STRESSES AND STRAINS - A REVIEW

1. INTRODUCTION

Rock mechanics, being an interdisciplinary field, borrows many concepts from the field of continuum mechanics and mechanics of materials, and in particular, the concepts of stress and strain. Stress is of importance to geologists and geophysicists in order to understand the formation of geological structures such as folds, faults, intrusions, etc...It is also of importance to civil, mining and petroleum engineers who are interested in the stability and performance of man-made structures (tunnels, caverns, mines, surface excavations, etc..), or the stability of boreholes. A list of activities requiring knowledge of stresses is given in Table 1. Stress terminology is shown in Figure 1.

Unlike man-made materials such as concrete or steel, natural materials such as rocks (and soils) are initially stressed in their natural state. Stresses in rock can be divided into *in situ* stresses and *induced* stresses. *In situ* stresses, also called natural, primitive or virgin stresses, are the stresses that exist in the rock prior to any disturbance. On the other hand, induced stresses are associated with man-made disturbance (excavation, drilling, pumping, loading, etc..) or are induced by changes in natural conditions (drying, swelling, consolidation, etc..). Induced stresses depend on many parameters such as the *in situ* stresses, the type of disturbance (excavation shape, borehole diameter, etc..), and the rock mass properties.

Stress is an enigmatic quantity which, according to classical mechanics, is defined at a point in a continuum and is independent of the constitutive behavior of the medium. The concept of stress used in rock mechanics is consistent with that formulated by Cauchy and generalized by St. Venant in France during the 19th century (Timoshenko, 1983). Because of its definition, rock stress is a fictitious quantity creating challenges in its characterization, measurement, and application in practice. A summary of the continuum mechanics description of stress is presented below. More details can be found in Mase (1970).

2. STRESS ANALYSIS

2.1 Cauchy Stress Principle

Consider for instance, the continuum shown in Figure 2 occupying a region R of space and subjected to body forces $\mathbf{b}$ (per unit of mass) and surface forces $\mathbf{f}_s$ (tractions). Let x,y,z be a Cartesian coordinate system with unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ parallel to the x, y, and z directions, respectively.

Consider a volume V in the continuum, an infinitesimal surface element ΔS located on the outer surface S of V, a point P located on ΔS, and a unit vector $\mathbf{n}$ normal to ΔS at P. Under the effect of the body and surface forces, the material within volume V interacts with the material outside of V. Let $\Delta\mathbf{f}$ and $\Delta\mathbf{m}$ be respectively the resultant force and moment exerted across ΔS by the material outside of V upon the material within V. The *Cauchy stress principle* asserts that the average force per unit area $\Delta\mathbf{f}/\Delta S$ tends to a limit $d\mathbf{f}/dS$ as ΔS tends to zero, whereas $\Delta\mathbf{m}$ vanishes in the limiting process. The limit is called the *stress vector* $\mathbf{t}_{(n)}$, i.e.

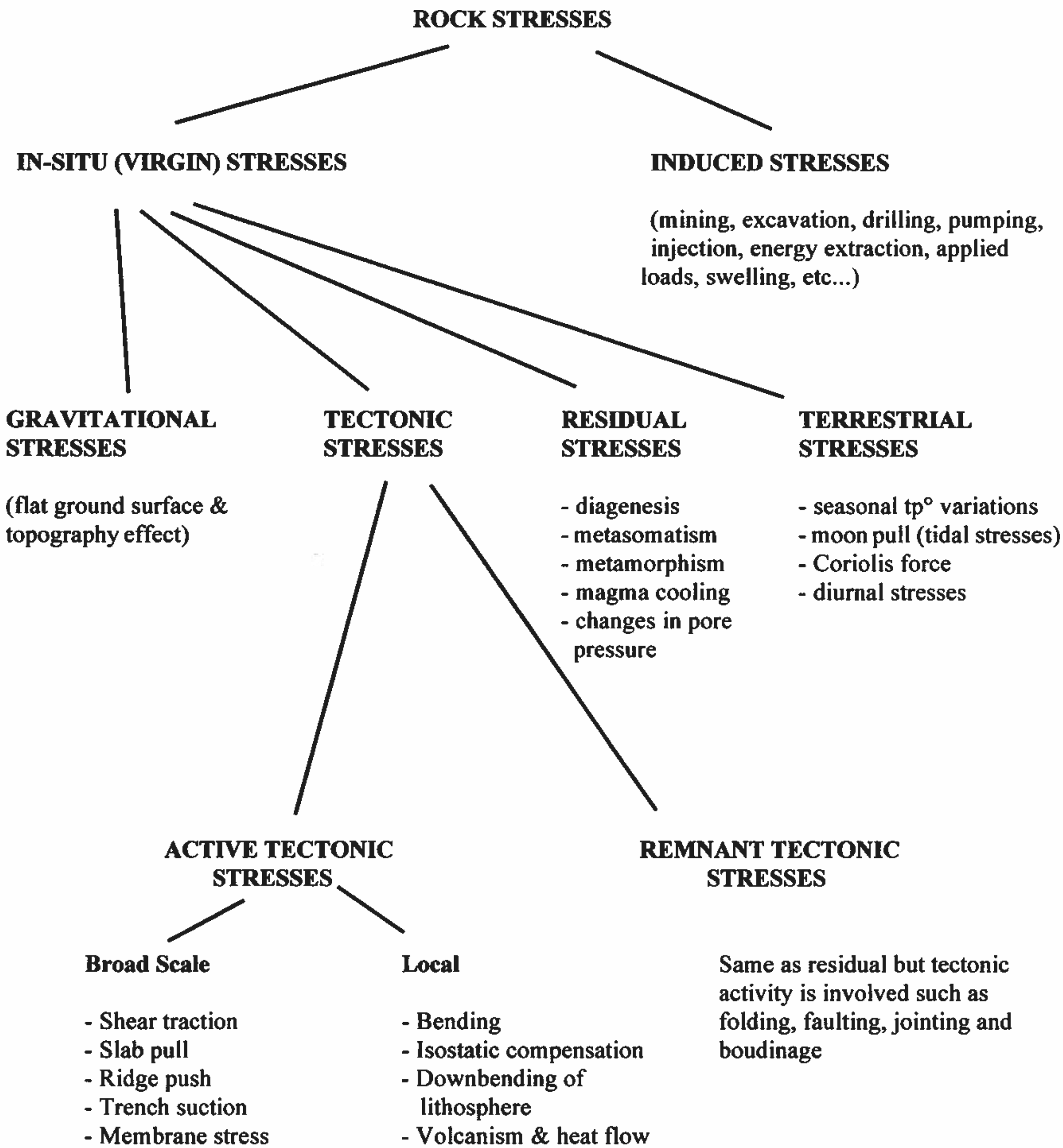

Figure 1 Stress terminology.

Civil & Mining Engineering
• Stability of Underground Excavations (Tunnels, Mines, Caverns, Shafts, Stopes, Haulages) • Drilling & Blasting • Pillar Design • Design of Support Systems • Prediction of Rock Bursts • Fluid Flow & Contaminant Transport • Dams • Slope Stability
Energy Development
• Borehole stability & deviation • Borehole deformation & failure • Fracturing & fracture propagation • Fluid flow & geothermal problems • Reservoir production management • Energy extraction and storage
Geology/Geophysics
• Orogeny • Earthquake Prediction • Plate Tectonics • Neotectonics • Structural Geology • Volcanology • Glaciation

Table 1. Activities requiring knowledge of *in-situ* stresses.

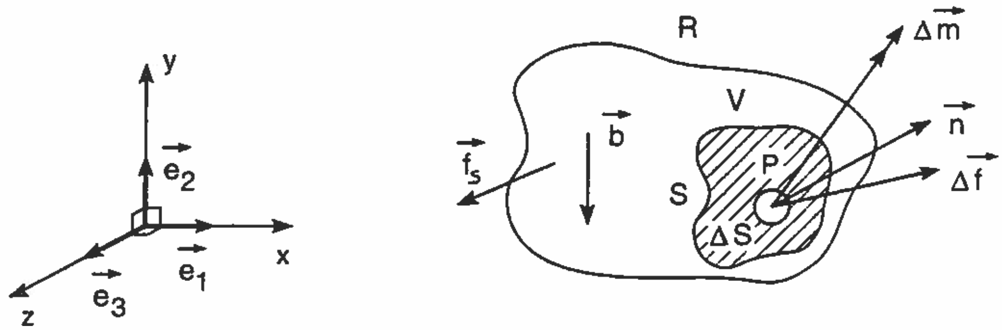

Figure 2. Material Continuum subjected to body and surface forces.

$$t_{(n)} = \lim_{\Delta S \to 0} \frac{\Delta f}{\Delta S} = \frac{df}{ds} \qquad (1)$$

The stress vector has three components in the x,y,z coordinate system which are expressed in units of force per unit area (MPa, psi, psf,..). It is noteworthy that the components of the stress tensor depend on the orientation of the surface element ΔS which is defined by the coordinates of its normal unit vector **n**.

