VISCOELASTICITY AND ROCKS

1. Introduction

Rocks show viscoelastic behavior. For example, creep occurs upon excavation in rocks such as shale, claystone and evaporites. Time dependent effects of rocks have also been observed in the laboratory. Figure 1 shows typical creep curves of salt tested at different temperatures ranging between 25 and 100°C.

Figure 2 shows a typical creep curve for rocks. The curve consists of four parts

- an instantaneous elastic strain ε_e
- a primary or transient creep phase. If the stress is released to zero suddenly then $PQ = \varepsilon_e$ and QR tends to zero (no permanent strain)
- steady state or secondary creep. If stress is released to zero, the rock shows a permanent strain
- tertiary creep that leads to failure

Note that rock creep depends largely on temperature, stress level, confining stress level, pore water pressure, humidity.

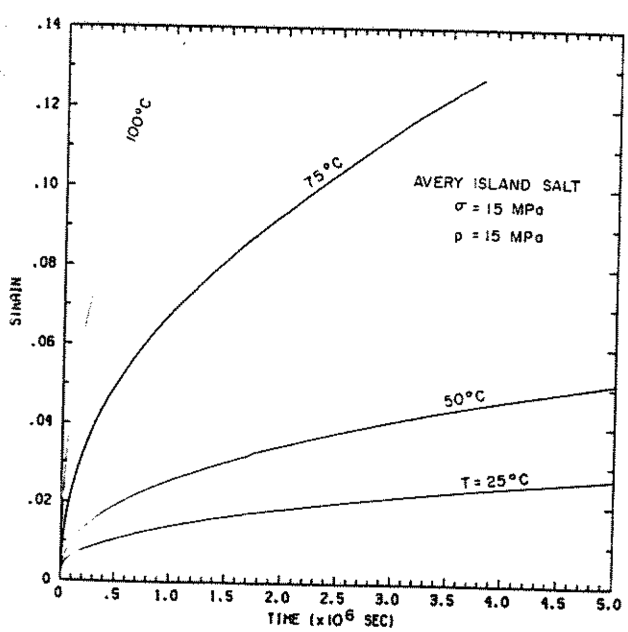


Figure 1. Viscoelastic response of salt (Senseny).

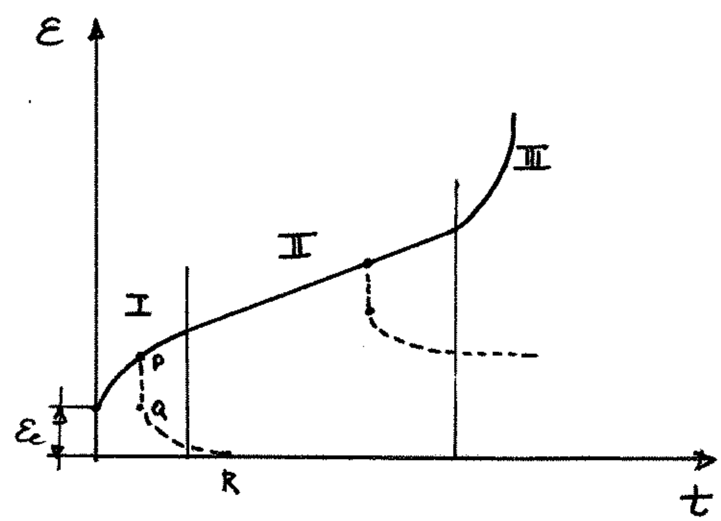


Figure 2. Typical Creep Curve for Rock

Two sets of creep laws are available in the literature : empirical laws and laws derived from rheological models.

Creep can be represented by the following equation

$$\varepsilon = \varepsilon_e + \varepsilon_1(t) + \varepsilon_2(t) + \varepsilon_3(t) \qquad (1)$$

where ε_e : instantaneous elastic strain

$\varepsilon_1(t)$: transient creep

- power law $\varepsilon_1(t) = At^n \quad (0<n<1)$
- logarithm law $\varepsilon_1(t) = A \ln t$

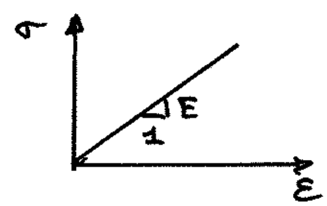
$\varepsilon_2(t)$: secondary creep with $\varepsilon_2(t) = Vt$ where $V = \dot{\varepsilon}$ is constant

$\varepsilon_3(t)$: tertiary creep.

