This chapter describes research that is attempting to coordinate a constructivist view of learning mathematics with the practice of teaching for the purpose of analyzing children's mathematical learning within the setting of the classroom. The chapter also is an attempt to transport research on learning from a constructivist perspective from the laboratory to the environment of the classroom. In so doing, the classroom also, unexpectedly, became a learning environment for the project teacher as well as the students. The teacher's experiences, that provided opportunities for her learning and transformed her beliefs about her role and the students' role, are described and interpreted. These experiences influenced the researchers such that teacher development has become a primary focus along with children's learning. The researchers' current perspective and their approach used with teachers, which differs significantly with traditional procedures, is described.

The focus of our research and the emphasis of our development work has been on second graders' construction of mathematical knowledge in the setting of classroom instruction. As such, our primary interest has been in the processes by which children create mathematical meaning in the course of classroom social interactions. Our work has been influenced in general by Piaget's and von Glasersfeld's constructivist epistemology that emphasizes the role of cognitive conflict, reflective abstraction, and conceptual reorganization in mathematical learning (Piaget, 1970a, 1980a; von Glasersfeld, 1988). At a more specific level, we have drawn on the cognitive models of young children's construction of arithmetical knowledge developed by Steffe (Steffe, Cobb, & von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983).

**Theoretical Perspective**

Although Piaget's theory provides a general explanation of cognitive development, it was intended to address epistemological issues (Fabricus, 1979) and, as a consequence, considers only broad areas of intellectual development. His theory therefore constitutes a general orienting framework but leaves much unsaid about the nature of cognitive development in specific conceptual domains. Not surprisingly this has posed major difficulties for educators who have attempted to develop pedagogical implications from Piaget's ideas about learning. The cognitive models developed by Steffe and colleagues (1983, 1988) extend the work of Piaget by offering an explanation of children's cognitive development in areas directly relevant to elementary school mathematics. In particular, these models specify ways in which children might construct increasingly sophisticated concepts of number, position, addition, subtraction, and place value numeration. This elaboration of Piaget's general theory of cognitive development makes it possible to consider children's construction of mathematical knowledge in a way relevant to instructional issues (Thompson, 1985). This constructivist approach to cognitive modeling, while offering an account of the psychological processes involved in children's mathematical development, has tended to down-play the importance of social interaction in the learning process. As Smedslund (1977) commented:

> In so far as Piagetian psychologists focus on logicality as a variable (e.g., conserver or non-conserver) and give only peripheral attention to the problem of determining children's understanding of instructions and situations, I think they are making an epistemological error and are out of step with everyday human life as well as with all useful psychological practice. (p. 4)

In this regard, our work has also been influenced to some extent by Vygotsky's (1962, 1978) analysis of the crucial role that social interaction plays in learning. Like Piaget, Vygotsky views learners as active organizers of their
experiences but, in contrast, he emphasizes the social and cultural dimensions of development. One of the most frequently quoted passages from Vygotsky's writings is his formulation of what Wertsch (1985) called the "general genetic law of cultural development" (p. 60).

Any function in the child's cultural development appears twice or on two planes. First it appears on the social plane, and then on the psychological plane. First it appears between people as an intersubpsychological category, and then within the child as an intrapsychological category...Social relations or relations among people genetically underlie all higher [cognitive] functions and their relationships. (Vygotsky, 1978, p. 57)

In this general characterization of development, internalization is a process involved in the transformation of social phenomena into psychological phenomena. Consequently, Vygotsky saw social reality as playing a primary role in determining the nature of intrapsychological functioning (Wertsch, 1985). Vygotsky has clearly made a profound contribution to our understanding of intellectual development by attempting to relate cognitive and social phenomena. However, the key explanatory process of internalization functions as an undefined primary construct in his theory and has resisted subsequent attempts to elaborate its workings. In contrast, the work of Blumer (1969), Mead (1934), and Schutz (1962) dispenses with the notion of internalization and instead focuses on the processes by which people interpret each other's actions and thus achieve compatible meanings. From this perspective, social interaction is not a source of processes to be internalized. Instead it is the process by which individuals create interpretations of situations that fit with those of others for the purposes at hand. In doing so, they negotiate and institutionalize meanings, resolve conflicts, mutually take others' perspectives and, more generally, construct consensual domains for coordinated activity (Bauersfeld, 1988; Bishop, 1985; Blumer, 1969; Maturana, 1980b; Perret-Clermont, 1980). These compatible meanings are continually modified by means of active interpretative processes as individuals attempt to make sense of situations while interacting with others. Social interaction therefore constitutes a crucial source of opportunities to learn mathematics in that the process of constructing mathematical knowledge involves cognitive conflict, reflection, and active cognitive reorganization (Piaget, 1970a). As such, mathematical learning is, from our perspective, an interactive as well as constructive activity (Cobb, 1988).

Research Emphases
As initially conceptualized, our research objective was to analyze young children's mathematical learning in a classroom where instruction was broadly compatible with constructivism. Our original intention was to extend the methodology of the "constructivist teaching experiment" (Cobb & Steffe, 1983; Steffe, 1983) to the complexity of a public school classroom by conducting a classroom teaching experiment. We planned to analyze individual children's construction of mathematical knowledge as they interacted with the teacher and their peers. In the process of undertaking these analyses we became aware that the classroom had simultaneously and unintentionally become a learning environment for the teacher. As the teacher used the instructional activities in her classroom and interacted with her students, her beliefs about her own role, the students' roles, and the nature of mathematical activity changed dramatically (Wood, Cobb, & Yackel, in press). It was by analyzing her learning that we developed an initial, tentative understanding of classrooms as learning environments for teachers. This chapter is our attempt both to provide an unsanitized account of the learning opportunities that arose for the teacher and to reflect on how our observations of her learning have influenced our current approach to teacher development.