The stress vector $\mathbf{t}_{(n)}$ at point P in Figure 2 is associated with the action of the material outside of V upon the material within V. Let $\mathbf{t}_{(-n)}$ be the stress vector at point P corresponding to the action across ΔS of the material within V upon the material outside of V. By Newton's law of action and reaction

$$t_{(n)} + t_{(-n)} = \mathbf{0} \qquad (2)$$

Equation (2) implies that the stress vectors acting on opposite sides of a same surface are equal in magnitude but opposite in direction.

2.2 State of Stress at a Point

The state of stress at point P in Figure 2 can be defined by using equation (1) for all possible infinitesimal surfaces ΔS having point P as an interior point. An alternative is to consider the stress vectors $\mathbf{t}_{(e1)}$, $\mathbf{t}_{(e2)}$, and $\mathbf{t}_{(e3)}$ acting on three orthogonal planes normal to the x-, y- and z-axes and with normal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, respectively. The three planes form an infinitesimal stress element around point P (Figures 3a and 3b).

The nine components of vectors $\mathbf{t}_{(e1)}$, $\mathbf{t}_{(e2)}$, and $\mathbf{t}_{(e3)}$ form the components of a second-order Cartesian tensor also known as the *stress tensor* σ_{ij} (i,j=1-3). The components σ_{11}, σ_{22} and σ_{33} represent the three *normal stresses* σ_x, σ_y and σ_z acting in the x, y, and z directions, respectively. The components σ_{ij} ($i \neq j$) represent six *shear stresses* τ_{xy}, τ_{yx}, τ_{xz}, τ_{zx}, τ_{yz} and τ_{zy} acting in the xy, xz and yz planes. Two sign conventions are considered below:

Engineering mechanics sign convention

Tensile normal stresses are treated as positive and the direction of positive shear stresses is as shown in Figure 3a. The stress vectors $\mathbf{t}_{(e1)}$, $\mathbf{t}_{(e2)}$, and $\mathbf{t}_{(e3)}$ have the following expressions

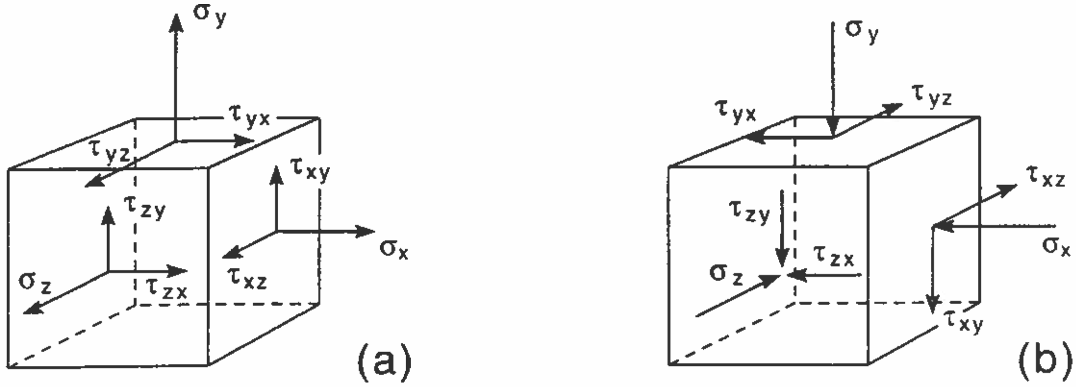

Figure 3. Direction of positive normal and shear stresses. (a) Engineering mechanics convention; (b) Rock Mechanics convention.

$$t_{(e1)} = \sigma_x e_1 + \tau_{xy} e_2 + \tau_{xz} e_3$$
$$t_{(e2)} = \tau_{yx} e_1 + \sigma_y e_2 + \tau_{yz} e_3 \quad (3)$$
$$t_{(e3)} = \tau_{zx} e_1 + \tau_{zy} e_2 + \sigma_z e_3$$

Rock mechanics sign convention

Compressive normal stresses are treated as positive and the direction of positive shear stresses is as shown in Figure 3b. The stress vectors $\mathbf{t}_{(e1)}$, $\mathbf{t}_{(e2)}$, and $\mathbf{t}_{(e3)}$ have the following expressions

$$t_{(e1)} = -\sigma_x e_1 - \tau_{xy} e_2 - \tau_{xz} e_3$$
$$t_{(e2)} = -\tau_{yx} e_1 - \sigma_y e_2 - \tau_{yz} e_3 \quad (4)$$
$$t_{(e3)} = -\tau_{zx} e_1 - \tau_{zy} e_2 - \sigma_z e_3$$

2.3 State of Stress on an Inclined Plane

Knowing the components of the stress tensor representing the state of stress at a point P, the components of the stress vector on any plane passing by P, and of known orientation with respect to the x-, y-, and z-axes, can be determined.

Consider again point P of Figure 2 and let σ_{ij} be the stress tensor representing the state of stress at that point. The components of the stress vector $\mathbf{t}_{(n)}$ acting on an inclined plane passing through P can be expressed in terms of the σ_{ij} components and the orientation of the plane using a limiting process similar to that used to introduce the stress vector concept. As shown in Figure 4, consider a plane ABC of area dS parallel to the plane of interest passing through P. Let $\mathbf{n}$ be the normal to the plane with components n_1, n_2, and n_3. The force equilibrium of the PABC tetrahedron leads to the following relation between the average stress vectors acting on its faces

$$t_{(n)} dS + t_{(-e_1)} n_1 dS + t_{(-e_2)} n_2 dS + t_{(-e_3)} n_3 dS = \mathbf{0} \quad (5)$$

where n_1dS, n_2dS and n_3dS are respectively the areas of faces CPB, CPA and APB of the tetrahedron. Using equation (2), $\mathbf{t}_{(n)}$ can be expressed as follows

$$t_{(n)} = t_{(e_1)} n_1 + t_{(e_2)} n_2 + t_{(e_3)} n_3 \quad (6)$$

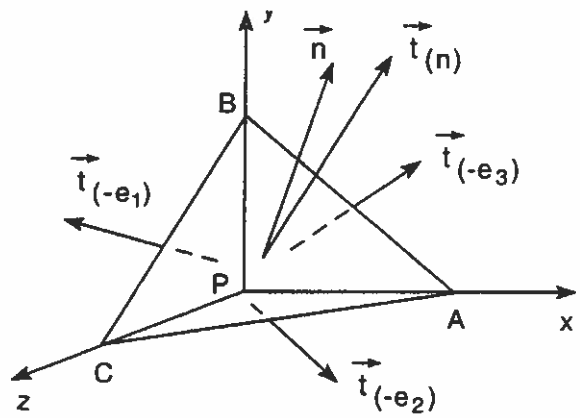

Figure 4. State of stress on an inclined plane passing through point P.

The stress acting on plane ABC will approach the stress on the parallel plane passing through P as the tetrahedron in Figure 4 is made infinitesimal. In that limiting process, the contribution of any body force acting in the PABC tetrahedron vanishes.

Equation (6) can also be expressed in terms of the normal and shear stress components at point P. Let t_x, t_y and t_z be the x, y, z components of the stress vector $\mathbf{t}_{(n)}$. When using the engineering mechanics sign convention, combining equations (3) and (6) yields

$$\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{7a}$$

On the other hand, for the rock mechanics sign convention, combining equations (4) and (6) yields

$$-\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{7b}$$

The (3 x 3) matrix in equations (7a) and (7b) is a matrix representation of the stress tensor σ_{ij}.

2.4 Force and Moment Equilibrium

For all differential elements in the continuum of Figure 2, force and moment equilibrium leads respectively to the equilibrium equations and the symmetry of the stress tensor σ_{ij}.

Equations of equilibrium

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho b_1 &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho b_2 &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho b_3 &= 0 \end{aligned} \tag{8}$$

where ρ is the density and ρb_1, ρb_2 and ρb_3 are the components of the body force per unit volume of the continuum in the x, y and z directions, respectively. The positive directions of those components are in the positive x, y and z directions if the engineering mechanics convention for stress is used, and in the negative x, y and z directions if the rock mechanics sign convention is used instead.