2. Rheological Models.

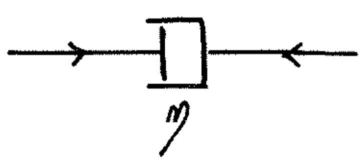
Since rocks are time dependent in their response to loading, the elastic solid (Hookean solid) and the viscous fluid (Newtonian fluid) represent two extreme models. Rheological models are based on simple physical idealizations and consist of combinations of massless linear springs and viscous dashpots.

$\sigma = E\varepsilon$

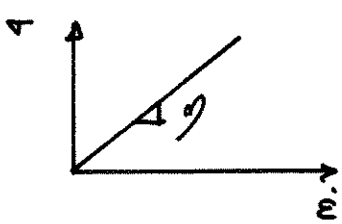


Hookean substance

E: spring constant



$\sigma = \eta\dot{\varepsilon}$



Newtonian substance $\quad \eta$: viscosity constant

$\dot{\varepsilon} = d\varepsilon/dt$ is the strain rate.

Several two, three and four parameter models obtained by combination of the two basic models are shown in Figure 3.

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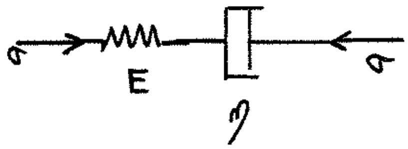

Maxwell model

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

$$\sigma + p_1\dot{\sigma} = q_1\dot{\varepsilon}$$

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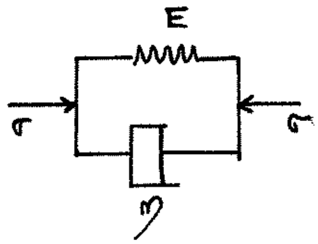
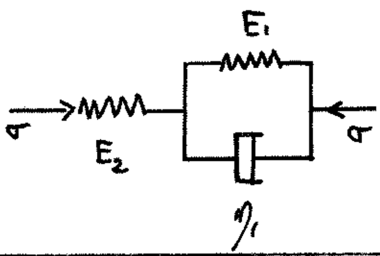


Kelvin-Voigt model

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

$$\sigma = q_0\varepsilon + q_1\dot{\varepsilon}$$

③



$$\sigma + \frac{\eta_1}{E_1+E_2}\dot{\sigma} = \frac{E_1E_2}{E_1+E_2}\varepsilon + \frac{E_2\eta_1}{E_1+E_2}\dot{\varepsilon}$$

$$\sigma + p_1\dot{\sigma} = q_0\varepsilon + q_1\dot{\varepsilon}$$

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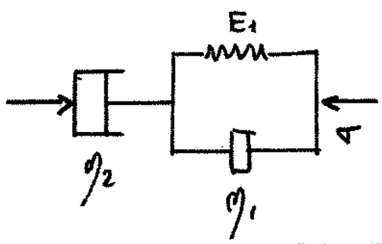
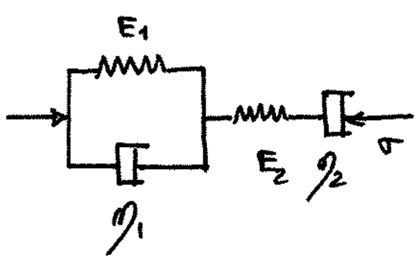


$$\dot{\sigma}\left(\frac{1}{E_1} + \frac{1}{\eta_2} + \frac{\eta_1}{E_1\eta_2}\right) = \dot{\varepsilon} + \frac{\eta_1}{E_1}\ddot{\varepsilon}$$

$$p_1\dot{\sigma} = q_1\dot{\varepsilon} + q_2\ddot{\varepsilon}$$

⑤



Burger model.

$$\sigma + \dot{\sigma}\frac{\eta_2}{E_1}\left(1 + \frac{E_1}{E_2} + \frac{\eta_1}{\eta_2}\right) + \ddot{\sigma}\frac{\eta_1\eta_2}{E_1E_2} = \eta_2\dot{\varepsilon} + \frac{\eta_1\eta_2}{E_1}\ddot{\varepsilon}$$

$$p_0\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} = q_1\dot{\varepsilon} + q_2\ddot{\varepsilon}$$

Figure 3. Sereral two, three and four parameter models used in viscoelasticity.

In general, stresses, strains and rates of stress and strain are related as follows

$$p_0\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} + \ldots\ldots = q_0\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon} + \ldots \qquad (2a)$$

or

$$\sum_{i=0}^{m} p_i \frac{d^i\sigma}{dt^i} = \sum_{i=0}^{n} q_i \frac{d^i\varepsilon}{dt^i} \qquad (2b)$$

or

$$\{P\}\sigma = \{Q\}\varepsilon \qquad (2c)$$

with

$$\{P\} = \sum_{i=0}^{m} p_i \frac{d^i}{dt^i} \quad \text{and} \quad \{Q\} = \sum_{i=0}^{n} q_i \frac{d^i}{dt^i} \qquad (2d)$$

3. <u>Creep and Relaxation</u>

Creep and relaxation tests are the two basic experiments of linear viscoelasticity.