The Classroom Teaching Experiment
The one-on-one constructivist teaching experiment extends Piaget's clinical interview methodology by including teaching episodes that enable the researcher as teacher to investigate more extensively the processes by which a single child constructs mathematical knowledge (Cobb & Steffe, 1983; Steffe, 1983). During the clinical interview, the researcher attempts to infer the child's current mathematical ways of knowing. Teaching episodes follow in which the researcher attempts to provide opportunities for the child to learn by judiciously selecting tasks, offering suggestions, and posing questions. In this situation, the researcher/teacher interprets the child's mathematical activity and thus elaborates and tests a provisional model of the child's cognitions. The tentative model is then used to guide the creation of new situations in which to further investigate the child's learning.

The constructivist teaching experiment methodology is ideally suited to the purpose of investigating the processes by which children might construct mathematical knowledge. However, it tends to emphasize the cognitions of individual children at the expense of social interaction. In the course of the analysis, for example, the researcher focuses almost exclusively on what the child might be thinking and implicitly takes the social process of mutually negotiating the interview situation for granted. Our research and development project in second grade was an attempt to extend the methodology of the one-on-one constructivist teaching experiment to the classroom and to coordinate cognitive and social analyses. To this end, a classroom teaching experiment was conducted for the entire school year.
during which we had to address all the objectives for second grade mathematics set by the participating school corporation. Throughout the experiment, the teacher was a full member of the project staff and made her own decisions about how to use the instructional activities in her classroom. We, for our part, visited the classroom each day to video-tape both small group work and whole class discussions. These recordings constitute the primary data source for our analysis of the children's construction of mathematical knowledge as they interacted with each other and the teacher during the mathematics lessons.

At the outset, we anticipated that the classroom teacher would conduct her mathematics lessons in a manner similar to that of the researcher in the one-on-one teaching episodes. Our initial expectation was that the teacher would construct models of her students' mathematical understandings as she interacted with them. She then would use these models to generate conjectures about the children's potential mathematical constructions and, on this basis, select instructional activities and interact with them in ways that might give rise to opportunities to construct mathematical knowledge. In the course of these teaching episodes, we anticipated that she would test and, when necessary, revise her interpretations of children's mathematical understandings (Steffe, 1986, 1988; Stevens & Collins, 1980). We initially believed that it might be feasible for the teacher to interact with twenty or more students in this manner in that the children typically attempted to solve the instructional activities in small groups and then participated in a teacher-orchestrated whole class discussion of their solutions. This instructional approach provides opportunities for the children to construct mathematical knowledge not found in traditional classrooms (Cobb, Wood, & Yackel, in press) and for the teacher to observe and discuss with the children their interpretations of and solutions to the instructional activities. We speculated that this might make it possible for the teacher to develop models of her students' mathematical understandings that could be used to inform her pedagogical interventions.

The instructional activities were developed in the course of the experiment on the basis of on-going observations of children's mathematical activity in the classroom. Given the central role we attribute to children's personal experiences, we were well aware that, historically, child-centered curriculum efforts have been strongly criticized for engaging children in activities in which the "subject matter" is lost (Thompson, 1985). It was here that the models developed by Steffe proved to be of greatest value in that they account for children's mathematical experiences rather than their cognitive behaviors. We therefore drew on the models in an attempt to develop instructional activities that might give rise to experientially-based opportunities for children to construct mathematical knowledge. In particular, we used the models to anticipate what might be problematic for children at qualitatively distinct conceptual levels as they interpreted and attempted to solve potential instructional activities. These personally experienced mathematical problems that we, hoped, would arise as the children attempted to achieve their goals in the classroom would constitute opportunities for them to learn (Confrey, 1985; von Glasersfeld, 1987b). In general, the activities were designed to make possible multiple solutions and thus both accommodate individual differences and facilitate sustained small group and whole class discussions about mathematics. Our intent was for children at various conceptual levels to complete the instructional activities in ways that they could explain and justify to others. In this regard, numerous research findings indicate that children enter school with a rich repertoire of conceptually-based self-generated algorithms and problem solving strategies (Baroody, 1987a; Carpenter, Hiebert, & Moser, 1983; Ginsburg, 1977; and Steffe et al., 1983). However, as a consequence of traditional instruction in the early grades, children learn to rely on instrumental procedures at the expense of sense making. Children can follow prescribed rules, but no longer give conceptually-based meaning to what they are doing (Burton, 1984; Ginsburg, 1982; Perry, Church, & Goldin-Meadow, 1988; Ross, 1986). The problem-centered instructional activities were designed to provide learning opportunities in which conceptual and procedural developments would, ideally, go hand in hand (Cobb, Yackel, & Wood, 1988).

In summary, although this approach to mathematics instruction is generally compatible with other child-centered approaches, it differs from typical approaches of this type in two important ways. First, the instructional activities were grounded in detailed analyses of children's mathematical experiences and the processes by which they construct mathematical knowledge. The activities were therefore designed to give rise to opportunities for children to reorganize their mathematical activity and thus develop increasingly sophisticated conceptual understandings. Second, the researchers were primarily responsible for the construction of the instructional activities in consultation with the project teacher. Consequently, the teacher was able to concentrate on the development of her classroom practice and was not distracted by the need to search for or develop from scratch instructional activities that may or may not offer opportunities for children to extend their current mathematical ways of knowing.

