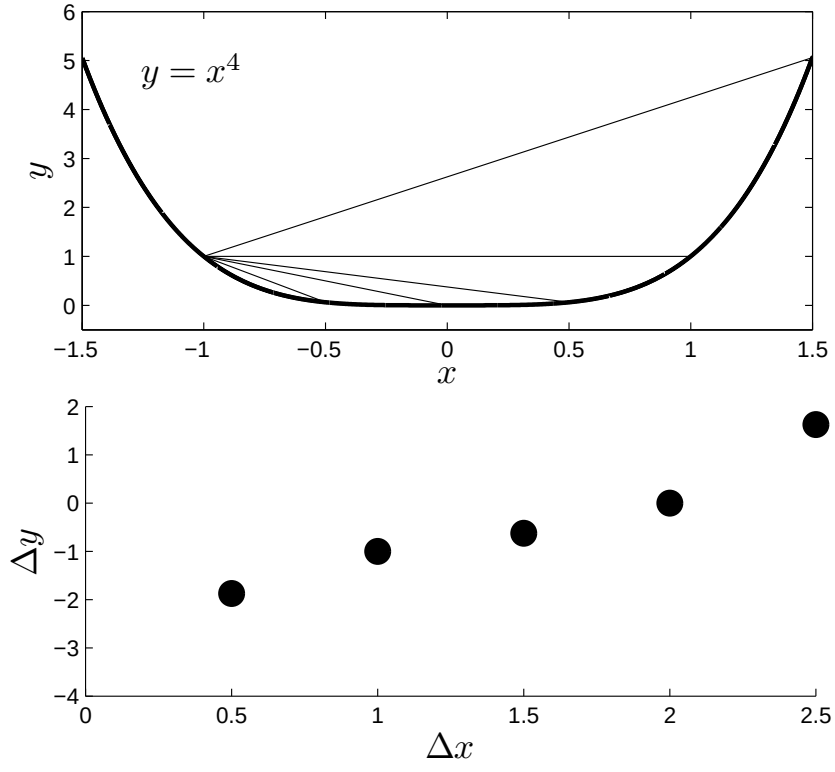

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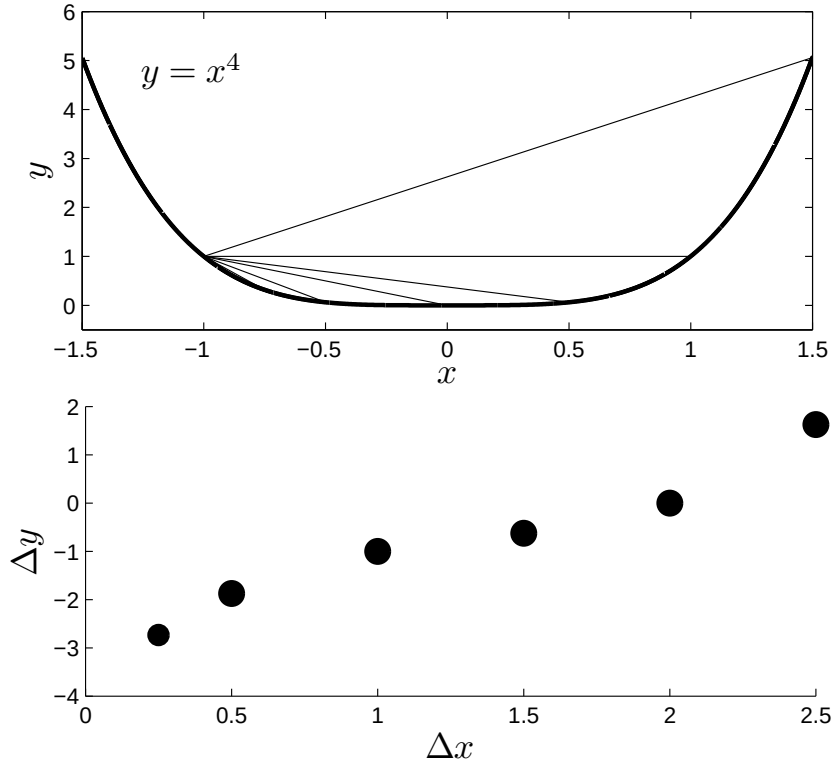

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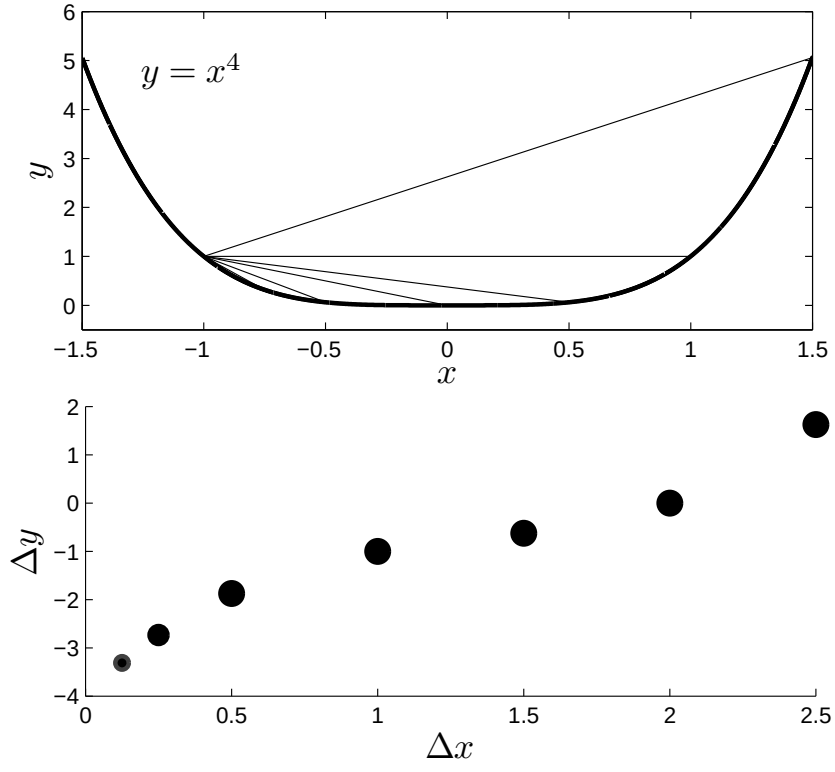

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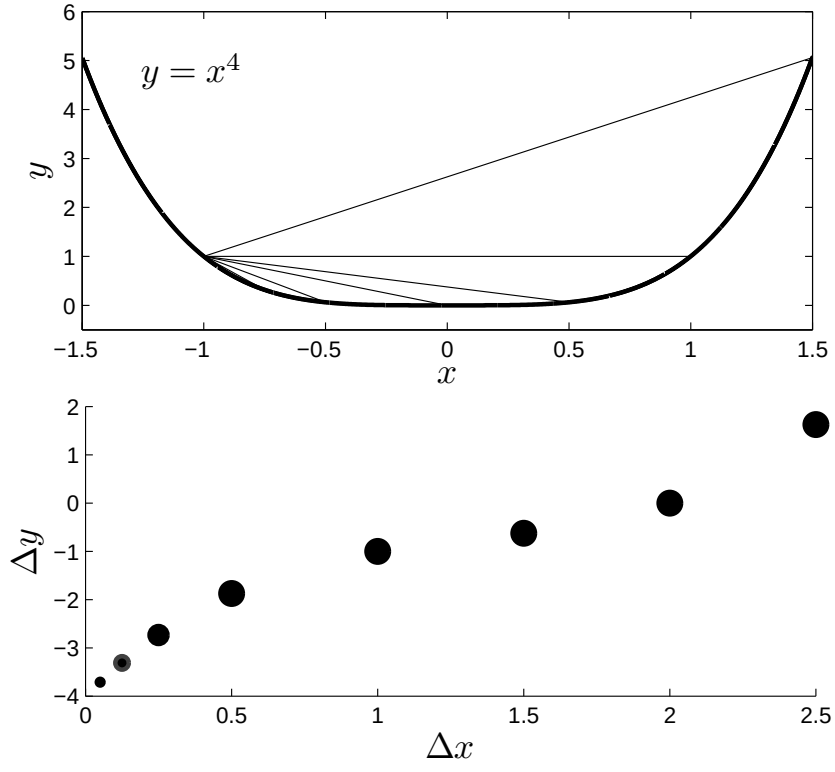