Symmetry of stress tensor

$$\tau_{xy} = \tau_{yx}; \quad \tau_{xz} = \tau_{zx}; \quad \tau_{yz} = \tau_{zy} \tag{9}$$

which implies that only six stress components are needed to describe the state of stress at a point in a continuum: three normal stresses and three shear stresses.

2.5 Stress Transformation Law

Consider now two rectangular coordinate systems x,y,z and x',y',z' at point P. The orientation of the x'-, y'-, z'-axes is defined in terms of the direction cosines of unit vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$ and $\mathbf{e}'_3$ in the x,y,z coordinate system, i.e.

$$\begin{aligned} e_1' &= l_{x'}e_1 + m_{x'}e_2 + n_{x'}e_3 \\ e_2' &= l_{y'}e_1 + m_{y'}e_2 + n_{y'}e_3 \\ e_3' &= l_{z'}e_1 + m_{z'}e_2 + n_{z'}e_3 \end{aligned} \tag{10}$$

Let [A] be a coordinate transformation matrix such that

$$[A] = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} \\ l_{y'} & m_{y'} & n_{y'} \\ l_{z'} & m_{z'} & n_{z'} \end{bmatrix} \tag{11}$$

Matrix [A] is an orthogonal matrix with $[A]^t = [A]^{-1}$. Using the coordinate transformation law for second order Cartesian tensors, the components of the stress tensor σ'_{ij} in the x',y',z' coordinate system are related to the components of the stress tensor σ_{ij} in the x,y,z coordinate system as follows

$$\begin{bmatrix}\sigma_{x'} & \tau_{x'y'} & \tau_{x'z'}\\ \tau_{x'y'} & \sigma_{y'} & \tau_{y'z'}\\ \tau_{x'z'} & \tau_{y'z'} & \sigma_{z'}\end{bmatrix}=\begin{bmatrix}l_{x'} & m_{x'} & n_{x'}\\ l_{y'} & m_{y'} & n_{y'}\\ l_{z'} & m_{z'} & n_{z'}\end{bmatrix}\begin{bmatrix}\sigma_{x} & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \sigma_{y} & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \sigma_{z}\end{bmatrix}\begin{bmatrix}l_{x'} & l_{y'} & l_{z'}\\ m_{x'} & m_{y'} & m_{z'}\\ n_{x'} & n_{y'} & n_{z'}\end{bmatrix} \tag{12}$$

Using (6x1) matrix representation of σ'_{ij} and σ_{ij}, and after algebraic manipulations, equation (12) can be rewritten in matrix form as follows

$$[\sigma]_{x'y'z'} = [T_\sigma][\sigma]_{xyz} \tag{13}$$

where $[\sigma]^t_{xyz}=[\sigma_x\ \sigma_y\ \sigma_z\ \tau_{yz}\ \tau_{xz}\ \tau_{xy}]$, $[\sigma]^t_{x'y'z'}=[\sigma_{x'}\ \sigma_{y'}\ \sigma_{z'}\ \tau_{y'z'}\ \tau_{x'z'}\ \tau_{x'y'}]$ and $[T_\sigma]$ is a (6x6) matrix whose components can be found in equation A1.23 in Goodman (1989). It can be written as follows

$$[T_\sigma] = \begin{bmatrix}l^2_{x'} & m^2_{x'} & n^2_{x'} & 2m_{x'}n_{x'} & 2l_{x'}n_{x'} & 2m_{x'}l_{x'}\\ l^2_{y'} & m^2_{y'} & n^2_{y'} & 2m_{y'}n_{y'} & 2l_{y'}n_{y'} & 2m_{y'}l_{y'}\\ l^2_{z'} & m^2_{z'} & n^2_{z'} & 2m_{z'}n_{z'} & 2l_{z'}n_{z'} & 2m_{z'}l_{z'}\\ l_{y'}l_{z'} & m_{y'}m_{z'} & n_{y'}n_{z'} & m_{y'}n_{z'}+m_{z'}n_{y'} & n_{y'}l_{z'}+n_{z'}l_{y'} & l_{y'}m_{z'}+l_{z'}m_{y'}\\ l_{x'}l_{z'} & m_{x'}m_{z'} & n_{x'}n_{z'} & m_{x'}n_{z'}+m_{z'}n_{x'} & n_{x'}l_{z'}+n_{z'}l_{x'} & l_{x'}m_{z'}+l_{z'}m_{x'}\\ l_{y'}l_{x'} & m_{y'}m_{x'} & n_{y'}n_{x'} & m_{y'}n_{x'}+m_{x'}n_{y'} & n_{x'}l_{y'}+n_{y'}l_{x'} & l_{x'}m_{y'}+l_{y'}m_{x'}\end{bmatrix}$$

Expressions for the direction cosines $l_{x'}$, $m_{x'}$, $n_{x'}$......are given below for two special cases shown in Figures 5a and 5b, respectively. In Figure 5a, the orientation of the x'-axis is defined by two angles β and δ and the z'-axis lies in the Pxz plane. In this case, the direction cosines are

$$\begin{gathered} l_{x'}=\cos\delta\cos\beta;\quad m_{x'}=\sin\delta;\quad n_{x'}=\cos\delta\sin\beta \\ l_{y'}=-\sin\delta\cos\beta;\quad m_{y'}=\cos\delta;\quad n_{y'}=-\sin\delta\sin\beta \\ l_{z'}=-\sin\beta;\quad m_{z'}=0;\quad n_{z'}=\cos\beta \end{gathered} \tag{14}$$

If we take $\beta=0$, $\delta=\theta$, and the z'-axis to coincide with the z-axis, the x'-, y'- and z'-axes coincide, for instance, with the radial, tangential and longitudinal axes of a cylindrical coordinate system r,θ,z (Figure 5b) with

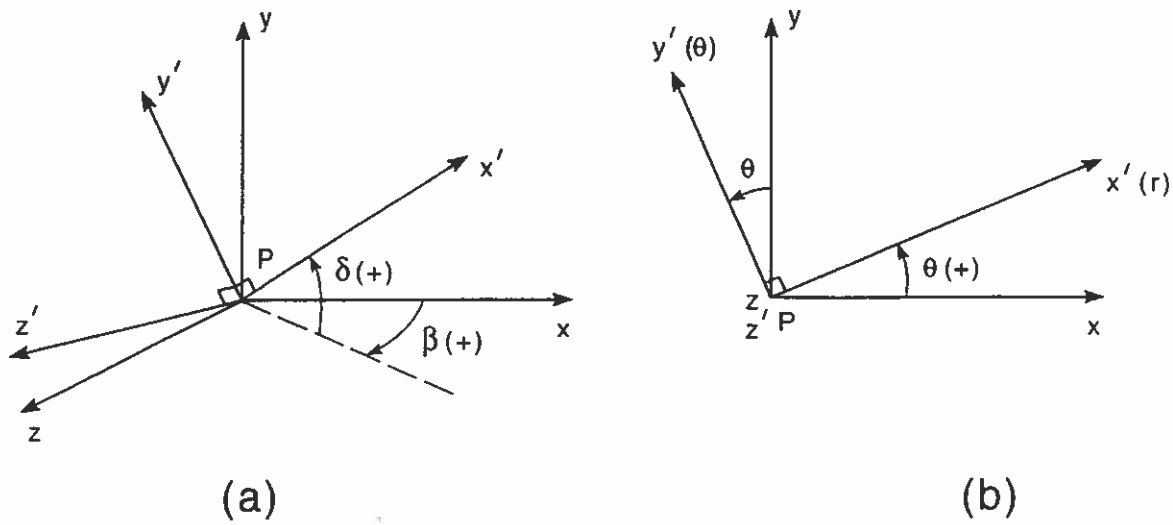

Figure 5. Two special orientations of x'-, y'- and z'-axes with respect to the x, y, z coordinate system.

$$l_{x'} = l_r = \cos\theta; \quad m_{x'} = m_r = \sin\theta; \quad n_{x'} = n_r = 0$$

$$l_{y'} = l_\theta = -\sin\theta; \quad m_{y'} = m_\theta = \cos\theta; \quad n_{y'} = n_\theta = 0 \tag{15}$$

$$l_{z'} = 0; \quad m_{z'} = 0; \quad n_{z'} = 1$$

Substituting these direction cosines into equation (12) gives a relationship between the stress components in the r, θ, z coordinate system and those in the x,y,z coordinate system as follows

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2 - \tau_{xy} \sin 2\theta$$

$$\tau_{\theta z} = \tau_{yz} \cos\theta - \tau_{xz} \sin\theta \tag{16}$$

$$\tau_{rz} = \tau_{yz} \sin\theta + \tau_{xz} \cos\theta$$

$$\tau_{r\theta} = (\sigma_y - \sigma_x) \sin\theta \cos\theta + \tau_{xy} \cos 2\theta$$

2.6 Normal and Shear Stresses on an Inclined Plane

Consider a plane passing through point P and inclined with respect to the x-, y- and z-axes. Let x',y',z' be a Cartesian coordinate system attached to the plane such that the x'-axis is along its outward normal and the y'- and z'-axes are contained in the plane. The x'-, y'- and z'-axes are oriented as shown in Figure 5 with the direction cosines defined in equation (14).