Initial Induction of the Project Teacher
As the teacher was to be the researcher/teacher in the experiment, we felt that it was important to help her understand the research-based cognitive models before commencing the classroom teaching experiment (Fennema, Carpenter & Pelenon, 1986; Osborne, Bell & Gilbert, 1982; Steffe, 1986). We anticipated that the teacher would change her general view of children's mathematical learning and
learn about children’s counting types, thinking strategies, and their various conceptions often. In the spring prior to the experiment, we met with her once a week to discuss the cognitive models and to watch video-recordings of clinical interviews that had been conducted with her current second-grade students at the beginning of the school year. As we watched the tapes, it became apparent to the project director that although the teacher was taking extensive notes about children’s cognitive levels, our conceptual analyses of children’s mathematical activity made little sense to her. She seemed to feel “on the spot” whenever we asked about her interpretation of a child’s solution and attempted to respond by giving one of the technical labels for a particular conceptual level. The social context we mutually constructed with the teacher during these initial sessions was such that she viewed us as evaluators of her answers. She seemed to ask herself, “What does he want me to say now?” As a consequence, the possibility of attempting to understand the children’s mathematical activity did not arise for her. In our view, she was rote learning a list of technical names that would have no relevance to her practice in the classroom. The development of this interaction pattern seemed to be influenced in part by the teacher’s view of the project director as a “math professor” who, by definition, knew a lot more than she did. The project director also contributed to the mutual construction of this unproductive context by explaining what he saw in the tapes. This only confirmed the teachers view of him as an authority who had all the answers.

In an attempt to renegotiate the social norms of their relationship, the project director initiated a dialogue about a topic within the domain of the teacher’s expertise—her mathematics textbook. The teacher questioned his suggestion that textbook-based instruction led many children to develop detrimental concepts of place value. The teacher referred to the ability of her students to complete textbook exercises correctly to support her claim that most of them did understand place value. In her view, most of her students were learning the mathematics they were supposed to learn.

In an attempt to make this taken-for-granted assumption about textbook instruction problematic, the project director suggested that she conduct her own interviews with some of her students to ascertain whether his claims were viable or not. She selected two of her better students and video-taped interviews in which she used the same tasks that had been given at the beginning of the school year. A description of these tasks can be found in Cobb and Wheatley (1988). Crucially, the tasks had face validity for the teacher in that successful performance seemed to involve the very concepts she assumed the children had learned as a consequence of her textbook instruction. In the course of the interviews, she began to realize that even though she had carefully taught them the algorithmic procedures specified in the textbook and although they could produce correct answers, neither of the students had reorganized their conceptions of place value since the earlier interviews. In retrospect, we see that our genuine collaboration with the project teacher began when she realized that her current instructional practices were problematic. She now viewed us as people with whom she could work to develop an alternative instructional practice. We had common problems and interests, and could engage in joint pedagogical problem solving.

Reflections on the Induction Process
We learned several important lessons in the course of our interactions with the project teacher. These reflections were of direct relevance to the issue of how to induct other teachers into the project. First, it seemed essential to initiate discussions with teachers on issues about which they considered themselves knowledgeable on the basis of their first-hand experiences. Researchers’ formal cognitive models obviously fail to meet this criterion. Second, and relatedly, it became apparent that the project teacher’s primary concern was, quite reasonably, the intellectual and social development of her students. The productive phases of her initiation into the project centered on what her students were learning in her classroom during mathematics instruction. Thus, the teacher’s classroom served as a learning environment for her even during her induction. From this we later concluded that our interactions with other teachers should focus on specific classroom events that could serve as paradigm cases. Third, the crucial point in our development of a collaborative relationship with the project teacher occurred when she began to realize that her current practice might be problematic. We were then no longer researchers or teachers but people with complementary domains of expertise working on problems of common interest. This strongly indicated to us that our first step when working with other teachers must be to help them become aware of and make problematic aspects of their textbook-based instruction. Only then would they have reason and motivation to attempt to modify their classroom practice while working with us.

The Classroom as an Environment for the Teacher’s Learning
When we began the classroom teaching experiment, we still clung to the belief that it would be crucial for the teacher to understand the cognitive models so that she could use them to inform her classroom practice. We assumed that this research-based knowledge would be essential when she analyzed her students’ mathematical activity and, more generally, that using the models to guide pedagogical interventions was the hallmark of teaching mathematics in a way compatible with constructivist theory. We therefore expected that the project teacher would draw on detailed inferences she made about each child’s thinking as she interacted with the children in small groups and as she orchestrated the whole class discussions. However it became evident to us that she was not applying the formal cognitive
models to her practice but instead she was trying to develop her own ways of making sense of her experiences as she interacted with her students. Although the detailed formal models did not appear not to be relevant to the teacher, her practice was compatible with more general aspects of constructivist theory that had been discussed in weekly meetings during her induction into the project. These included the beliefs that children's actions are rational to them and that as teachers we must try to make sense of their meanings (Labinowicz, 1985, 1987).

In the course of analyzing this and other aspects of the teacher's pedagogical actions in the classroom, we assumed that she, like her students, was rational, given her premises. We therefore accepted that she had sound if unarticulate reasons for not attempting to apply the cognitive models to her practice. Once we adopted this stance, we began to realize that researchers construct formal models in contexts that are incompatible with those in which teachers construct the knowledge that informs their practice. Formal models are a product of a series of abstractions and formalizations made by researchers who operate in the context of academic reasoning and attempt to satisfy the current standards of their research community. In contrast, teachers operate in the context of pragmatic pedagogical problem solving in which they have to make on-the-spot decisions as they interact with their students in specific situations. The distinction between the academic and pragmatic ways of knowing is, in many ways, analogous to that between the principled methods of formal mathematics and the informal, out-of-school mathematics that people construct to resolve the pragmatic mathematical problems they encounter in the course of their everyday lives (cf. Lave, 1988; Rogoff & Lave, 1984).