<u>Creep</u> consists of applying a constant stress σ_0 that is maintained constant with time while $\varepsilon(t)$ is recorded. In mathematical form

$$\sigma = \sigma_0 H(t) \qquad\qquad \varepsilon = \sigma_0 J(t) \qquad (3)$$

where $H(t)$ is the Heaviside function and $J(t)$ is the creep function or creep compliance (Figure 4)

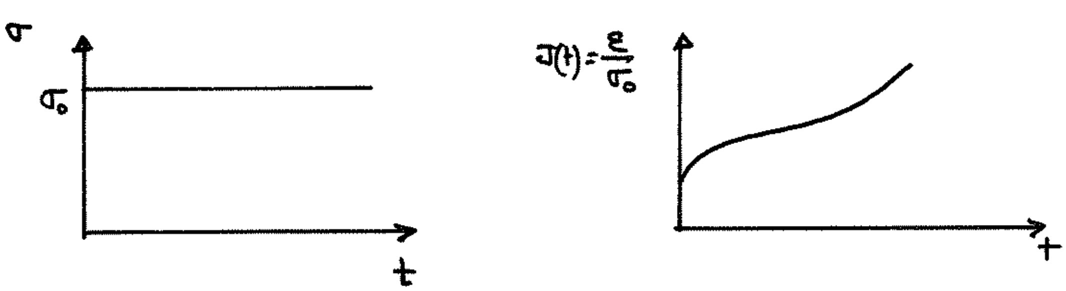


Figure 4 : Creep test.

On the other hand, a <u>relaxation</u> experiment consists of applying a constant strain ε_o while $\sigma(t)$ is recorded. In mathematical form

$$\varepsilon = \varepsilon_o H(t) \qquad \sigma = \varepsilon_o E(t) \qquad (4)$$

where $E(t)$ is the relaxation modulus (Figure 5)

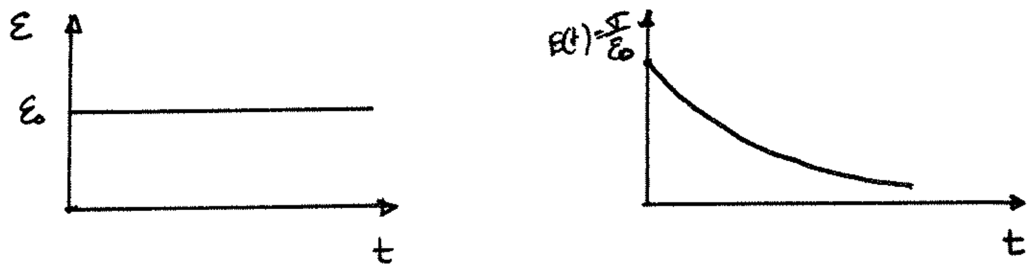


Figure 5. Relaxation test.

Creep and relaxation curves for the Maxwell and Kelvin models are shown in Figure 6.

4. Superposition Integrals

For a stress history $\sigma = \sigma(t)$, it can be shown using the superposition principle that the creep response is given by

$$\varepsilon(t) = \sigma_o J(t) + \int_{0^+}^{t} \frac{d\sigma(t')}{dt'} J(t-t')\,dt' \qquad (5a)$$

where σ_o is the stress jump at time $t=0$. This equation is also equal to

$$\varepsilon(t) = \sigma(t)J(0) + \int_{0^+}^{t} \sigma(t') \frac{dJ(t-t')}{d(t-t')}\,dt' \qquad (5b)$$

The stress as a function of time may also be represented by a superposition integral involving the strain history and the relaxation modulus $E(t)$ as follows