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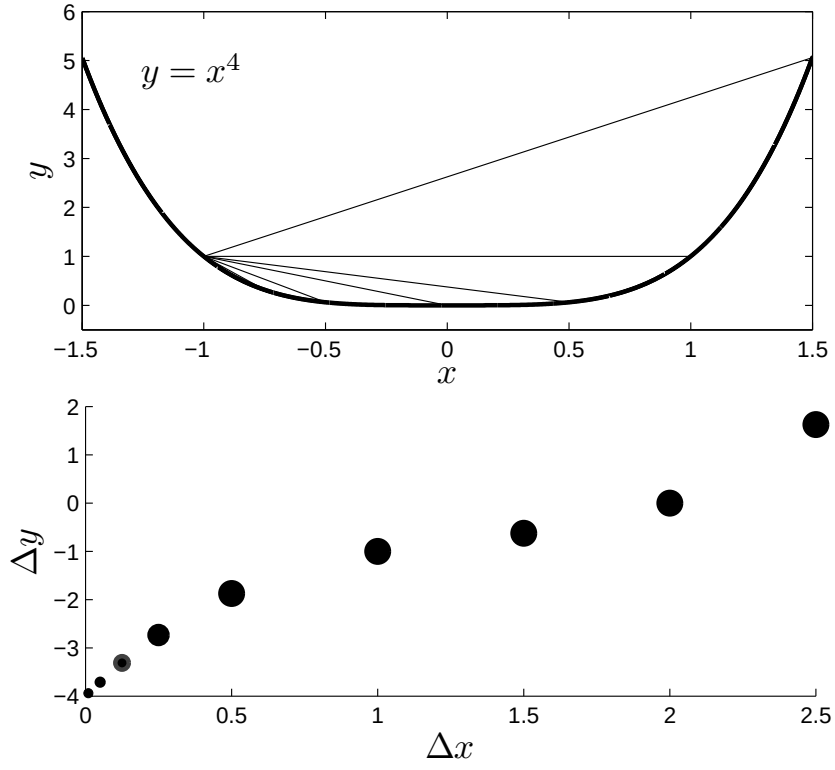
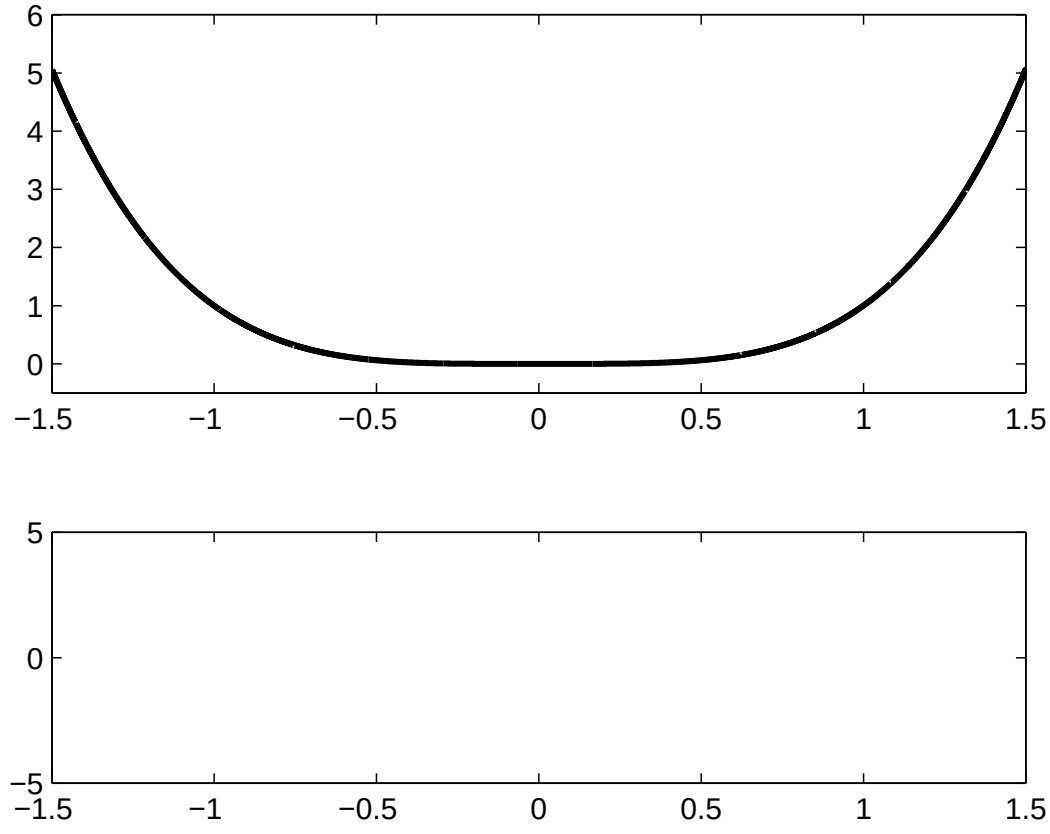


$$\lim_{\Delta_x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = -4$$

Tangent Lines



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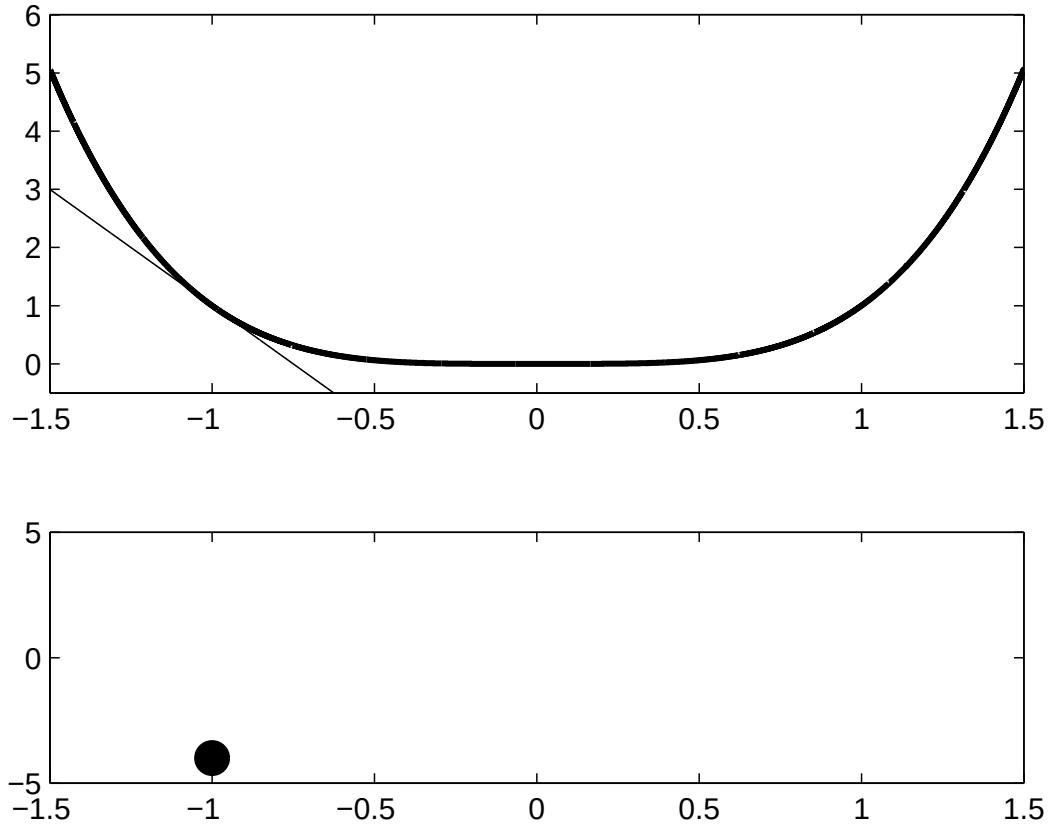