The Negotiation of Social Norms as an Opportunity to Learn
The teacher's immediate concern at the beginning of the teaching experiment was (in our language) to initiate and guide the mutual construction of classroom social norms that would make it possible for the children to work productively in small groups and express their thinking in whole class discussions. We have argued elsewhere that the processes of negotiating classroom social norms and of negotiating mathematical meanings constitute two distinct levels of discourse (Cobb, Yackel, & Wood, 1988, 1989). The currently established norms form a taken-for-granted framework within which to both engage in mathematical activity and communicate about mathematics. The teacher appeared to be aware before the teaching experiment began of conflicts between her prior traditional form of practice and classroom norms she now believed were desirable. Her attempts to resolve these conflicts in the context of her practice gave rise to opportunities for her to learn. Further learning opportunities arose for the teacher as she encountered unanticipated problems and made observations that were generally surprising to her. We will discuss both types of learning opportunities first in the whole class setting and then in the small group setting.

Whole Class Interactions
The teacher wanted the children to feel "psychologically safe" to explain how they had actually solved problems when they participated in whole class discussions. However, this conflicted with the traditional teacher elicitation, student response, teacher valuation pattern that she had been comfortable with. She was concerned that she would be unable to anticipate their responses if she initiated discussions in which the focus was on the children's mathematical activity rather than on an answer or solution method that she had in mind all along. From her perspective, this would create uncertainty and unpredictability in a situation in which she had previously felt in control. Her desire to facilitate and respect children's mathematical thinking was in conflict with her need to maintain control of events in her classroom (Gunstone & Northfield, 1988; Harlen & Osborne, 1985).

Despite her concerns, the teacher began to initiate and guide the renegotiation of social norms in the first mathematics lesson of the school year. This renegotiation was essential in that the expectations she had for the children during whole class discussions were incompatible with the beliefs about their own and the teacher's role that they had constructed in the course of their kindergarten and first grade mathematics instruction (Wood, Cobb, & Yackel, in press). In particular, the children tacitly assumed that they were expected to figure out the response that the teacher considered appropriate rather than to express their own thinking (Voigt, 1985; Weber, 1986). However, talking about expectations was not enough. If the children were to accept the obligation of making public their mathematical thinking, then the teacher had to accept certain complementary obligations for her own actions. From the children's point of view, a definite risk was involved in attempting to fulfill her expectations. For them, it was one thing to think privately about how to solve a problem, but quite another to express those thoughts to their peers. Their thinking would be subject to public scrutiny and evaluation at the risk of feelings of embarrassment and incompetence. If they were to express their thoughts, then they expected the teacher both to respect their thinking and to place other children under the obligation of doing so. As a consequence, the teacher was obligated not to overly evaluate their solutions or to try to impose her ways of doing mathematics on them (Cobb, Wood, & Yackel, in press; Wood, in press).

As part of the process of initiating and guiding the renegotiation of classroom norms, the teacher capitalized on particular classroom events by framing them as paradigm cases in which to discuss her expectations with the students.
For example, she initiated a discussion about erroneous solutions when some of the children became embarrassed after realizing that their answers were incorrect. In the course of the discussion, she emphasized that sharing such solutions was appropriate in every way in her classroom. More generally, her creative use of paradigm cases enabled her to help the children realize that her primary interest was to understand their solutions and facilitate a dialogue rather than to judge the correctness of their answers. This flexible use of paradigmatic events did not appear to be a consciously applied pedagogical strategy. Rather, it expressed knowledge-in-action that greatly contributed to her effectiveness in achieving a pedagogical agenda compatible with constructivism.

Her fears about losing control of the course of events during whole class discussions were almost immediately alleviated because the children were able to come to a consensus about answers in the very first lesson of the school year without the need for her to steer or funnel the discussions. She also observed that children would frequently revise their thinking in the course of discussions. Thus, renegotiating social norms to make it possible for the students and her to act as a community of validators did not result in an "anything goes" atmosphere. She was fulfilling her obligations as a teacher in that the children did eventually agree on correct answers.

The major surprise that occurred for the teacher as she listened to her students' explanations in whole class discussions was her realization that beginning second graders' mathematical thinking was far more sophisticated than she had previously assumed. She commented, "I have been teaching all this time, and I never knew second graders knew so much about math!" The whole class discussions constituted the first opportunity she had in the course of her teaching career to actually listen to her students as they expressed their mathematical thinking. More generally, her initially surprising observations about her students' capabilities also brought home to her the value of actually listening to what students had to say about mathematics. At the end of the year she commented:

I have become a better listener. Teachers are basically talkers who feel a strong desire to share their knowledge with other people. Children are no different. If we really make an effort to listen to our students, we will become richer for it.

It was precisely because she learned the importance of listening while interacting with her students in the classroom that she and the students were able to engage in genuine conversations about mathematics. The manner in which she initiated and guided the renegotiation of classroom social norms made this learning possible.

Small Group Interactions
As was the case with whole class discussions, the teacher came to more fully appreciate that her previously taken-for-granted agenda for mathematics instruction was incompatible with mathematical sense making when she initiated the renegotiation of social norms. Her initial concern as she interacted with her students during small group work reflected her assumption that part of her responsibility as a teacher was to constantly monitor her students to ensure that they stayed on task (Maher, 1986). However, she now wanted time to observe and interact with them as they worked in small groups. She was therefore faced with the challenge of initiating and guiding the development of the social norms that would make it possible for the children to work cooperatively without her close supervision as they solved problems and completed the instructional activities (Harlen & Osborne, 1985).