The state of stress across the plane is defined by one normal component $\sigma_{x'} = \sigma_n$ and two shear components $\tau_{x'y'}$ and $\tau_{x'z'}$ such that (see Figure 6)

$$\begin{bmatrix} \sigma_{x'} \\ \tau_{x'y'} \\ \tau_{x'z'} \end{bmatrix} = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} \\ l_{y'} & m_{y'} & n_{y'} \\ l_{z'} & m_{z'} & n_{z'} \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} l_{x'} \\ m_{x'} \\ n_{x'} \end{bmatrix} \tag{17}$$

Equation (17) is the matrix representation of the first, fifth and sixth lines of equation (13). The resultant shear stress, τ, across the plane is equal to

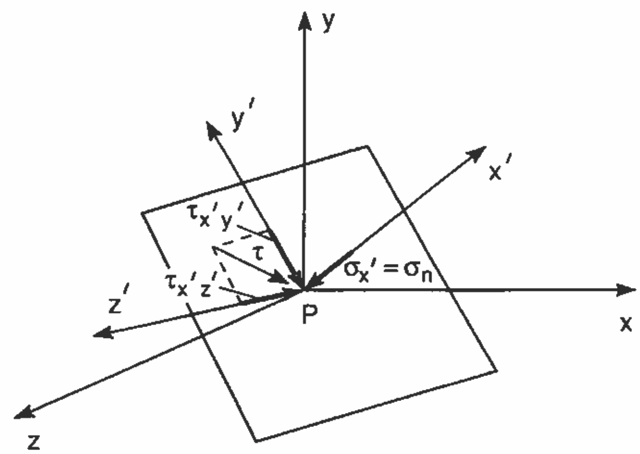

Figure 6. Normal and shear components of the stress vector acting on a plane passing through point P.

$$\tau^2 = \tau_{x'y'}^2 + \tau_{x'z'}^2 \tag{18}$$

The stress vector $\mathbf{t}_{(n)}$ acting on the plane is such that

$$|t_{(n)}|^2 = \sigma_n^2 + \tau^2 = \sigma_{x'}^2 + \tau_{x'y'}^2 + \tau_{x'z'}^2 \tag{19}$$

2.7 Principal Stresses

Among all the planes passing by point P, there are three planes (at right angles to each other) for which the shear stresses. These planes are called *principal planes* and the normal stresses acting on those planes are called *principal stresses* and are denoted σ_1, σ_2 and σ_3 with $\sigma_1 > \sigma_2 > \sigma_3$. Finding the principal stresses and the principal stress directions is equivalent to finding the eigenvalues and eigenvectors of the stress tensor σ_{ij}. Since this tensor is symmetric, the eigenvalues are real.

The eigenvalues of σ_{ij} are the values of the normal stress σ such that the determinant of $\sigma_{ij} - \sigma\delta_{ij}$ vanishes, i.e.

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0 \tag{20}$$

Upon expansion, the principal stresses are the roots of the following cubic polynomial

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \tag{21}$$

where I_1, I_2, and I_3 are respectively the first, second and third *stress invariants* and are equal to

$$\begin{aligned} I_1 &= \sigma_x + \sigma_y + \sigma_z \\ I_2 &= \sigma_y\sigma_z + \sigma_x\sigma_z + \sigma_x\sigma_y - (\tau_{yz}^2 + \tau_{xz}^2 + \tau_{xy}^2) \\ I_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - (\sigma_x\tau_{yz}^2 + \sigma_y\tau_{xz}^2 + \sigma_z\tau_{xy}^2) \end{aligned} \tag{22}$$

For each principal stress σ_k (σ_1, σ_2, σ_3), there is a principal stress direction for which the direction cosines $n_{1k}=\cos(\sigma_k,x)$, $n_{2k}=\cos(\sigma_k,y)$ and $n_{3k}=\cos(\sigma_k,z)$ are solutions of

$$(\sigma_x-\sigma_k)n_{1k}+\tau_{xy}n_{2k}+\tau_{xz}n_{3k}=0$$
$$\tau_{xy}n_{1k}+(\sigma_y-\sigma_k)n_{2k}+\tau_{yz}n_{3k}=0 \tag{23}$$
$$\tau_{xz}n_{1k}+\tau_{yz}n_{2k}+(\sigma_z-\sigma_k)n_{3k}=0$$

with the normality condition

$$n_{1k}^2+n_{2k}^2+n_{3k}^2=1 \tag{24}$$

2.8 Stress Decomposition

The stress tensor σ_{ij} can be separated into a *hydrostatic* component $\sigma_m\delta_{ij}$ and a *deviatoric* component s_{ij}. Using (3x3) matrix representations, the decomposition can be expressed as follows

$$\begin{bmatrix}\sigma_x & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & \sigma_y & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & \sigma_z\end{bmatrix}=\begin{bmatrix}\sigma_m & 0 & 0\\ 0 & \sigma_m & 0\\ 0 & 0 & \sigma_m\end{bmatrix}+\begin{bmatrix}\sigma_x-\sigma_m & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & \sigma_y-\sigma_m & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & \sigma_z-\sigma_m\end{bmatrix} \tag{25}$$

with $\sigma_m=(\sigma_x+\sigma_y+\sigma_z)/3$. As for the stress matrix, three principal deviatoric stresses s_k (k=1,2,3) can be calculated by setting the determinant of $s_{ij}-s\delta_{ij}$ to zero. Equation (21) is then replaced by the following cubic polynomial

$$s^3-J_1s^2-J_2s-J_3=0 \tag{26}$$

where J_1, J_2, and J_3 are respectively the first, second and third invariants of the deviatoric stress tensor and are equal to

$$J_1=0$$
$$J_2=-(s_ys_z+s_xs_z+s_ys_x)+\tau_{yz}^2+\tau_{xz}^2+\tau_{xy}^2 \tag{27}$$
$$J_3=s_xs_ys_z+2\tau_{xy}\tau_{xz}\tau_{yz}-(s_x\tau_{yz}^2+s_y\tau_{xz}^2+s_z\tau_{xy}^2)$$

with $s_x=\sigma_x-\sigma_m$, $s_y=\sigma_y-\sigma_m$, and $s_z=\sigma_z-\sigma_m$. Note that J_2 can also be written as follows

$$J_2 = \frac{1}{6}[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2] \tag{28}$$

2.9 Octahedral Stresses

Let assume that the x, y, and z directions of the x,y,z coordinate system coincide with the principal stress directions, i.e. $\sigma_x=\sigma_1$, $\sigma_y=\sigma_2$, and $\sigma_z=\sigma_3$. Consider a plane that makes equal angles with the three coordinate axes and whose normal has components $n_1=n_2=n_3=1/\sqrt{3}$. This plane is an *octahedral* plane. The normal stress across the plane is called the *octahedral normal stress*, σ_{oct}, and the shear stress is called the *octahedral shear stress*, τ_{oct}. The stresses are equal to

$$\sigma_{oct} = \frac{(\sigma_1+\sigma_2+\sigma_3)}{3} = \frac{I_1}{3}$$
$$\tau_{oct}^2 = \frac{1}{9}[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2] = \frac{2}{3}J_2 \tag{29}$$

2.10 References

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3. STRAIN ANALYSIS

3.1 Deformation and Finite Strain Tensors

Consider a material continuum which at time t=0 can be seen in its initial or undeformed configuration and occupies a region R_o of Euclidian 3D-space (Figure 7). Any point P_o in R_o can be described by its coordinates X_1, X_2, X_3 with reference to a suitable set of coordinate axes (*material coordinates*). Upon deformation and at time t=t, the continuum will now be seen in its deformed configuration, R being the region it now occupies. Point P_o will move to a position P with coordinates x_1, x_2, x_3 (*spatial coordinates*). The X_1,X_2,X_3 and x_1,x_2,x_3 coordinate systems are assumed to be superimposed. The deformation of the continuum can be defined with respect to the initial configuration *(Lagrangian* formulation) or with respect to the current configuration (*Eulerian* formulation). The vector **u** joining points P_o and P is known as the *displacement vector* and is equal to

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} \tag{31}$$

where **x=OP** and **X=OP$_o$**. It has the same three components u_1, u_2 and u_3 in the x_1,x_2,x_3 and X_1,X_2,X_3 coordinate systems (since both coordinate systems are assumed to coincide).