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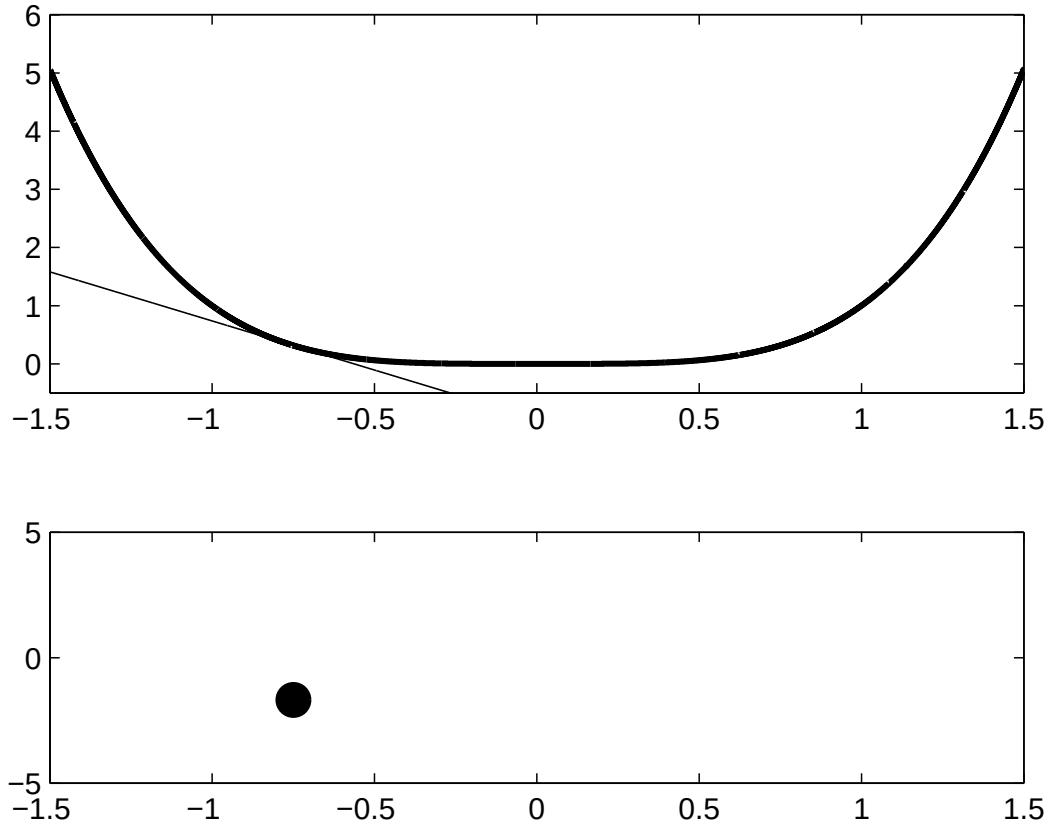

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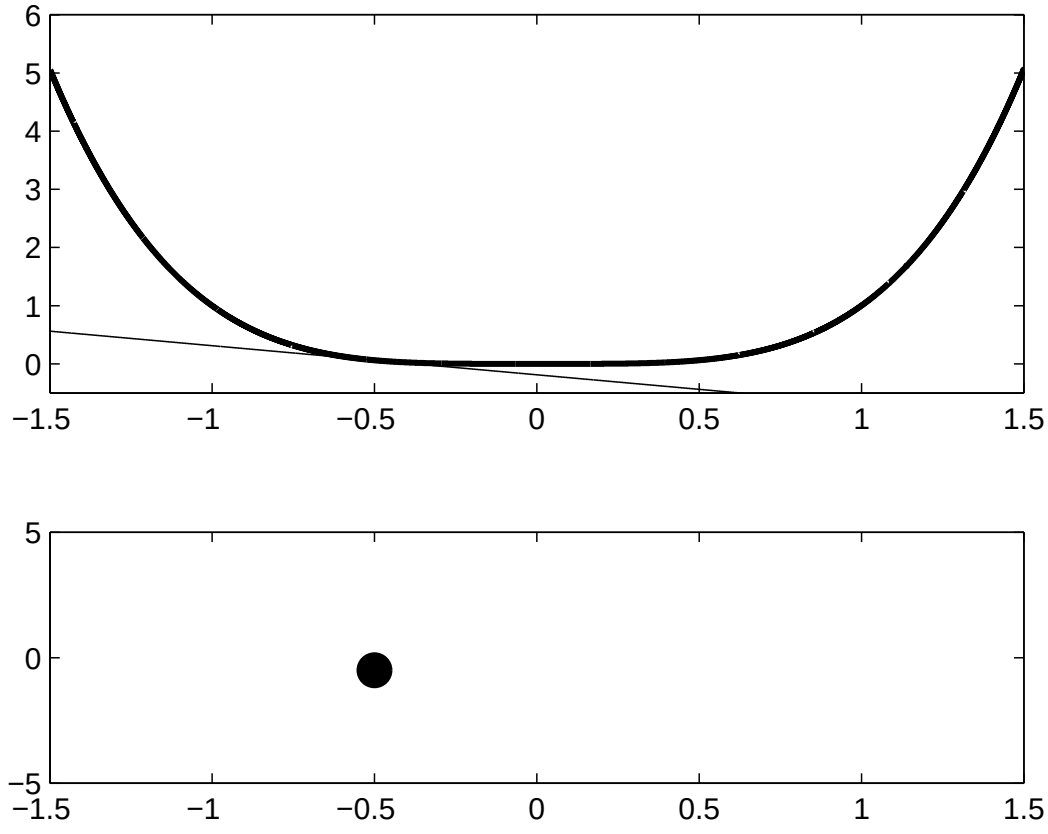

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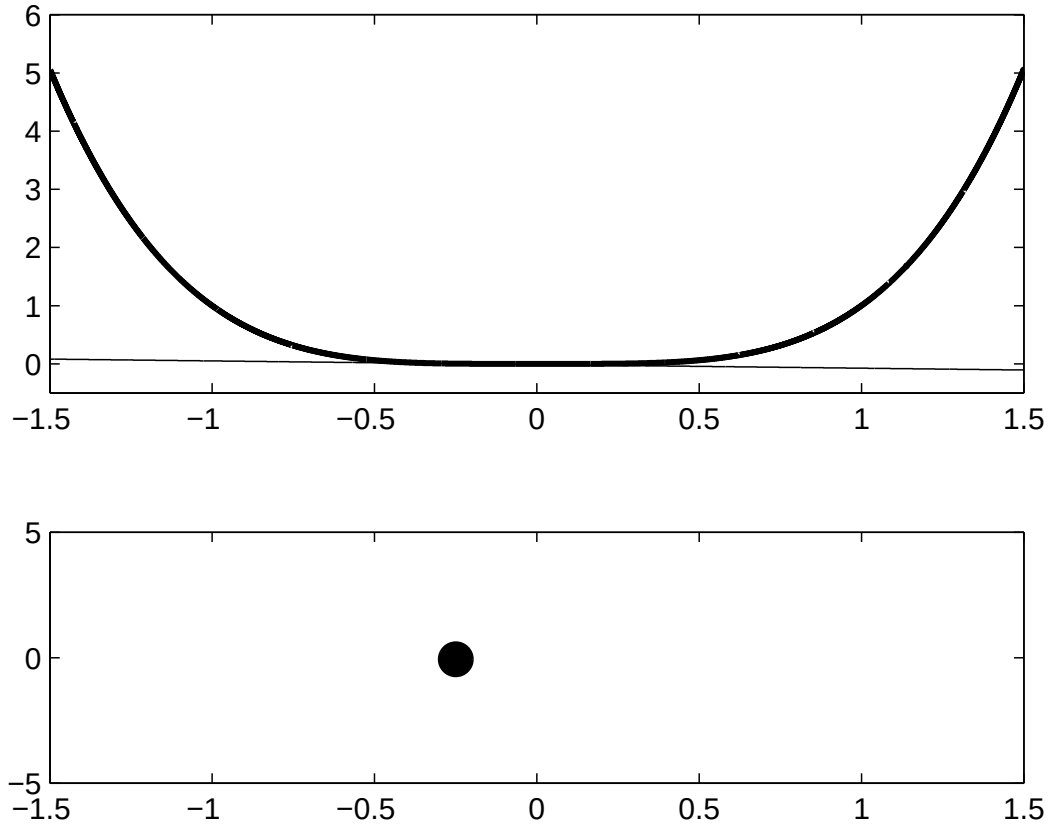

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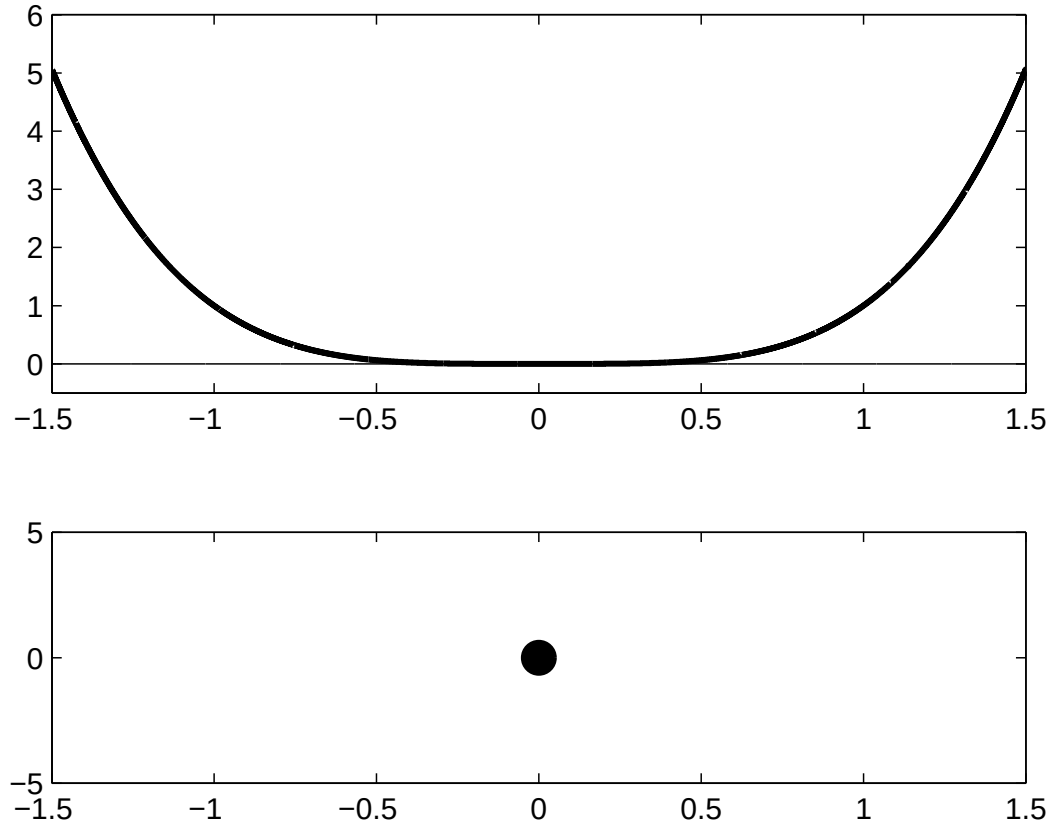