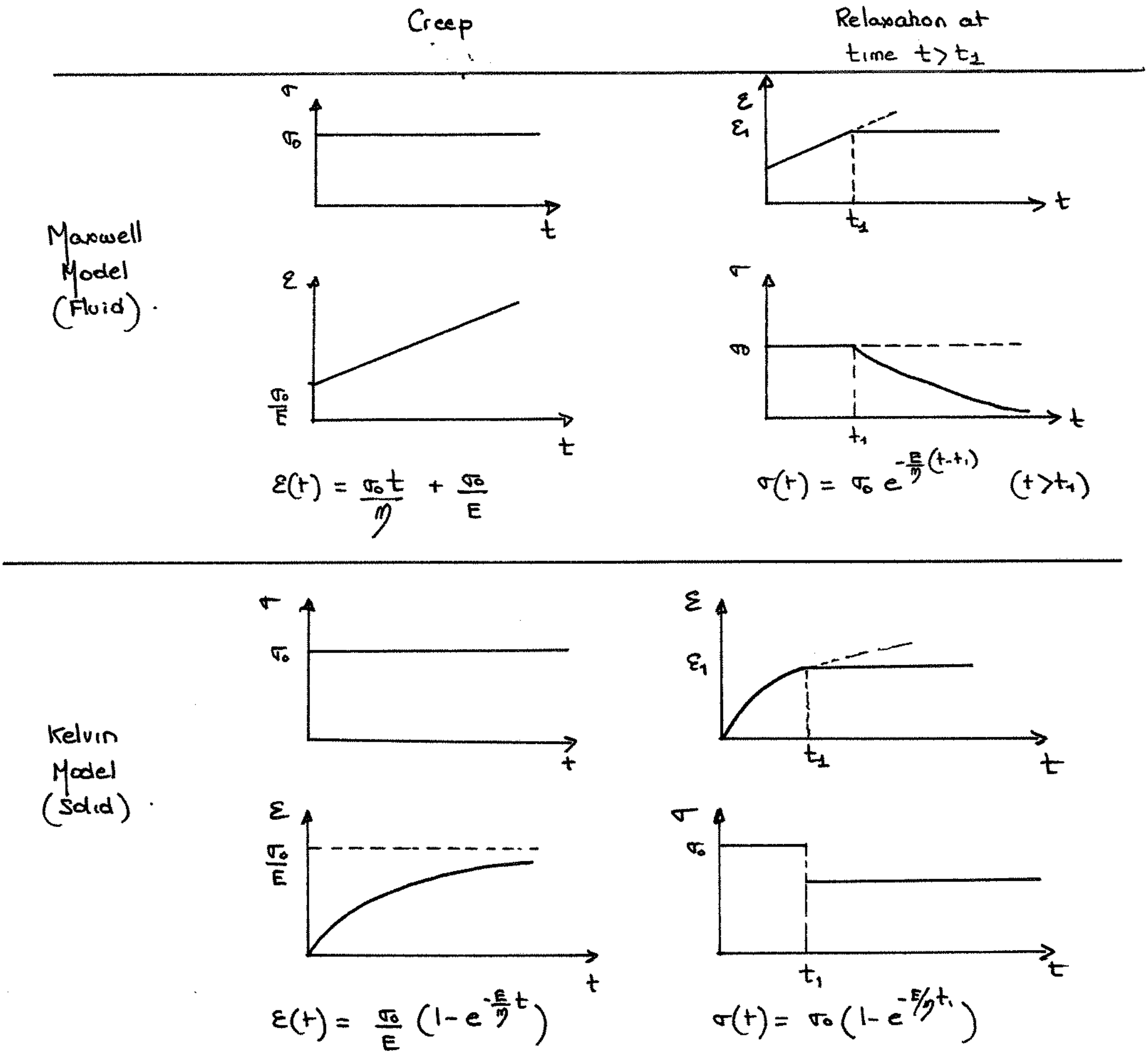


Figure 6. Creep and Relaxation Curves for the Maxwell and Kelvin Models.

$$\sigma(t) = \varepsilon_0 E(t) + \int_{0^+}^{t} \frac{d\varepsilon(t')}{dt'} E(t-t')\,dt' \qquad (6a)$$

or
$$\sigma(t) = \varepsilon(t) E(0) + \int_{0^+}^{t} \varepsilon(t') \frac{d\,E(t-t')}{d(t-t')}\,dt' \qquad (6b)$$

Creep and relaxation functions are related as follows

$$\int_{0^+}^{t} E(t-t')\,J(t')\,dt' = t \qquad (7)$$

5. Burger Model

Consider the Burger model of Figure 10 subject to a constant stress of magnitude σ_0. Let ε_1 and ε_2 be the strains in the Kelvin and Maxwell parts of this model respectively. Thus,

$$\sigma = E_1\varepsilon_1 + \eta_1\dot{\varepsilon}_1 \quad \text{and} \quad \frac{\dot{\sigma}}{E_2} + \frac{\sigma}{\eta_2} = \dot{\varepsilon}_2 \qquad (8)$$

with $\varepsilon = \varepsilon_1 + \varepsilon_2$. and $\sigma = \sigma_0\, H(t)$ (9)