Partial differentiation of the spatial coordinates with respect to the material coordinates $\partial x_i/\partial X_j$ defines the *material deformation gradient*. Likewise, partial differentiation of the material coordinates with respect to the spatial coordinates $\partial X_i/\partial x_j$ defines the *spatial deformation gradient*. Both gradients can be expressed using (3x3) matrices and are related as follows

$$\frac{\partial x_i}{\partial X_j}\frac{\partial X_j}{\partial x_k} = \frac{\partial X_i}{\partial x_j}\frac{\partial x_j}{\partial X_K} = \delta_{ik} \tag{32}$$

Partial differentiation of the displacement vector u_i with respect to the coordinates gives either the *material displacement gradient* $\partial u_i/\partial X_j$ or the *spatial displacement gradient* $\partial u_i/\partial x_j$. Both gradients can be written in terms of (3x3) matrices and are related as follows

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij} \tag{33}$$

In general, two strain tensors can be introduced depending on which configuration is used as reference. Consider, for instance, Figure 7 where two neighboring particles P_o and Q_o before deformation move to points P and Q after deformation. The square of the linear element of length

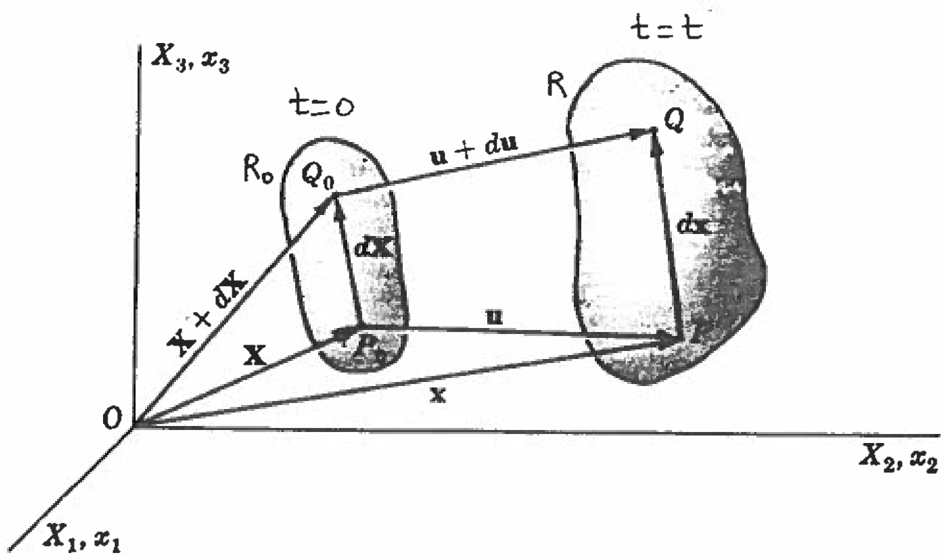


Figure 7. Initial and final (deformed) configurations of a continuum (after Mase, 1970).

between P_o and Q_o is equal to

$$(dX)^2 = dX_i dX_i = \delta_{ij} dX_i dX_j = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} dx_i dx_j = C_{ij} dx_i dx_j \tag{34}$$

where C_{ij} is called the *Cauchy's deformation tensor.* Likewise, in the deformed configuration, the square of the linear element of length between P and Q is equal to

$$(dx)^2 = dx_i dx_i = \delta_{ij} dx_i dx_j = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} dX_i dX_j = G_{ij} dX_i dX_j \tag{35}$$

where G_{ij} is the *Green's deformation tensor.* The two deformation tensors represent the spatial and material description of deformation measures. The relative measure of deformation that occurs in the neighborhood of two particles in a continuum is equal to $(dx)^2 - (dX)^2$. Using the material description, the relative measure of deformation is equal to

$$(dx)^2 - (dX)^2 = \left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) dX_i dX_j = 2L_{ij} dX_i dX_j \tag{36}$$

where L_{ij} is the *Lagrangian (or Green's) finite strain tensor.* Using the spatial description, the relative measure of deformation is equal to

$$(dx)^2 - (dX)^2 = \left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) dx_i dx_j = 2E_{ij} dx_i dx_j \tag{37}$$

where E_{ij} is the *Eulerian (or Almansi's) finite strain tensor.*

Both L_{ij} and E_{ij} are second-order symmetric strain tensors that can be expressed in terms of (3x3) matrices. They can also be expressed in terms of the displacement components by combining equation (36) or (37) with equation (31). This gives,

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \tag{38}$$

and

$$E_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}) \tag{39}$$

3.2 Small Deformation Theory

Infinitesimal Strain Tensors

In the small deformation theory, the displacement gradients are assumed to be small compared to unity, which means that the product terms in equations (38) and (39) are small compared to the other terms and can be neglected. Both equations reduce to

$$l_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}) \tag{40}$$

which is called the *Lagrangian infinitesimal strain tensor*, and

$$\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \tag{41}$$

which is called the *Eulerian infinitesimal strain tensor*.

If the deformation gradients and the displacements themselves are small, both infinitesimal strain tensors may be taken as equal.

Examples

Consider first, the example of a prismatic block of initial length l_o, width w_o, and height h_o. The block is stretched only along its length by an amount $l-l_o$. The corresponding *engineering strain* ϵ is then equal to $(l-l_o)/l_o$. The deformation of the block can be expressed as $x_1=X_1+\epsilon X_1$; $x_2=X_2$ and $x_3=X_3$. Thus, the displacement components are $u_1=\epsilon X_1$, $u_2=u_3=0$. For this deformation, the matrix representation of the Lagrangian finite strain tensor L_{ij} is equal to

$$[L_{ij}] = \frac{1}{2}\begin{bmatrix} 2\epsilon+\epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{42}$$

For any vector **dX** of length dX and components dX_1, dX_2, and dX_3, equation (36) can be written as follows

$$dx^2 - dX^2 = [dX_1 \; dX_2 \; dX_3] \begin{bmatrix} 2\epsilon + \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix} \tag{43}$$

If **dX** is parallel to the X_1-axis with $dX_1 = dX = l_o$, $dX_2 = dX_3 = 0$, then equation (43) yields

$$\epsilon_{lag} = \frac{1}{2} \frac{dx^2 - dX^2}{dX^2} = \epsilon + \frac{1}{2}\epsilon^2 \tag{44}$$

The block does not experience any deformation along the X_2 and X_3 -axes. Equation (44) shows that the longitudinal Lagrangian strain, ϵ_{lag}, differs from the engineering strain, ϵ, by the amount $0.5\epsilon^2$. For small deformations, the square term is very small and can be neglected.

As a second example, consider again the same prismatic block deforming such that $x_1 = X_1$; $x_2 = X_2 + AX_3$ and $x_3 = X_3 + BX_2$. The corresponding displacement components are $u_1 = 0$; $u_2 = AX_3$ and $u_3 = BX_2$. For this deformation, the matrix representation of the Lagrangian finite strain tensor L_{ij} is equal to

$$[L_{ij}] = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & B^2 & A+B \\ 0 & A+B & A^2 \end{bmatrix} \tag{45}$$

For any vector **dX** of length dX and components dX_1, dX_2, and dX_3, equation (36) can be written as follows

$$dx^2 - dX^2 = [dX_1 \; dX_2 \; dX_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & B^2 & A+B \\ 0 & A+B & A^2 \end{bmatrix} \begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix} \tag{46}$$

If **dX** is parallel to the X_1-axis with $dX_1 = dX = l_o$, $dX_2 = dX_3 = 0$, then dx=dX, i.e the prismatic block does not deform in the X_1 direction.

If **dX** is parallel to the X_2-axis with $dX_2=dX=h_o$, $dX_1=dX_3=0$, then equation (46) yields $dx^2=(1+B^2)dX^2$, i.e the dip of vector **dX** is displaced in the X_3 direction by an amount Bh_o.

If **dX** is parallel to the X_3-axis with $dX_3=dX=w_o$, $dX_2=dX_3=0$, then equation (46) yields $dx^2=(1+A^2)dX^2$, i.e the dip of vector **dX** is displaced in the X_2 direction by an amount Aw_o.

Overall, the prismatic block is deformed in the X_2-X_3 plane with the rectangular cross-section becoming a parallelogram. This deformation can also be predicted by examining the components of L_{ij} in equation (45); there is a finite shear strain of magnitude 0.5(A+B) in the X_2-X_3 plane and finite normal strains of magnitude $0.5B^2$ and $0.5A^2$ in the X_2 and X_3 directions, respectively. Note that if A and B are small (small deformation theory), those normal strains can be neglected.