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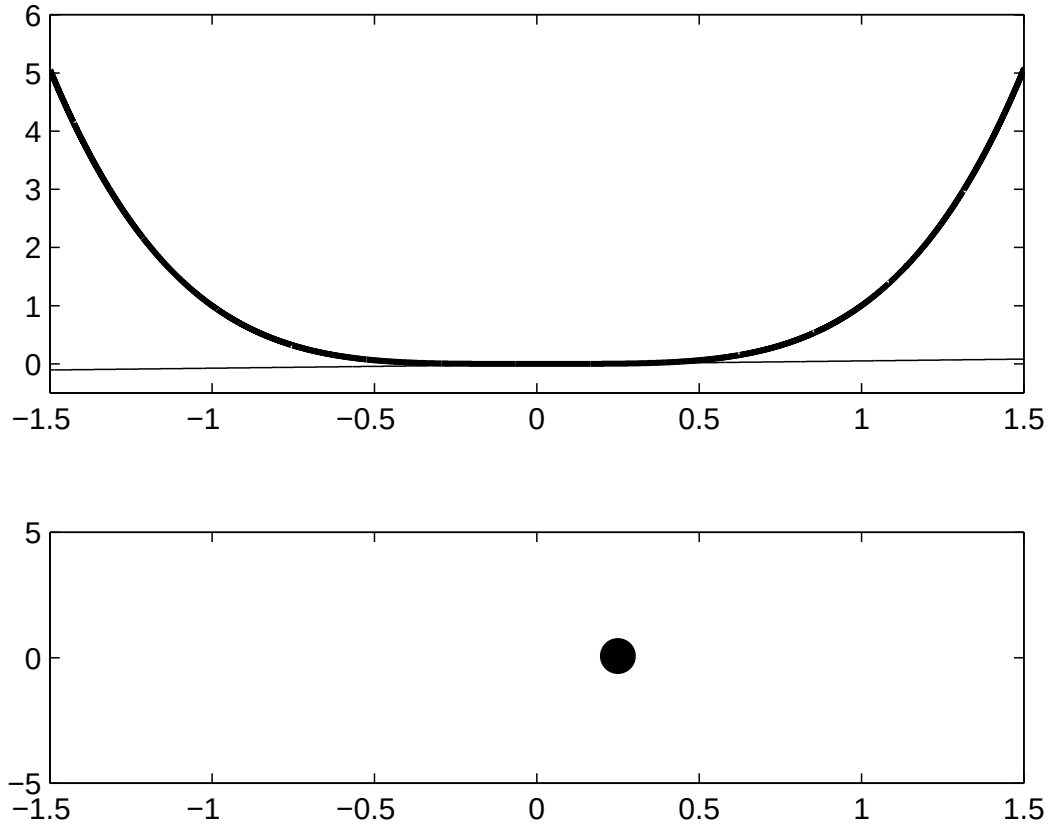

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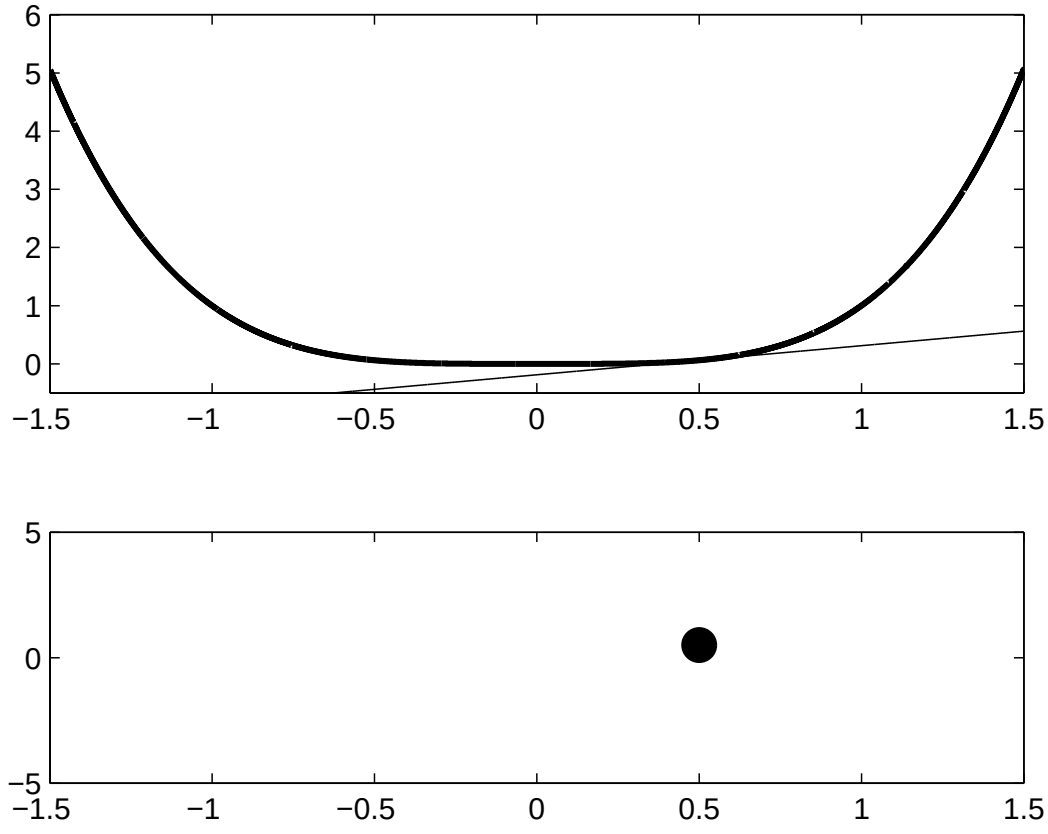

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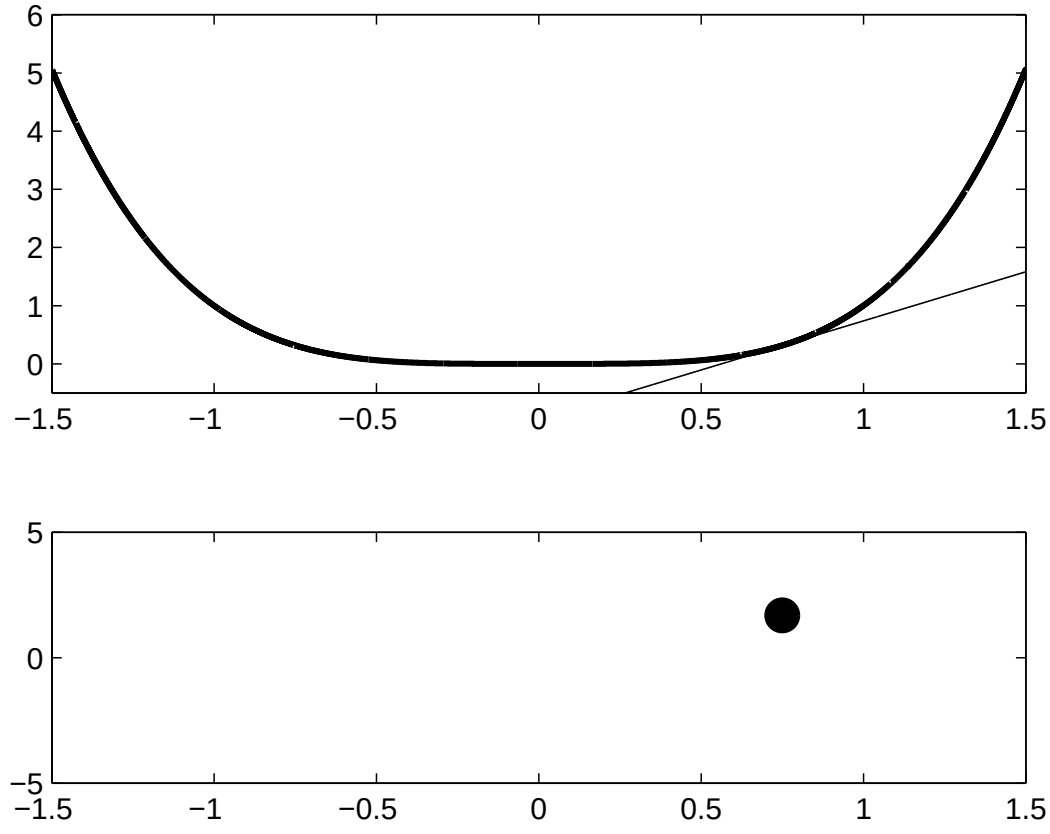