The initial renegotiation of social norms that occurred in the whole class setting was crucial to the development of increasingly productive small group interactions. The children realized that they would be expected to explain and justify how they had solved problems and this facilitated the development of the obligation of making sense of things when they worked in small groups. In addition, the teacher capitalized on events that occurred as the children worked in groups to further discuss their expectations for them. The obligations she attempted to negotiate included respecting each other's thinking, figuring things out for themselves, and working collaboratively to complete the instructional activities. The development of these norms for small group work was facilitated by the use of instructional activities designed to give rise to experientially-based mathematics problems, which in turn generated opportunities for dialogue and communication about mathematics. The instructional activities therefore played a crucial role in making it possible for the small groups to engage in productive mathematical activity. This, together with the renegotiation of social norms, provided an opportunity for the teacher to relinquish her traditional responsibility as an overseer who ensured that the children stayed on task. She commented on several occasions that the children were, for the most part, talking about mathematics as they worked in groups.

Not surprisingly, observations that her students stayed on task as they worked together in a somewhat noisy atmosphere and without the promise of tangible rewards while accepting personally challenging (i.e., hard) problems called into question some of her previous assumptions. To make sense of what she actually saw happening in her classroom, she had to reorganize her beliefs about what motivated her students to engage in mathematical activity. In the process of doing so, she seemed to construct a notion similar to that of task involvement in the achievement motivation literature (Nicholls, 1983, 1989).
For example, she wrote in notes for other teachers:

Students were motivated to work hard during math time because of the personal satisfaction they felt. Teacher reinforcement was not as necessary as before. I never gave stickers or happy faces for their work and they never asked for those types of rewards either. Further, work that was too easy often meant more behavioral problems. Twenty easy problems were not as self-satisfying as one or two difficult ones.

These comments indicate that doing work was of great importance to the teacher. However, she radically revised her understanding of what it meant to be on task:

When a child does not appear to be doing any productive thinking, do not be too hasty to judge or criticize the behavior. The student may be reflecting in a non-traditional way which teachers interpret as "goofing off." In reality, this reflection time may be part of thinking through or taking a time-out for a few moments.

The Teacher's Reconceptualization of Her Role

As the whole class and small group social norms necessary for the relatively smooth flow of classroom life became established, the teacher's and children's obligations for their own activity and their expectations for others' activities gradually achieved a fit. In the process, the children began to take increasing responsibility for their own conduct and learning. This did not escape the teacher's attention as she reconceptualized her role:

The teacher is not the only decision-maker in the classroom. Each student has leadership qualities that can be encouraged. They are responsible for the classroom and its materials. Students can learn a great deal from one another, the teacher is a 'facilitator of learning.'...The teacher can set up the physical layout for the room and the students maintain that order—not just the teacher.

This last comment indicates that the teacher seemed to be aware that she and the students together created the classroom social context. The teacher initiated and guided the development of intellectual and social autonomy by negotiating with her students the obligations of explaining and justifying their solutions, resolving conflicts involving solutions and answers, and developing productive small group relationships. It was no longer the teacher's responsibility to provide or sanction the official way to solve problems. Instead, she used her authority to guide and sustain mathematical communication in both whole class and small group settings by listening, offering suggestions, and clarifying children's meanings. In describing the change that took place in her role, she commented:

My teaching role is pleasantly different. Rather than being the "person with all the answers," the children have been given the opportunity to count on themselves and each other...Giving them responsibilities gives them the feeling that they are needed and are important in our classroom. They do have ownership in what they are learning.

In short, the teacher's reconceptualization of her role went hand in hand with the children's increasing autonomy. Each was made possible by the other and, in the process, mathematics instruction became more "psychically rewarding" (Lortie, 1975) for the teacher. This sustained the teacher's commitment to continue to develop her practice.

Negotiating Mathematical Meanings as an Opportunity to Learn

From the constructivist perspective, learning is an interactive as well as a constructive process (Bruner, 1986; Cobb, in press; von Glaserfeld, 1988) Opportunities for children to construct mathematical knowledge arise as they interact with both the teacher and their peers. As a consequence, their mathematical constructions are not purely arbitrary—anything does not go in the classroom. Instead, their constructions are constrained by an obligation to develop interpretations that fit with those of other members of the classroom community (Bauersfeld, 1988; Blumer, 1969). It is this fit between personal interpretations that make possible mathematical communication and the subjective experience of a shared, objective mathematical reality (Pierce, 1935; Schutz, 1962; Wittgenstein, 1964). In the course of a mathematical communication, meanings are negotiated and particular mathematical practices are institutionalized and taken for granted as beyond justification by members of the classroom community (Cobb, in press). Mathematics is therefore both an individual constructive activity and a human social activity—a community project (de Millo, Lipton, & Perlis, 1986). The latter aspect of mathematics was most apparent when we focused on the teacher's and children's discussions about mathematics rather than on individual children's construction of mathematical knowledge.

Facilitating Mathematical Learning and Communication

As we have noted, the teacher's primary intention when interacting with her students during both small group and whole class discussions became to facilitate their engagement in meaningful mathematical activity. A major problem began to take shape for the teacher in the course of these interactions. She had concluded during her induction into the project that simply pointing out children's mistakes and telling them what to do did not work. On the other hand, children's mathematical explanations were frequently unacceptable with respect to the institutionalized
mathematical practices of the wider community. If direct telling was inappropriate, then so was a blind acceptance of all solutions. The result would be a chaotic "anything goes" in which each child would be allowed to pursue his or her own interests. The teacher gradually developed a form of practice that avoided these twin dangers as she interacted with her students. It is one thing to be non-evaluative during whole class discussion and another to do nothing more than regulate individual children's explanations and ensure that only one child is speaking at a time. The teacher seemed to realize that it was not sufficient for her to regulate the separate explanations of a series of children; she had to foster communication about mathematics among the children and thus make mathematics a community project. This realization was indicated by the manner in which she frequently framed incompatible solutions as problems for the children to resolve. In doing so, she was implicitly communicating to the children her belief that mathematical solutions should be questioned and, when necessary, justified. Thus, she was subtly acculturating the children into her own interpretive stance with regard to mathematical knowledge (Bruner, 1986). This process was most apparent when the teacher encouraged the children to say whether they agreed or disagreed with others' solutions and to settle the ensuing disputes by discussing their reasons.