3.3 Interpretation of Strain Components

Relative Displacement Vector

Throughout the rest of these notes we will assume that the small deformation theory is valid and that, for all practical purposes, the Lagrangian and Eulerian infinitesimal strain tensors are equal.

Consider the geometry of Figure 8 and the displacement vectors $\mathbf{u}^{(Po)}$ and $\mathbf{u}^{(Qo)}$ of two neighboring particles P_o and Q_o. The relative displacement vector **du** between the two particles is taken as $\mathbf{u}^{(Qo)}-\mathbf{u}^{(Po)}$. Using a Taylor series expansion for the displacement components in the neighborhood of P_o and neglecting higher order terms in the expansion gives

$$\begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \tag{47}$$

The displacement gradients (material or spatial) appearing in the (3x3) matrix in equation (47) can be decomposed into a symmetric and an anti-symmetric part, i.e.

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right) \tag{48}$$

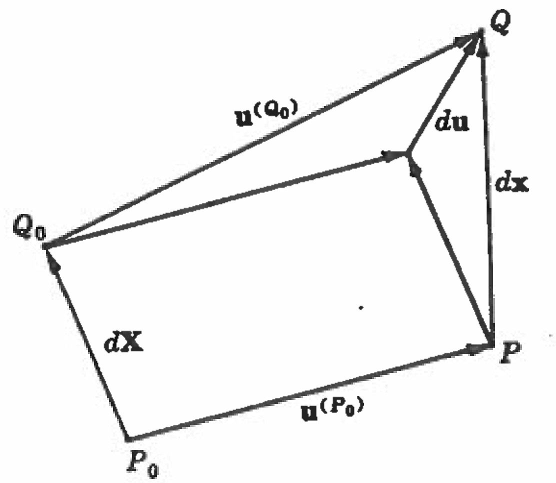


Figure 8. Definition of relative displacement vector between two neighboring particles (after Mase, 1970)

The first term in (48) is the infinitesimal strain tensor, ϵ_{ij}, defined in section 3.2. The second term is called the *infinitesimal rotation tensor* w_{ij} and is denoted as

$$w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{49}$$

This tensor is anti-(or skew) symmetric with $w_{ji}=-w_{ij}$ and corresponds to rigid body rotation around the coordinate system axes.

Strain Components

In three dimensions, the *state of strain* at a point P in an arbitrary x_1,x_2,x_3 Cartesian coordinate system is defined by the components of the strain tensor. Since that tensor is symmetric, only six components defined the state of strain at a point: three *normal strains* ϵ_{11}, ϵ_{22}, and ϵ_{33} and three *shear strains* $\epsilon_{12}=0.5\gamma_{12}$, $\epsilon_{13}=0.5\gamma_{13}$, and $\epsilon_{23}=0.5\gamma_{23}$ with

$$\epsilon_{11}=\frac{\partial u_1}{\partial x_1};\quad \epsilon_{22}=\frac{\partial u_2}{\partial x_2};\quad \epsilon_{33}=\frac{\partial u_3}{\partial x_3}$$

$$\epsilon_{12}=\frac{1}{2}\gamma_{12}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)$$

$$\epsilon_{13}=\frac{1}{2}\gamma_{13}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \tag{50}$$

$$\epsilon_{23}=\frac{1}{2}\gamma_{23}=\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)$$

In equation (50), γ_{12}, γ_{13}, and γ_{23} are called the *engineering shear strains* and are equal to twice the tensorial shear strain components.

From a physical point of view, the normal strains ϵ_{11}, ϵ_{22}, and ϵ_{33} represent the change in length of unit lines parallel to the x_1, x_2, and x_3 directions, respectively. The shear strain components ϵ_{12}, ϵ_{13}, and ϵ_{23} represent one-half the angle change (γ_{12}, γ_{13}, and γ_{23}) between two line elements originally at right angles to one another and located in the (x_1,x_2), (x_1,x_3), and (x_2,x_3) planes.

Note that two sign conventions are used when dealing with strains. In both cases, the displacements u_1, u_2, and u_3 are assumed to be positive in the the $+x_1$, $+x_2$, and $+x_3$ directions, respectively. In *engineering mechanics*, positive normal strains correspond to extension, and positive shear strains correspond to a decrease in the angle between two line elements originally at right angles to one another. In *rock mechanics*, however, positive normal strains correspond to contraction (since compressive stresses are positive), and positive shear strains correspond to an increase in the angle between two line elements originally at right angles to one another. When using the rock mechanics sign convention, the displacement components u_1, u_2, and u_3 in equation (50) must be replaced by $-u_1$, $-u_2$, and $-u_3$, respectively.

3.4 Strain Transformation Law

The components of the strain tensor ϵ'_{ij} in an x',y',z' (x_1',x_2',x_3') Cartesian coordinate system can be determined from the components of the strain tensor ϵ_{ij} in an x,y,z (x_1,x_2,x_3) Cartesian coordinate system using the same coordinate transformation law for second order Cartesian tensors used in the stress analysis. The direction cosines of the unit vectors parallel to the x'-,y'- and z'-axes are assumed to be known and to be defined by equation (10). Equation (12) is replaced by

$$\begin{bmatrix} \epsilon_{x'x'} & \epsilon_{x'y'} & \epsilon_{x'z'} \\ \epsilon_{x'y'} & \epsilon_{y'y'} & \epsilon_{y'z'} \\ \epsilon_{x'z'} & \epsilon_{y'z'} & \epsilon_{z'z'} \end{bmatrix} = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} \\ l_{y'} & m_{y'} & n_{y'} \\ l_{z'} & m_{z'} & n_{z'} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} l_{x'} & l_{y'} & l_{z'} \\ m_{x'} & m_{y'} & m_{z'} \\ n_{x'} & n_{y'} & n_{z'} \end{bmatrix} \tag{51}$$

Using (6x1) matrix representation of ϵ'_{ij} and ϵ_{ij}, and after algebraic manipulations, equation (51) can be rewritten in matrix form as follows

$$[\epsilon]_{x'y'z'} = [T_\epsilon][\epsilon]_{xyz} \tag{52}$$

where $[\epsilon]^t_{xyz} = [\epsilon_{xx}\ \epsilon_{yy}\ \epsilon_{zz}\ \gamma_{yz}\ \gamma_{xz}\ \gamma_{xy}]$, $[\epsilon]^t_{x'y'z'} = [\epsilon_{x'x'}\ \epsilon_{y'y'}\ \epsilon_{z'z'}\ \gamma_{y'z'}\ \gamma_{x'z'}\ \gamma_{x'y'}]$ and $[T_\epsilon]$ is a (6x6) matrix with components similar to those of matrix $[T_\sigma]$ in equation (13). It can written as follows:

$$[T_\epsilon] = \begin{bmatrix} l_{x'}^2 & m_{x'}^2 & n_{x'}^2 & m_{x'}n_{x'} & l_{x'}n_{x'} & m_{x'}l_{x'} \\ l_{y'}^2 & m_{y'}^2 & n_{y'}^2 & m_{y'}n_{y'} & l_{y'}n_{y'} & m_{y'}l_{y'} \\ l_{z'}^2 & m_{z'}^2 & n_{z'}^2 & m_{z'}n_{z'} & l_{z'}n_{z'} & m_{z'}l_{z'} \\ 2l_{y'}l_{z'} & 2m_{y'}m_{z'} & 2n_{y'}n_{z'} & m_{y'}n_{z'}+m_{z'}n_{y'} & n_{y'}l_{z'}+n_{z'}l_{y'} & l_{y'}m_{z'}+l_{z'}m_{y'} \\ 2l_{x'}l_{z'} & 2m_{x'}m_{z'} & 2n_{x'}n_{z'} & m_{x'}n_{z'}+m_{z'}n_{x'} & n_{x'}l_{z'}+n_{z'}l_{x'} & l_{x'}m_{z'}+l_{z'}m_{x'} \\ 2l_{y'}l_{x'} & 2m_{y'}m_{x'} & 2n_{y'}n_{x'} & m_{y'}n_{x'}+m_{x'}n_{y'} & n_{x'}l_{y'}+n_{y'}l_{x'} & l_{x'}m_{y'}+l_{y'}m_{x'} \end{bmatrix}$$

$[T_\sigma]$ and $[T_\epsilon]$ are related as follows

$$[T_\epsilon]^t = [T_\sigma]^{-1}; \quad [T_\epsilon]^{-1} = [T_\sigma]^t \tag{53}$$