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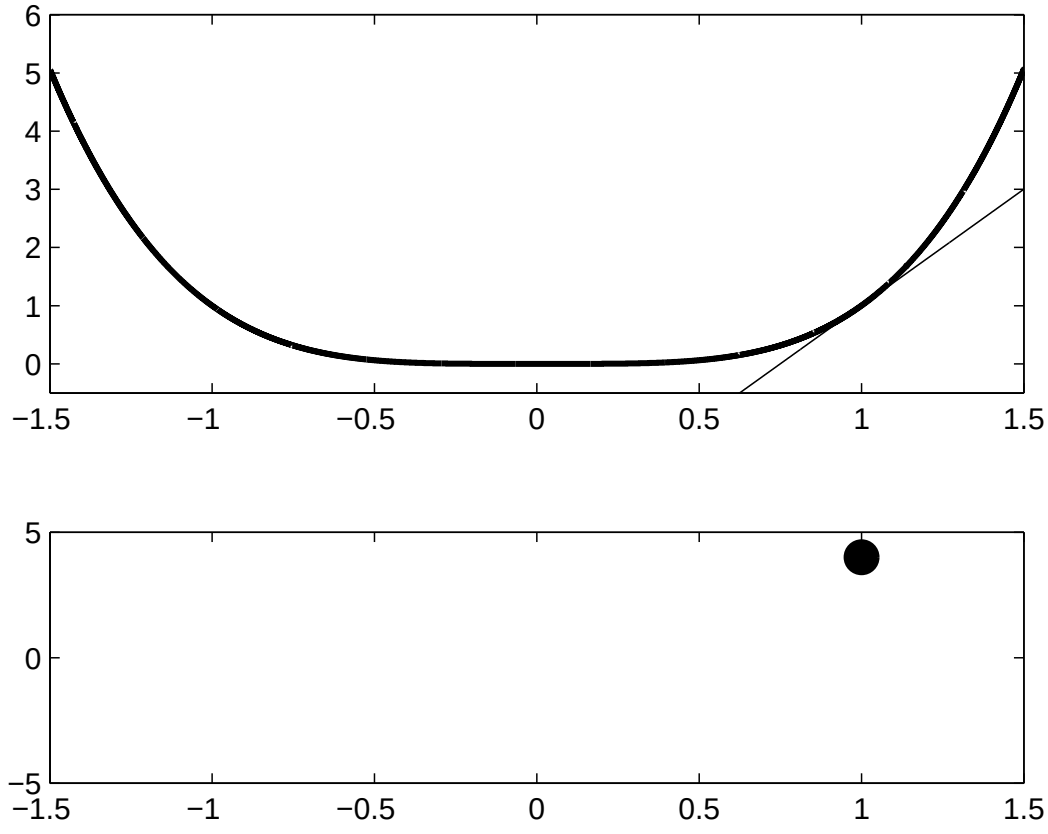

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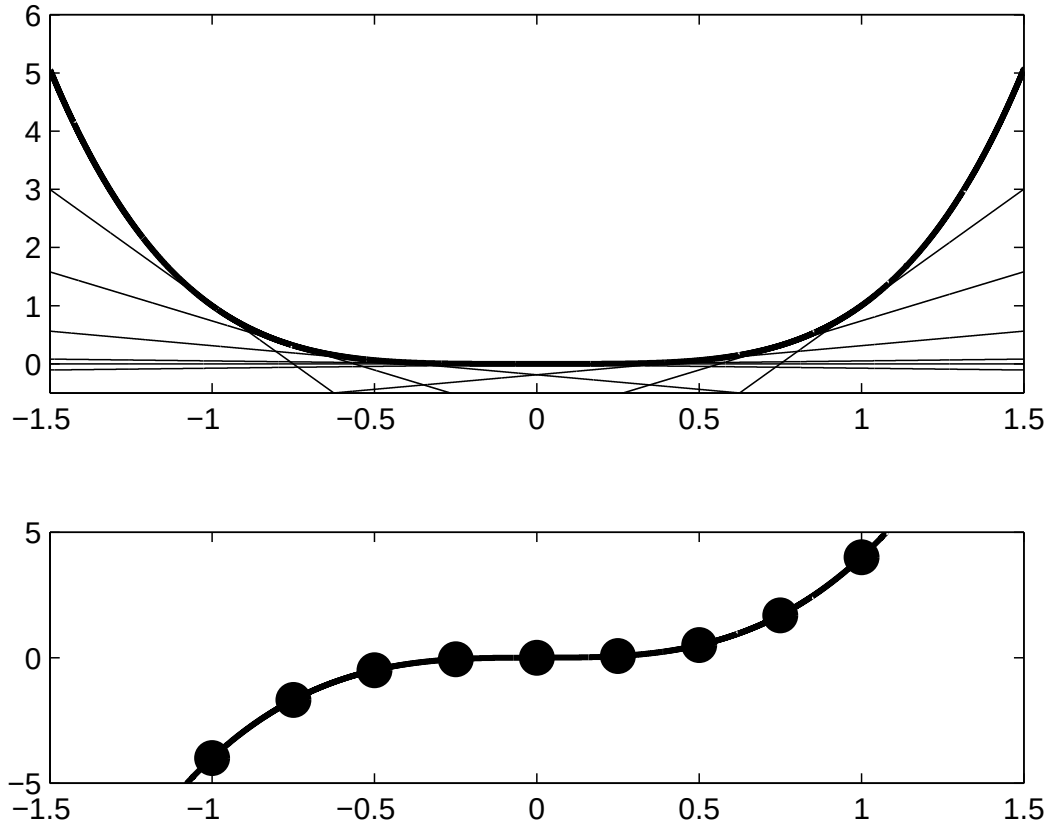

tangere = to touch

The Derivative

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\frac{df(x)}{dx} = \frac{dy}{dx}$$

Simple Differentiation Rules

$$\frac{d}{dx}[C] = 0$$
$$\frac{d}{dx}[ax] = a$$
$$\frac{d}{dx}[af(x)] = af'(x)$$
$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$
$$\frac{d}{dx}[x^n] = nx^{n-1}$$
$$\frac{d}{dx}[e^x] = e^x$$
$$\frac{d}{dx}[\log(x)] = \frac{1}{x}$$
$$\frac{d}{dx}[\sin(x)] = \cos(x)$$
$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$
$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

The Antiderivative

So $F'(x) = \frac{dF(x)}{dx}$

is THE function that gives the slope of $F(x)$ at x

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If $\frac{dF(x)}{dx}$ goes forward, can we go backward?

$$\int \frac{dF}{dx} = F$$

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If $\frac{dF(x)}{dx}$ goes forward, can we go backward?

$$\int \frac{dF}{dx} = F + C$$

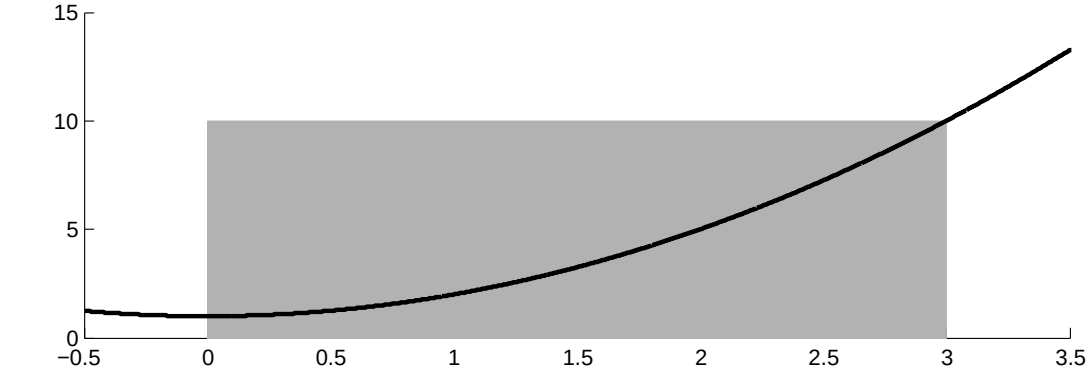
Similarly, differentiation reverses antidifferentiation