The teacher experienced greater difficulties in transcending the tension between traditional and laissez faire forms of practice when she interacted with the children as they worked in small groups. On one occasion she was working with her weakest student to help him solve a problem that involved tens. The child had been using multilinks arranged in bars of ten. He had been counting individual cubes as he solved the problems. The teacher, in an attempt to help, counted each bar as a unit of ten. After a few minutes of watching, the child told her, "You're confusing me." This made such a strong impression on her that in the subsequent whole class discussion she announced:

Rick had two sets of multilinks he was adding up. And you know what I did? And I shouldn't have done this, because it kind a confused Rick. I added up all the tens first, and you know what it did to Rick? It confused him. I'm glad he spoke up and said, "I can't do it that way." He was counting by ones. I did more to confuse Rick than to help him even though I thought I was [helping him].

In this incident, she openly expressed to the class the tension that she felt between offering suggestions that would help the children solve their mathematical problems and directing them to produce the predetermined response she desired (Voigt, 1985; Wood, Cobb, & Yackel, in press). As the year progressed, she became increasingly sensitive in her interactions with the children and became adept at recognizing when her suggestions were fruitful and when the children merely searched for responses that would fit with her expectations.

The teacher's learning in the classroom illustrates the self-organizing nature of classroom life. It was the teacher who initiated the renegotiation of social norms to allow children to express their mathematical thinking. In the course of listening to their solutions the teacher modified her beliefs about mathematics and extended her understanding of children's learning of mathematics. By drawing on this knowledge, the teacher could better facilitate the children's construction of mathematical knowledge. In doing so, she created further opportunities to listen to creative solutions and thus further elaborated her understanding of second grade mathematics. In a very real sense the teacher and students mutually constructed a social context within which they could learn from each other. Mathematics was a community project. As the teacher and children engaged in and talked about mathematical activity, they created a "microcosm of mathematical culture" (Schoenfeld, 1987). In this setting the children were viewed as having mathematical ideas that were worth knowing. There was a change from the elementary school mathematics tradition of the teacher as the sole validator of official knowledge to one characterized by interaction and the negotiation of mathematical meanings.

**Our Current Views on Teacher Development**

The project is now in its third year. Thirty second-grade teachers are using the problem-centered instructional activities in their classrooms, twenty-two of them for the second year. Our experiences of interacting with the project teacher who participated in the classroom teaching experiment profoundly influenced the way in which we inducted the other teachers into the project. We first conducted a one-week summer institute with the teachers and then visited their classrooms at least once every two weeks during the first year in which they participated in the project. The teachers also met once a week in small groups to discuss their classroom experiences. In addition, the teachers participated in four after-school working sessions during the school year. Our continued interaction with the teachers throughout the year reflects our belief that classrooms are learning environments for teachers.

**The Summer Institute**

Our initial goal in the summer institute was to develop situations that would make it possible for the teachers to begin to question their current practices and thus have a reason to consider an alternative approach. In light of our work with the project teacher, we chose children's understanding of place value and their use of the standard two-digit addition algorithm as an initial setting for discussion. This was of immediate interest to the teachers because they considered it a central topic in second grade
mathematics and they reported that their students have difficulty in learning to add and subtract with regrouping. We began the week-long session by showing video recordings of children solving textbook and non-textbook tasks. In one of these recordings, children were first shown solving non-textbook tasks which consisted of number sentences such as:

\[ 22 + 13 = \_ \text{ and } 16 + 9 = \_. \]

The same children were then shown solving addition tasks involving the same number combinations presented in the traditional textbook vertical format. The teachers expected that children who could do the non-textbook tasks would also be able to complete the textbook tasks. Consequently, they were surprised when they found that their assumptions about children's learning were unwarranted. Like the project teacher, they then began to differentiate between correct adherence to accepted procedures and mathematical activity that expressed conceptual understanding.

As the teachers began to question the adequacy of textbook instructional activities and their current ways of teaching, they were willing to consider alternative instructional activities designed to encourage meaningful mathematical activity. In doing so, they demonstrated the value they placed on children's mathematical sense-making. We did not have to convince them that children should learn with understanding. Rather, they had assumed that this kind of learning was occurring in their classrooms. A shared desire to facilitate meaningful learning and a general concern for children's intellectual and social welfare constituted the foundation upon which we and the teachers began to mutually construct a consensual domain. We began to discuss our rationale for an alternative instructional approach by focusing on the crucial role the teacher plays in developing a "problem-solving atmosphere." From our point of view, it was essential that the teachers understand that the instructional activities did not constitute the curriculum. Learning opportunities for the students were not embedded in the activities, but were instead realized as the teachers used the activities in their classrooms while interacting with their students. In the last analysis, it was the teachers' responsibility to initiate and guide the mutual construction of situations conducive to learning. In the course of this discussion with the teachers, we showed video-recorded episodes of small group work and whole class interactions. The teachers frequently asked questions about the pragmatics of the instructional approach and we gave them specific answers about concerns such as techniques for organizing the manipulative materials and using the overhead projector. In addition, we gave relatively direct advice about how to initiate and guide the development of classroom social norms crucial to the establishment of a problem-solving atmosphere (Cobb, Yackel, & Wood, 1988; Cobb, Wood, & Yackel, in press). Our purpose was not to program the teachers to act in a predetermined way but rather, to help them find a way of coping with these concerns as rapidly as possible. In doing so they would have greater opportunity to focus on children's mathematical activity when they used the instructional activities in their classrooms. We were prescriptive to make it possible for classrooms to be learning environments for teachers as well as children (Cobb, Yackel, & Wood, 1988).