Using the Laplace transforms of eq. (8) and eq. (9) we have

$$\bar{\varepsilon} = \frac{\bar{\sigma}}{E_2} + \frac{\bar{\sigma}}{s\eta_2} + \frac{\bar{\sigma}}{E_1 + \eta_1 s} \qquad (10)$$

where $\bar{\varepsilon}$ and $\bar{\sigma} = \sigma_0/s$ are the Laplace transforms of ε and σ, respectively. Then

$$\frac{\bar{\varepsilon}}{\sigma_0} = \frac{1}{E_2 s} + \frac{1}{s^2\eta_2} + \frac{1}{s(E_1+\eta_1 s)} = \frac{1}{E_2 s} + \frac{1}{s^2\eta_2} + \frac{1}{E_1 s} - \frac{1}{E_1\left(\frac{E_1}{\eta_1} + s\right)} \qquad (11)$$

The inverse Laplace transform of eq. (11) gives

$$\frac{\varepsilon(t)}{\sigma_0} = J(t) = \frac{1}{E_2} + \frac{t}{\eta_2} + \frac{1}{E_1}\left(1 - e^{-E_1/\eta_1 t}\right) \qquad (12)$$

In this equation:

σ_0/E_2 represents the instantaneous elastic response
$\sigma_0 t/\eta_2$ represents viscous flow (steady state or II creep)
$\sigma_0\left(1-e^{-E_1/\eta_1 t}\right)/E_1$ represents the delayed elastic response (transient creep).
The creep curve corresponding to eq. (12) is shown in Figure 7.

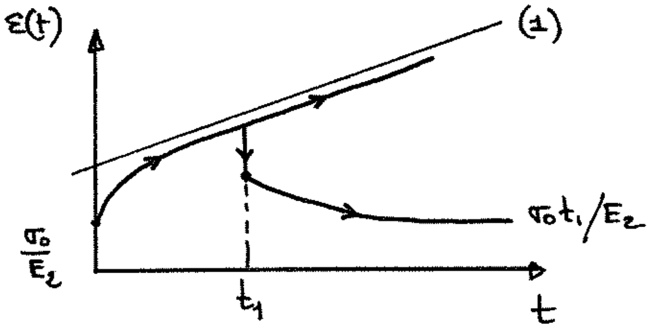


Figure 7. Creep curve for a Burger model.

Note that the creep curve becomes asymptotic to a line with equation $\varepsilon(t) = \frac{\sigma_0}{E_1} + \frac{\sigma_0}{E_2} + \frac{\sigma_0}{\eta_2} t$ shown as line (1) in Figure 7.

If stress is released to zero at time $t = t_1$, a permanent strain exists and can be calculated using eq. (5b). Since $\sigma(t) = 0$ for $t > t_1$, eq. (5 b) becomes

$$\varepsilon(t) = \int_{0^+}^{t_1} \sigma(t') \frac{d J(t-t')}{d(t-t')} dt' = \sigma_0 \left[J(t) - J(t-t_1)\right] \qquad (13)$$

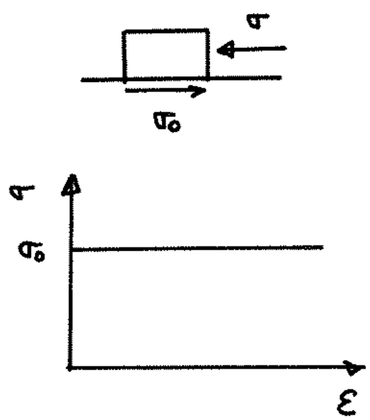
Using eq. (12), $\varepsilon(t)$ is equal to

$$\varepsilon(t) = \frac{\sigma_0 t_1}{E_2} + \frac{\sigma_0}{E_1}\left(e^{-\frac{E_1}{\eta_1}(t-t_1)} - e^{-\frac{E_1}{\eta_1} t}\right) \qquad (14)$$

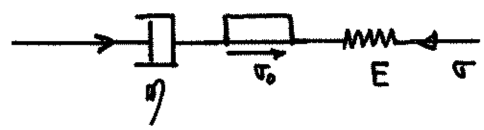
As $t \to \infty$, $\varepsilon(t)$ approaches a finite strain equal to $\sigma_0 t_1/E_2$. (see Figure 7)

6. Bingham Model

The Bingham model is a viscoelastoplastic model that consists of a Maxwell model in series with a frictional element (Figure 8).



(a)



$\sigma < \sigma_0 \qquad \varepsilon = \sigma / E$

$\sigma > \sigma_0 \qquad \varepsilon = (\sigma - \sigma_0)\dfrac{t}{\eta} + \dfrac{\sigma}{E}$

(b)

Figure 8. (a) Perfectly plastic element, (b) Bingham model.