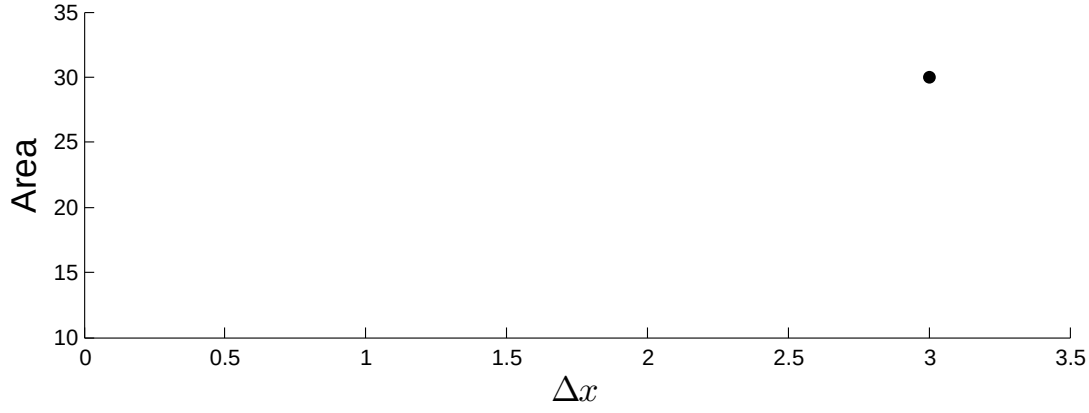
$$\frac{d \int f}{dx} = f$$

Simple Antiderivatives

$\frac{d}{dx}[C] = 0$	$\int 0 = C$
$\frac{d}{dx}[ax] = a$	$\int a = ax + C$
$\frac{d}{dx}[af(x)] = af'(x)$	$\int af(x) = a \int f(x)$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)] = \int f(x) \pm \int g(x)$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x = e^x + C$
$\frac{d}{dx}[\log(x)] = \frac{1}{x}$	$\int \frac{1}{x} = \log \lvert x \rvert + C$
$\frac{d}{dx}[\sin(x)] = \cos(x)$	$\int \cos(x) + \sin(x) + C$
$\frac{d}{dx}[\cos(x)] = -\sin(x)$	$\int \sin(x) = -\cos(x) + C$
$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$	NotUseful