In the remainder of the one week institute, the teachers solved mathematical problems in small groups, familiarized themselves with the instructional materials, and visited a simulation of the project classroom. One set of mathematical tasks the teachers solved in groups was created by translating some of the second grade arithmetic activities into base eight (cf. Steffe, 1987). In the course of completing these activities and discussing their solutions, the teachers appeared to appreciate further that computational tasks can be solved in multiple ways. We also discussed the similarity between the teachers' solution methods and the children's methods that had been viewed earlier in the week. Finally, we asked the teachers to reflect on the difficulties that they experienced when constructing eight as a unit. Our hope was that they would begin to question the apparent obviousness of base ten numeration and begin to appreciate the intellectual challenge that second graders have to cope with.

During their first visit to the project classroom, teachers observed a demonstration mathematics lesson conducted by the initial project teacher with those of her students who were available to participate during their summer vacation. After the demonstration, we discussed the teachers' questions about and interpretations of classroom events. The next day the children returned to the classroom and the teachers worked with an individual child to investigate his or her mathematical interpretations and solutions. Again a discussion followed in which the teachers shared their observations. The teachers' comments indicated that they were learning about children's ways of solving mathematical problems and beginning to become aware of limitations in their understanding of their own students' mathematical thinking. As this brief discussion of the summer institute makes clear, we attempted to develop situations in which the teachers could engage in experientially based problem solving relevant to their practice. In general, we have come to believe that attempts to influence teachers knowledge and beliefs will not be at their most effective unless they draw on teachers' firsthand experiences of interacting with their students during mathematics instruction (Bush, 1986; Carpenter & Fennema, 1988; Cooney, 1985). For this reason, we did not discuss formal models of early number development during the summer institute.

The School Year

At the beginning of the school year, we discussed the potential value of the teachers meeting once a week in small groups at their school to discuss problems, concerns, and
insights. A member of the project staff visited their classrooms each week at the beginning of the school year and then gradually decreased the frequency of visits to once every two weeks for the remainder of the year. The primary purposes of the visits were to address teachers’ pragmatic concerns (e.g., how to involve all children in discussions) and to help them make problematic certain aspects of their practice that were outside their awareness (e.g., responding to children’s solutions in an evaluative if subtle manner). During these visits, we encouraged the teachers to think through problems themselves rather than to rely on us to tell them what to do (see Cobb, Yackel, & Wood, 1988, for a more detailed discussion). Consequently, we had intellectual autonomy as a developmental goal for them as well as for their students. There were two indications of having had some success in this regard: (1) the manner in which the teachers increasingly relied on their own judgments when selecting from the many instructional activities and materials, and (2) the decisions by which teachers determined activities to focus upon during class instruction.

Within the first few weeks of the school year, the teachers encountered difficulties when they attempted to interpret their students’ mathematical solutions. Some specifically requested assistance in judging the relative sophistication of particular children’s mathematical solutions. It was at this point that the teachers began to appreciate the relevance of relatively detailed knowledge of children’s mathematical cognition to their practice. We therefore conducted a series of working sessions that focused on various methods children use as they attempt to solve arithmetical problems. Initial sessions dealt with counting by ones and thinking strategies, and later sessions with units of ten, non-standard computational algorithms, and multiplicative and divisional concepts. These sessions were designed to dovetail with the instructional activities the teachers were using in their classrooms. Our purpose in orchestrating the discussions was to encourage the teachers to construct mutually acceptable interpretations of a particular child’s solution. Although we asked questions and drew attention to aspects of a solution that contradicted particular interpretations, we did not attempt to steer the teachers to an interpretation in terms of the formal cognitive models. Nor did we share our technical vocabulary with them; it was not relevant to their purposes. We thus attempted to walk the pedagogical tightrope as we interacted with the teachers in much the same way that we hoped they would when interacting with their students during mathematics instruction. This was one of the fundamental lessons we learned when interacting with the project teacher. More generally, we became aware of the hypocrisy involved in failing to apply our developing conceptions of the learning-teaching process reflectively to guide our own practice. We, like the teachers, encountered problematic and surprising situations that challenged us to question some of our taken-for-granted assumptions. Both our own and the teachers’ pedagogical knowledge and beliefs developed as we struggled with the problems of practice.

Beliefs and Practice
We can clarify the way we currently work with teachers by relating it to two alternative approaches. One approach assumes that a change in teachers’ beliefs will lead to specific changes in classroom practices which, in turn, will result in improved student learning. In other words, changes in beliefs are assumed to come before changes in practice. An extreme example of this approach would be to conduct intensive workshops with teachers and then to leave them to their own devices to figure out what it might mean for their practice. Guskey (1986) observed that “current research on teacher change indicates that the assumptions of this model may be inaccurate, at least under the specific conditions of staff development for experienced teachers” (p. 6).

Although we do not subscribe to this approach, Guskey’s outright dismissal seems overly hasty. We did attempt to influence teachers’ beliefs during the summer institute. We devised situations in which they could draw on their first-hand experiences to question the beliefs that their students were learning with understanding, that there is typically one way to solve second-grade mathematics tasks, and that certain conceptual developments such as constructing an understanding of place value numeration ought to be relatively simple even if students do experience difficulties. In doing so, we encouraged the teachers to make aspects of their current practice problematic. In effect, we asked the teachers to reconsider what they thought they knew. Only then would they have both an initial awareness of other possibilities and reason and motivation to pursue these possibilities by developing a new form of practice.