Note that equation (53) is valid as long as engineering shear strains (and not tensorial shear strains) are used in $[\epsilon]_{xyz}$ and $[\epsilon]_{x'y'z'}$

The direction cosines defined in equation (15) can be used to determine the strain components in the r, θ, z cylindrical coordinate system of Figure 5b. After algebraic manipulation, the strain components in the r, θ, z and x,y,z coordinate systems are related as follows

$$\begin{aligned}
\epsilon_{rr} &= \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta \\
\epsilon_{\theta\theta} &= \epsilon_{xx}\sin^2\theta + \epsilon_{yy}\cos^2 - \frac{1}{2}\gamma_{xy}\sin 2\theta \\
\gamma_{\theta z} &= \gamma_{yz}\cos\theta - \gamma_{xz}\sin\theta \\
\gamma_{rz} &= \gamma_{yz}\sin\theta + \gamma_{xz}\cos\theta \\
\gamma_{r\theta} &= (\epsilon_{yy} - \epsilon_{xx})\sin 2\theta + \gamma_{xy}\cos 2\theta
\end{aligned} \tag{54}$$

3.5 Principal Strains

The principal strain values and their orientation can be found by determining the eigenvalues and eigenvectors of the strain tensor ϵ_{ij}. Equation (20) is replaced by

$$\begin{vmatrix} \epsilon_{xx}-\epsilon & \epsilon_{yx} & \epsilon_{zx} \\ \epsilon_{xy} & \epsilon_{yy}-\epsilon & \epsilon_{zy} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz}-\epsilon \end{vmatrix} = 0 \tag{55}$$

Upon expansion, the principal strains are the roots of the following cubic polynomial

$$\epsilon^3 - I_{\epsilon 1}\epsilon^2 + I_{\epsilon 2}\epsilon - I_{\epsilon 3} = 0 \tag{56}$$

where $I_{\epsilon 1}$, $I_{\epsilon 2}$, and $I_{\epsilon 3}$ are respectively the first, second and third *strain invariants* and are equal to

$$I_{\epsilon 1} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$$
$$I_{\epsilon 2} = \epsilon_{yy}\epsilon_{zz} + \epsilon_{xx}\epsilon_{zz} + \epsilon_{xx}\epsilon_{yy} - (\epsilon_{yz}^2 + \epsilon_{xz}^2 + \epsilon_{xy}^2) \tag{57}$$
$$I_{\epsilon 3} = \epsilon_{xx}\epsilon_{yy}\epsilon_{zz} + 2\epsilon_{xy}\epsilon_{xz}\epsilon_{yz} - (\epsilon_x\epsilon_{yz}^2 + \epsilon_y\epsilon_{xz}^2 + \epsilon_z\epsilon_{xy}^2)$$

For each principal strain ϵ_k (ϵ_1, ϵ_2, ϵ_3), there is a principal strain direction which can be determined using the same procedure as for the principal stresses.

Let the x-, y-, and z-axes be parallel to the directions of ϵ_1, ϵ_2, and ϵ_3 respectively, and consider a small element with edges dx, dy and dz whose volume V_o=dxdydz. Assuming no rigid body displacement, the components of the relative displacement vector **du** are equal to $\epsilon_1$dx, $\epsilon_2$dy and $\epsilon_3$dz. After deformation the volume of the element is equal to

$$V = (1+\epsilon_1)dx(1+\epsilon_2)dy(1+\epsilon_3)dz \tag{58}$$

or

$$V = (1+I_{\epsilon 1}+I_{\epsilon 2}+I_{\epsilon 3})V_o \tag{59}$$

For small strains, the second and third strain invariants can be neglected with respect to the first strain invariant. Equation (59) yields

$$\frac{\Delta V}{V} = \frac{V-V_o}{V_o} = I_{\epsilon 1} \tag{60}$$

Equation (60) indicates that the first strain invariant can be used as an approximation for the *cubical expansion* of a medium. If the rock mechanics sign convention is used instead, the first strain invariant is an approximation for the *cubical contraction*. The ratio ΔV/V is called the *volumetric strain*.

3.6 Strain Decomposition

The strain tensor ϵ_{ij} can be separated into a *hydrostatic* part $e_m\delta_{ij}$ and a *deviato*ric part e_{ij}. Using (3x3) matrix representations and an x,y,z coordinate system, the strain decomposition can be expressed as follows

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{yx} & \epsilon_{zx} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{zy} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} e_m & 0 & 0 \\ 0 & e_m & 0 \\ 0 & 0 & e_m \end{bmatrix} + \begin{bmatrix} \epsilon_{xx}-e_m & \epsilon_{yx} & \epsilon_{zx} \\ \epsilon_{xy} & \epsilon_{yy}-e_m & \epsilon_{zy} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz}-e_m \end{bmatrix} \tag{61}$$

with $e_m=(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz})/3$.

3.7 Compatibility Equations

The six components of strain are related to the three components of displacement through equation (50). These relations can be seen as a system of six partial differential equations with three unknowns. The system is therefore over-determined and will not, in general, possess a unique solution for the displacements for an arbitrary choice of the six strain components.

Continuity of the continuum as it deforms requires that the three displacement components be continuous functions of the three coordinates and be single valued. It can be shown that this requires the strain components to be related by six equations called *equations of compatibility*. In an arbitrary x,y,z Cartesian coordinate system, these equations can be written as follows

$$\begin{aligned}
\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\
\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\
\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} &= \frac{\partial^2 \gamma_{zx}}{\partial x \partial z} \\
2\frac{\partial^2 \epsilon_x}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right) \\
2\frac{\partial^2 \epsilon_y}{\partial x \partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{xz}}{\partial y}\right) \\
2\frac{\partial^2 \epsilon_z}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)
\end{aligned} \tag{62}$$

3.8 Strain Measurements

Consider an (x,y) plane and a point P in that plane. The state of strain at point P is defined by three components ϵ_{xx}, ϵ_{yy}, and ϵ_{xy}. The longitudinal strain ϵ_l in any direction making an angle θ with the x-axis is, according to equation (54), equal to

$$\epsilon_l = \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + \epsilon_{xy}\sin 2\theta \tag{63}$$

The state of strain at (or in the near vicinity of) point P can be determined by measuring three longitudinal strains, ϵ_{l1}, ϵ_{l2}, and ϵ_{l3} in three different directions with angles θ_1, θ_2, and θ_3. This gives the following system of three equations and three unknowns

$$\begin{bmatrix}\epsilon_{l1}\\ \epsilon_{l2}\\ \epsilon_{l3}\end{bmatrix} = \begin{bmatrix}\cos^2\theta_1 & \sin^2\theta_1 & \sin 2\theta_1\\ \cos^2\theta_2 & \sin^2\theta_2 & \sin 2\theta_2\\ \cos^2\theta_3 & \sin^2\theta_3 & \sin 2\theta_3\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{xy}\end{bmatrix} \tag{64}$$

which can be solved for ϵ_{xx}, ϵ_{yy}, and ϵ_{xy}.

Longitudinal strains can be measured using strain gages (invented in the United States in 1939). A strain gage consists of many loops of thin resistive wire glued to a flexible backing (Figure 9a). It is used to measure the longitudinal strain of a structural member to which it is attached. As the material deforms, the wire becomes somewhat longer and thinner (or shorter and thicker) thereby changing its resistance by a small amount.

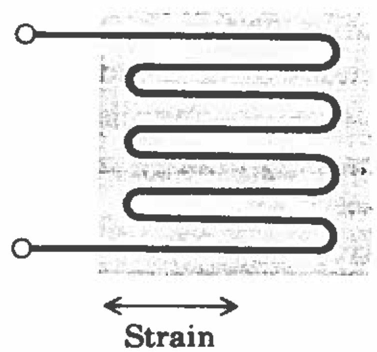
Recall that the electrical resistance, R, of a wire of length l, sectional area A, and resistivity ρ is equal to

$$R = \frac{\rho l}{A} \tag{65}$$

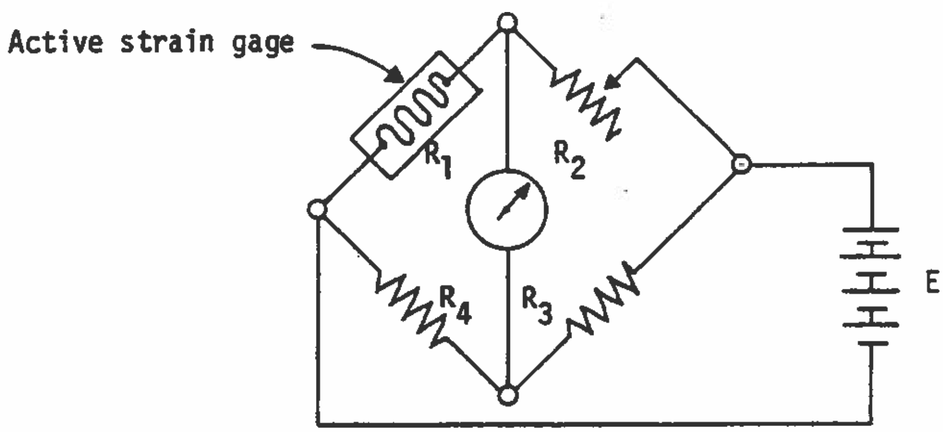
Let $\epsilon_l = \Delta l / l$ be the longitudinal strain of the wire. As the wire stretches, its diameter decreases due to the Poisson's effect. The change in resistance, ΔR, of the wire is related to ϵ_l as follows

$$\frac{1}{\epsilon_l}\frac{\Delta R}{R} = GF = \frac{1}{\epsilon_l}\frac{\Delta\rho}{\rho} + (1+2\nu) \tag{66}$$

where ν is the Poisson's ratio of the wire and GF is the so-called *gage factor* whose value is given by the gage manufacturer. For instance for Cr-Ni gages, GF=2.05. Thus,