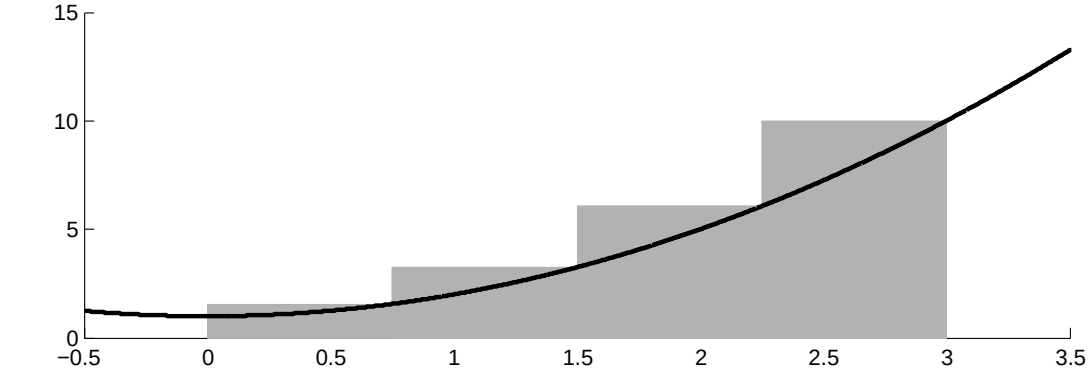
Integration of $x^2 + 1$

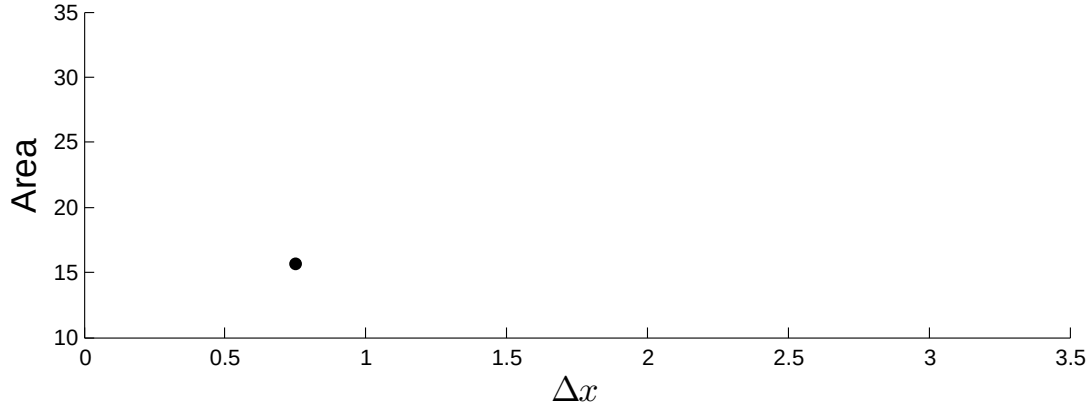




$$f(x_1)\Delta_1$$

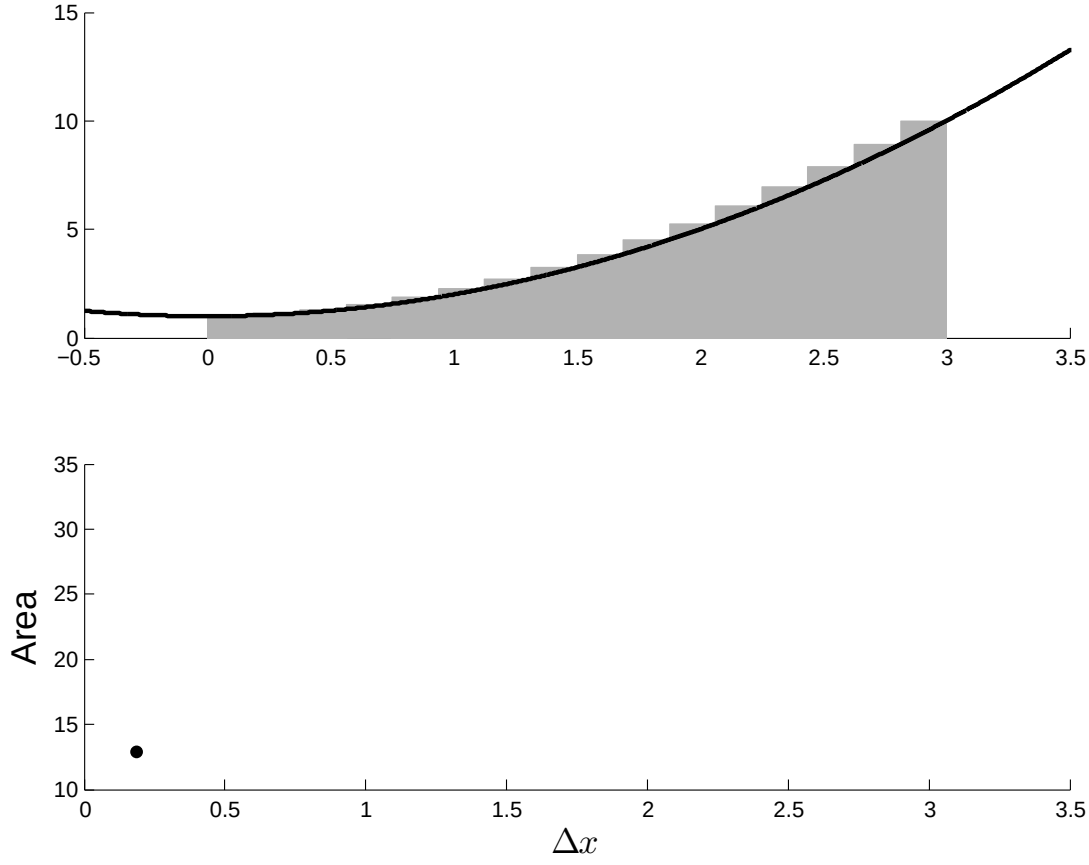
Integration of $x^2 + 1$





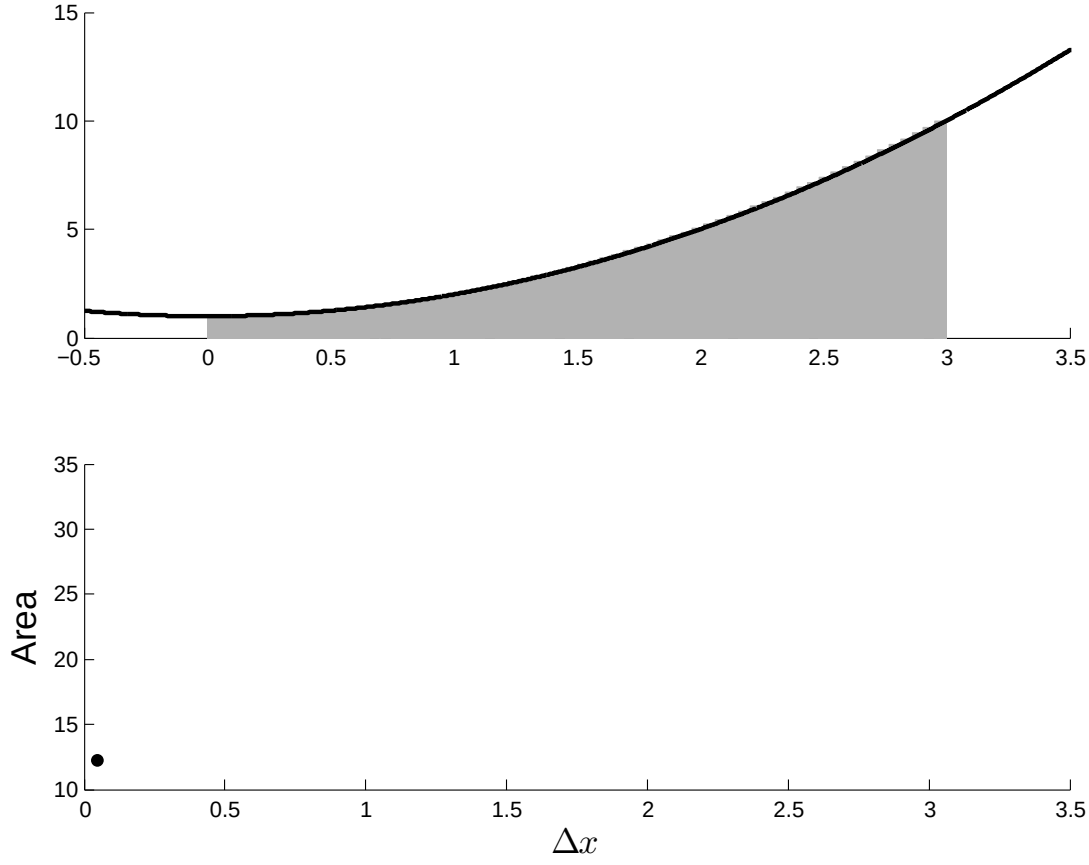
$$f(x_1)\Delta_1 + \ldots f(x_4)\Delta_4$$

Integration of $x^2 + 1$



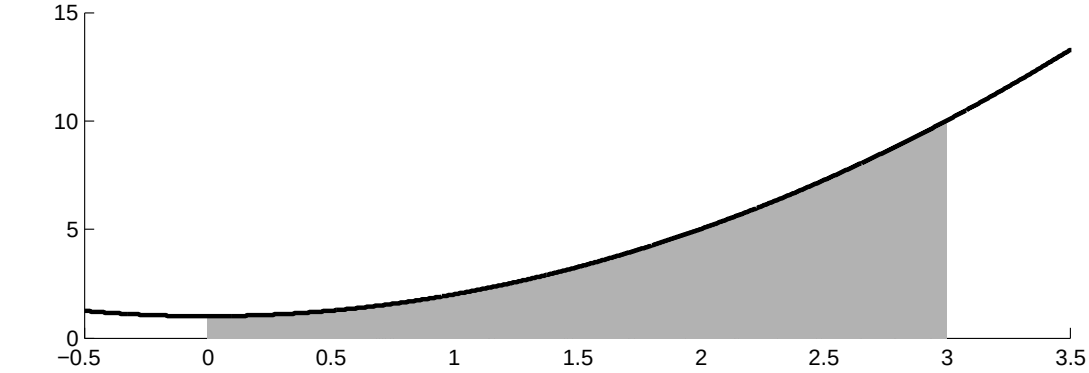
$$f(x_1)\Delta_1 + \ldots f(x_{16})\Delta_{16}$$