Guskey proposed a second approach to teacher development that is premised on the assumption that “significant changes in teachers’ beliefs and attitudes are likely to take place only after changes in student learning outcomes are evidenced” (p. 7). This assumption led Guskey to argue that “change is a learning process for teachers that is developmental and primarily experientially based” (p. 7). It follows that since “change occurs mainly after implementation of a new program takes place and evidence of improved student learning is gained, it is continued support following the initial training that is most crucial” (p. 10). This is because “no matter how much advanced staff development occurs, it is when teachers try to implement the new approach that they have the most specific concerns and doubts” (p. 10).

We will consider Guskey’s approach in some detail because it is easy to jump to the conclusion that it is highly compatible with the way we attempt to work with teachers. First, Guskey dismisses the value of gaining an initial sense of commitment from teachers. In contrast, our experience
indicates the importance of helping teachers develop personal, experientially-based reasons and motivations for reorganizing their classroom practice. Second, he seems to view a new form of practice as a collection of "concrete and practical ideas" (p. 6) devised by researchers or staff developers and then given ready-made to teachers. He suggests, for example, that "to be effective a staff development program must offer teachers practical ideas that can be efficiently used to directly enhance desired student learning outcomes" (p. 6).

Further, "if a staff development effort is to be successful, it must clearly illustrate how the new practices can be implemented" (p. 9). The emphasis seems to be on ensuring that teachers teach in the way the staff developer thinks they should rather than on helping them develop a new form of practice. In short, the staff developer is an authority rather than a collaborator when it comes to pedagogical issues. To be sure, we were relatively directive once the teachers began to see their current form of practice as problematic. But this was to make it possible for them to learn in their classrooms rather than to ensure that they taught the way we wanted them to. Third, although Guskey acknowledges that "teachers' knowledge of teaching is validated very pragmatically" (p. 7), he takes this to mean that they focus solely on learning outcomes. There seems to be no room for the possibility that teachers will reflect on what they are doing and develop a rationalization for their activity. Instead, they merely check to see if what they have been told to do works. In fact, Guskey believes it essential that staff developers "ensure that teachers receive regular feedback on student learning progress" (p. 9).

Apparently, teachers do not attempt to interpret students' thinking and learning in the course of their classroom interactions but instead rely on outcome measures of one sort or another. Our experience directly contradicts this assumption. Finally, the very notion of what constitutes a desirable learning outcome seems beyond question in Guskey's approach. In contrast, the issue of what should be our goals as mathematics educators was addressed repeatedly in our discussions with teachers. Is it improved test scores and more correct answers, or are we more concerned that students become increasingly autonomous and task-involved as they engage in meaningful mathematical activity?

In general, our primary goal when working with teachers has been to help them develop forms of practice that they can justify. In doing so, we have attempted to encourage teacher autonomy (Kamii, 1985). This, we believe, is the key to whatever success we have had in initiating and guiding relatively radical reorganizations in both teachers' beliefs and their classroom practices. It is perhaps because Guskey's approach encourages teachers to view themselves as recipients of researchers' wisdom that he is led to assert that teachers generally oppose radical alterations to their present instructional procedures:

Programs or innovations that are dramatically different from teachers' current practices or that require teachers to make major revisions in the way they presently teach are unlikely to be implemented well, if at all. (p. 9)

In presenting the rationale for his approach, Guskey challenged the assumption that changes in practices follow changes in beliefs and instead suggests that beliefs depend on practice. We have difficulty with both contentions in that they are premised on the underlying assumption that the relationship between beliefs and practices is one of linear causality. In our view, arguments about the direction of the assumed causality miss the point; the very nature of the relationship needs to be reconceptualized. Our current work with teachers is based on the alternative assumption that beliefs and practice are dialectically related. Beliefs are expressed in practice, and problems or surprises encountered in practice give rise to opportunities to reorganize beliefs. For example, we argued when analyzing the project teacher's learning that her beliefs and practices were interdependent and developed together. And it is precisely because of this interdependency that her classroom was her primary learning environment.

Conclusion

Throughout this chapter, we have attempted to demonstrate that teachers and students mutually constructed the social contexts within which to learn from each other. At another level, we and the teachers mutually constructed a social context in the course of our interactions that made it possible for us to learn from them and vice versa. In the course of these interactions, we radically revised our beliefs about how we could help teachers reorganize their practice. At the outset of the project, we took for granted the goal of attempting to transform the teachers into constructivists who thought just like we did. It was only when working with teachers that we became aware of the gross hypocrisy implicit in this goal. Clearly, our tolerance for a diversity of ideas did not extend to our epistemology.

Our goal, as we now see it, is to help teachers develop forms of pedagogical practice that improve the quality of their students' mathematical education, not to spread a particular philosophical doctrine. We are well aware that there are significant differences in the ways we and the teachers rationalize during their mathematics instruction. Few, if any, of the teachers would agree completely with a statement such as "learning is the process by which students reorganize their sensory-motor and conceptual activity to resolve experientially-based problematic situations" even if they could unravel the terminology. The crucial point is that our own and the teachers' interpretations of classroom events need only be compatible for the purposes at hand. We have learned to discuss differences between our own and teachers' interpretations only if they are differences that make a
difference in terms of classroom practice. In this regard, we agree with Kilpatrick's (1987) contention that there is not a one-to-one correspondence between background theories and forms of pedagogical practice. It is, in fact, for this reason that at we speak of "forms of teaching compatible with constructivism" rather than "constructivist teaching."


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