(a)



(b)

Figure 9. (a) Schematic representation of a strain gage, (b) Wheastone bridge.

$$\epsilon_l = \frac{1}{GF}\frac{\Delta R}{R} \tag{67}$$

Equation (67) shows that the strain can be determined once the change in resistance, ΔR, is measured. This can be done by mounting the strain gage on a *Wheastone bridge*. Figure 9b shows a Wheastone bridge where the active strain gage has a resistance R_1. The bridge is equilibrium when $R_1R_3=R_2R_4$. If R_1 changes by ΔR_1, the bridge will be in equilibrium only if

$$\Delta R_1 = \frac{R_4}{R_3}\Delta R_2$$

where ΔR_2 is changed by means of a *potentiometer*. Equation (68) indicates that in order to obtain a high precision, i.e. a large variation of R_2 for a given change of R_1 (corresponding to a certain strain), the ratio R_4/R_3 needs to be as small as possible.

In general, the variable potentiometer used for the experiment is calibrated so that the readings are immediately in microstrains (μ-strains).

Note that a single strain gage can only be used to measure the longitudinal deformation in one direction. Thus, in order to solve equation (64) for ϵ_{xx}, ϵ_{yy}, and ϵ_{xy}, three independent gages need to be used. Another option is to use strain gage rosettes which consist of three strain gages attached to the same flexible backing. Different strain gage arrangements are available as shown in Figure 10. Strain rosettes commonly used in rock mechanics include: 45° rosettes (Fig. 10a) where θ_1=0, θ_2=45 and θ_3=90; 60° rosettes (Fig. 10b) where θ_1=0, θ_2=60 and θ_3=120; and 120° rosettes (Fig. 10c) where θ_1=0, θ_2=120 and θ_3=240.

It is noteworthy that in the usual strain rosettes, the three separate electrical resistances are not exactly mounted at the same point. Consequently, a small error is introduced when determining the state of strain at a point.

The advantages of strain gages are as follows:

- high sensitivity (about 10^{-6}),
- large domain of variation (about 15×10^{-3}),
- negligible weight and inertia,
- neither mechanical nor electrical response delay,
- minimum space requirements,
- direct reading of strain instead of displacement.

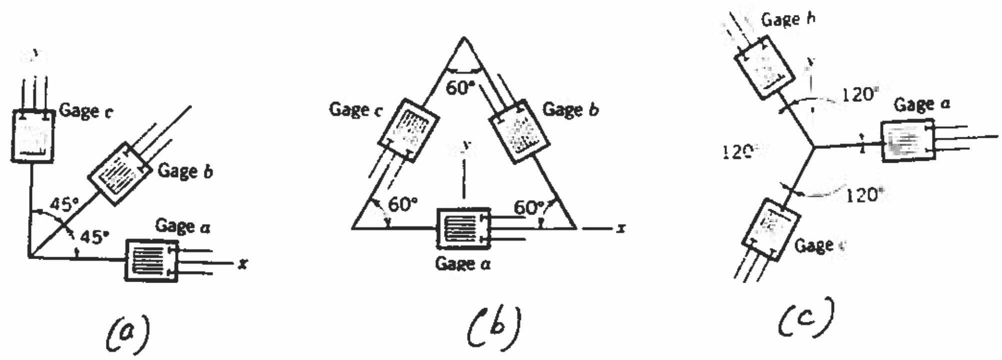


Figure 10. (a) 45° rosette; (b) 60° rosette; and (c) 120° rosette.

The main disadvantages include:

- lengthy and delicate mounting procedure,
- costly since they serve only once,
- sensitive to humidity unless encapsulated,
- important temperature effects since $R_\theta = R(1+\alpha\theta)$ where α is the thermal expansion coefficient of the strain gage.

Note that the effect of temperature can be compensated by using special temperature compensated strain gages, Another compensation method consists of substituting the resistance R_4 in Figure 9b by a strain gage identical to the one corresponding to R_1. The R_4 gage is glued onto the same material as R_1 and is exposed to the same environment but is not strained. Thus, the Wheastone bridge will always be thermally equilibrated.

3.9 References

Goodman, R.E. (1989) *Introduction to Rock Mechanics*, Wiley, 2nd Edition.

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APPENDIX

Components of matrix $[T_\sigma]$ in equation (13) and matrix $[T_\epsilon]$ in equation (52)

$$[T_\sigma] = \begin{bmatrix} l_{x'}^2 & m_{x'}^2 & n_{x'}^2 & 2m_{x'}n_{x'} & 2l_{x'}n_{x'} & 2m_{x'}l_{x'} \\ l_{y'}^2 & m_{y'}^2 & n_{y'}^2 & 2m_{y'}n_{y'} & 2l_{y'}n_{y'} & 2m_{y'}l_{y'} \\ l_{z'}^2 & m_{z'}^2 & n_{z'}^2 & 2m_{z'}n_{z'} & 2l_{z'}n_{z'} & 2m_{z'}l_{z'} \\ l_{y'}l_{z'} & m_{y'}m_{z'} & n_{y'}n_{z'} & m_{y'}n_{z'}+m_{z'}n_{y'} & n_{y'}l_{z'}+n_{z'}l_{y'} & l_{y'}m_{z'}+l_{z'}m_{y'} \\ l_{x'}l_{z'} & m_{x'}m_{z'} & n_{x'}n_{z'} & m_{x'}n_{z'}+m_{z'}n_{x'} & n_{x'}l_{z'}+n_{z'}l_{x'} & l_{x'}m_{z'}+l_{z'}m_{x'} \\ l_{y'}l_{x'} & m_{y'}m_{x'} & n_{y'}n_{x'} & m_{y'}n_{x'}+m_{x'}n_{y'} & n_{x'}l_{y'}+n_{y'}l_{x'} & l_{x'}m_{y'}+l_{y'}m_{x'} \end{bmatrix}$$

and

$$[T_\epsilon] = \begin{bmatrix} l_{x'}^2 & m_{x'}^2 & n_{x'}^2 & m_{x'}n_{x'} & l_{x'}n_{x'} & m_{x'}l_{x'} \\ l_{y'}^2 & m_{y'}^2 & n_{y'}^2 & m_{y'}n_{y'} & l_{y'}n_{y'} & m_{y'}l_{y'} \\ l_{z'}^2 & m_{z'}^2 & n_{z'}^2 & m_{z'}n_{z'} & l_{z'}n_{z'} & m_{z'}l_{z'} \\ 2l_{y'}l_{z'} & 2m_{y'}m_{z'} & 2n_{y'}n_{z'} & m_{y'}n_{z'}+m_{z'}n_{y'} & n_{y'}l_{z'}+n_{z'}l_{y'} & l_{y'}m_{z'}+l_{z'}m_{y'} \\ 2l_{x'}l_{z'} & 2m_{x'}m_{z'} & 2n_{x'}n_{z'} & m_{x'}n_{z'}+m_{z'}n_{x'} & n_{x'}l_{z'}+n_{z'}l_{x'} & l_{x'}m_{z'}+l_{z'}m_{x'} \\ 2l_{y'}l_{x'} & 2m_{y'}m_{x'} & 2n_{y'}n_{x'} & m_{y'}n_{x'}+m_{x'}n_{y'} & n_{x'}l_{y'}+n_{y'}l_{x'} & l_{x'}m_{y'}+l_{y'}m_{x'} \end{bmatrix}$$