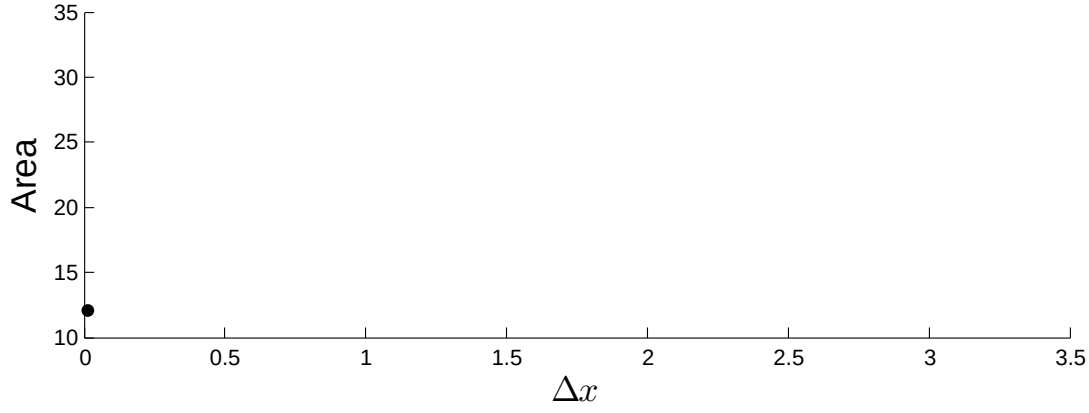
Integration of $x^2 + 1$



$$\sum_{i=1}^{64} f(x_i)\Delta_i$$

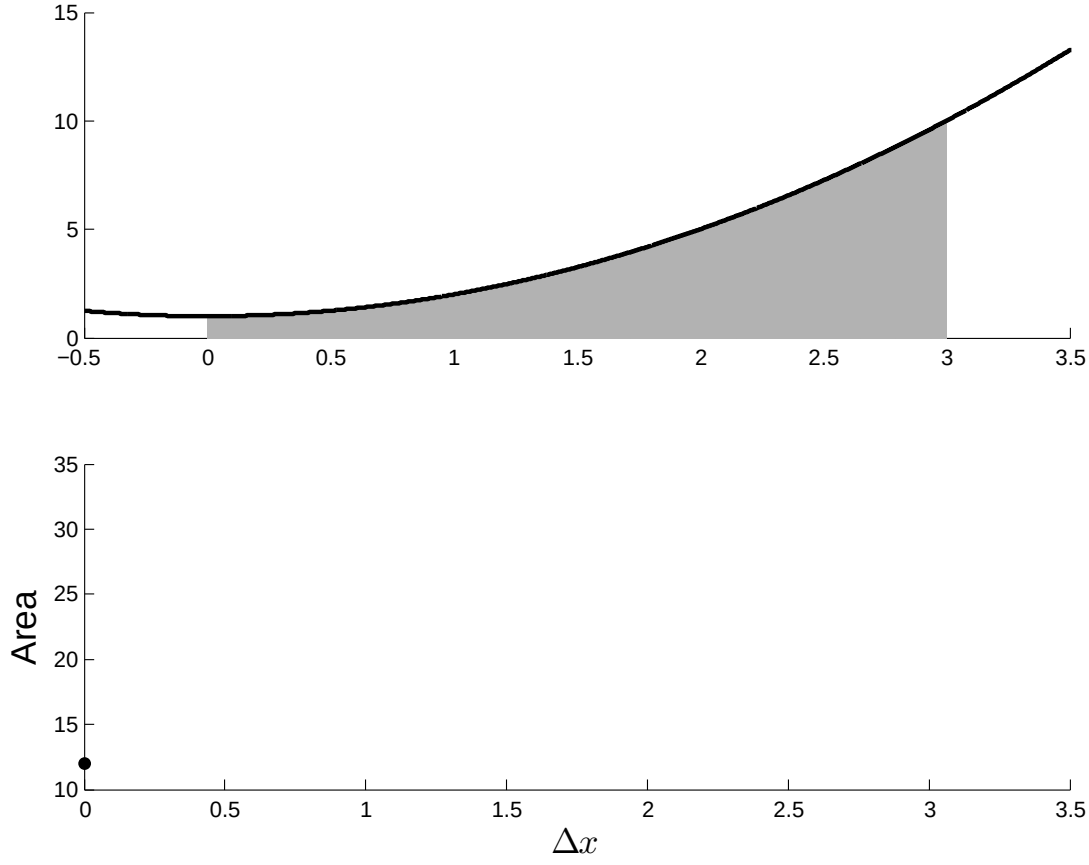
Integration of $x^2 + 1$





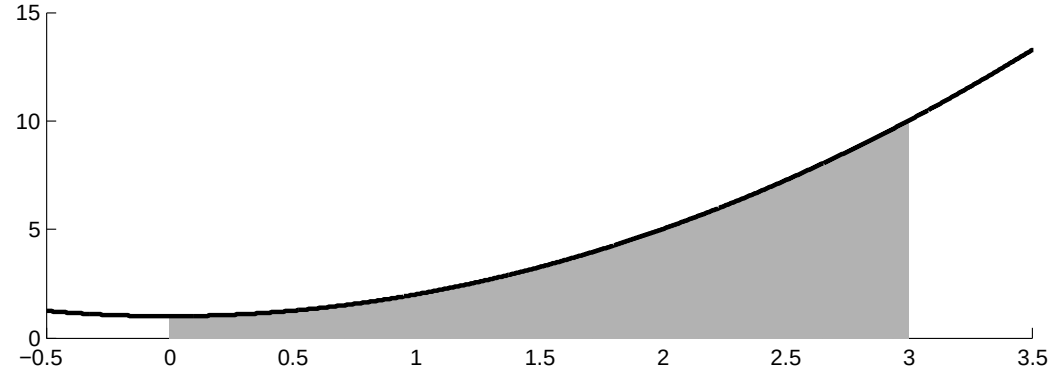
$$\sum_{i=1}^{256} f(x_i)\Delta_i$$

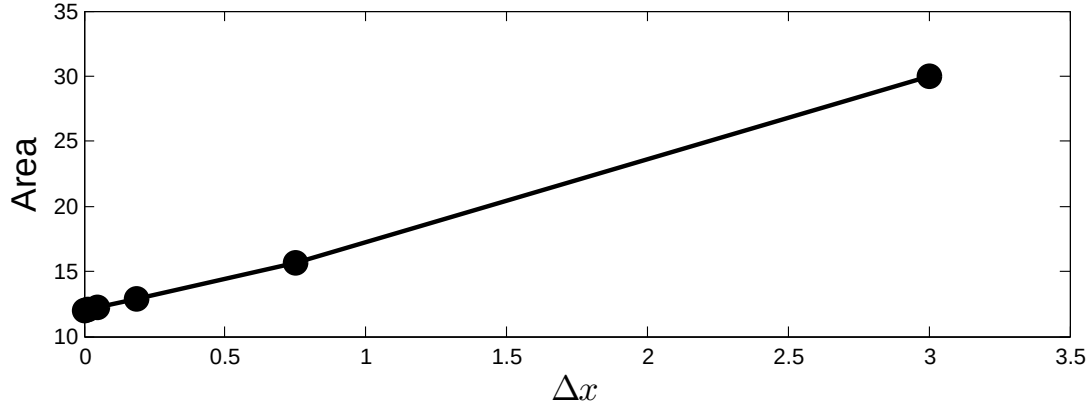
Integration of $x^2 + 1$



$$\sum_{i=1}^{1024} f(x_i)\Delta_i$$

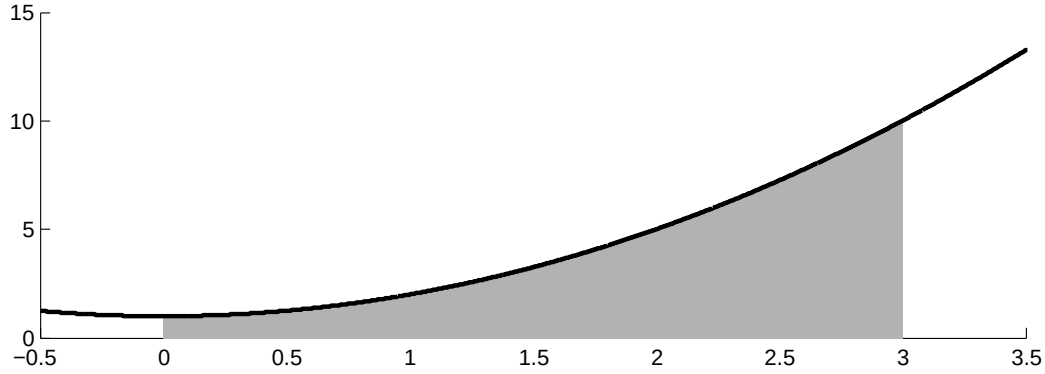
Integration of $x^2 + 1$

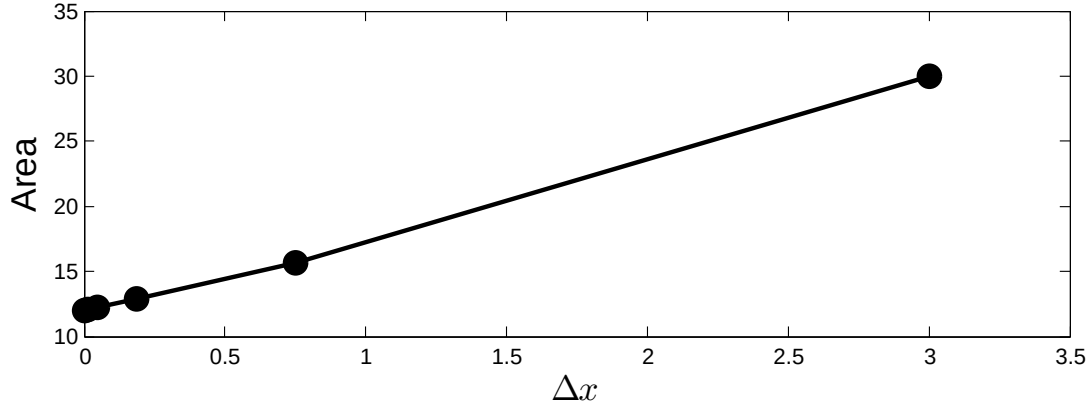




$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta_i = 12$$

Integration of $x^2 + 1$





$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta_i = 12 = \int_0^3 x^2 + 1$$

Fundamental Theorem of Calculus - I

Recall: $\int f = F + C$
antidifferentiation isn't precise
antidifferentiation takes a function and returns a family of functions

Recall: $\int_a^b f = \text{a number}$

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Recall: $\int_a^b f = \text{a number}$

But what about $\int_a^x f$?

$$\int_a^x f = G(x)$$

And even more astonishing

$$\frac{dG(x)}{dx} = \frac{d\int_a^x f}{dx} = f(x)$$

Fundamental Theorem of Calculus - II

$\int_a^x f = G(x)$

G is part of the $F + C$ family, but which one we do not know

It is not difficult to show that

$$\int_a^b f = G(b) - G(a)$$

Fundamental Theorem of Calculus - II

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Drumroll Please

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Drumroll Please

$$\int_a^b f = F(b) - F(a)$$

The C's, (those irksome C's), don't matter

Why is that astonishing?

$$\int_a^b f = F(b) - F(a)$$

Integration is a global property
The area under a curve depends on what the curve looks like everywhere between a and b

The Theorem says that we don't need to calculate functions everywhere, just its antiderivative at just the 2 endpoints.

We can evaluate an integral by looking at its local properties at the endpoints a and b.

Integration of $x^2 + 1$ – again

$$\int [x^2 + 1] = \int x^2 + \int 1$$

$$\int x^2 = \frac{x^3}{3} + C$$

$$\int 1 = x + C$$

$$F(3) = \frac{3^3}{3} + 3 = 12$$

$$F(0) = \frac{0^3}{3} + 0 = 0$$

$$F(3) - F(0) = 12$$

Integration of $x^2 + 1$ – again

This is often written in shorthand as

$$\left(\frac{x^3}{3} + x\right]_0^3$$

Read "$\frac{x^3}{3} + x$ evaluated at $x = 3$
minus $\frac{x^3}{3} + x$ evaluated at $x = 0$"

The Chain Rule

Learn it, Love it, Live it

Chain Rule : if $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x))g'(x)$

Differentiate $y = \frac{1}{1+e^x}$

Let $f = \frac{1}{g} = g^{-1},\ f' = -g^{-2}$,
Let $g = 1 + e^x,\ g' = e^x$

$$\frac{dy}{dx} = -g^{-2}e^x = -(1+e^x)^{-2}e^x = -\frac{e^x}{(1+e^x)^2}$$