Modal Analysis of MDOF Forced Damped Systems
# Lecture 22: Modal Analysis of MDOF Forced Damped Systems

## TABLE OF CONTENTS

| §22.1. | Introduction | 22–3 |
| §22.2. | What is Mechanical Damping? | 22–3 |
| §22.2.1. | Internal vs. External Damping | 22–3 |
| §22.2.2. | Distributed vs. Localized Damping | 22–3 |
| §22.3. | Modeling Structural Damping | 22–4 |
| §22.4. | Matrix Equations of Motion | 22–5 |
| §22.4.1. | Equations of Motion Using Undamped Modes | 22–5 |
| §22.4.2. | Three Ways Out | 22–6 |
| §22.5. | Diagonalization by Modal Damping | 22–7 |
| §22.5.1. | Damping Factor Guessing | 22–8 |
| §22.5.2. | Energy Equivalent Damping Factor | 22–8 |
| §22.6. | Diagonalization by Rayleigh Damping | 22–8 |
| §22.7. | Remainder of Lecture | 22–9 |
| §22.8. | Damped matrix EOM | 22–9 |
| §22.9. | Diagonalization By Energy Balance, aka RQ Diagonalization | 22–10 |
| §22.10. | Comparison of Exact Versus Diagonalized Responses | 22–11 |
§22.1. Introduction

The present lecture introduces damping within the context of dynamic modal analysis. After a brief overview of a very complex but “fuzzy” subject, the lecture focuses on a very specific damping model. Namely *linearly viscous* and *light*.

Linearly viscous damping is *proportional to the velocity*. Light damping means a damping factor (also called damping ratio and damping coefficient) that is small compared to unity. In the terminology of Lecture 17, a lightly damped mechanical system is said to be *underdamped*.

Accounting for damping effects brings good and bad news. All real dynamic systems experience damping because energy dissipation mechanisms are like death and taxes:* inevitable. Hence inclusion makes the dynamic model more physically realistic. The bad news is that it can seriously complicate the analysis process. Here is where the assumption of light viscous damping helps: it allows the *reuse* of major parts of the modal analysis techniques introduced in the previous 3 lectures.

§22.2. What is Mechanical Damping?

Damping is the (generally irreversible) conversion of mechanical energy into heat as a result of motion. For example, as we scratch a match against a rough surface, its motion generates heat and ignites the sulphur content. When shivering under cold, we rub palms against each other to warm up. Those are two classical examples of the thermodynamic effect of friction. In structural systems, damping is more complex, appearing in several forms. These may be broadly categorized into *internal* versus *external*, and *distributed* versus *localized*.

§22.2.1. Internal vs. External Damping

Internal damping is due to the structural material itself. Sources are varied: microstructural defects, crystal grain relative motions, eddy currents in ferromagnetic media, dislocations in metals, and molecular chain movements in polymers. The key *macroscopic* effect is the production of a *hysteresis* loop in stress-strain plots. The hysteresis loop area represents the energy dissipated per unit volume of material and per stress cycle. This kind of damping is intimately related to cyclic motions such as vibration.

External damping come from boundary effects. An important form is *structural damping*, which is produced by rubbing friction: stick-and-slip contact or impact. That may happen between structural components such as joints, or between a structural surface and non-structural solid media such as soil. This form is often modeled by *Coulomb damping*, which describes the energy dissipation of rubbing dry-friction.

Another form of external damping is *fluid damping*. When a material is immersed in a fluid, such as air or water, and there is relative motion between the structure and the fluid, a drag force appears. This force causes an energy dissipation through internal fluid mechanisms such as viscosity, convection or turbulence. This dissipation is collectively known as fluid damping. One well known instance is a vehicle shock absorber: a fluid (liquid or air) is forced through a small opening by a piston.

* One exception: tax-exempt vampires.
§22.2.2. Distributed vs. Localized Damping

All damping ultimately comes from frictional effects, which may however take place at different scales. If the effects are distributed over volumes or surfaces at macro scales, we speak of distributed damping. But occasionally the engineer uses damping devices designed to produce beneficial effects. For example: shock absorbers, airbags, parachutes, motion mitigators for buildings or bridges in seismic or hurricane zones, active piezoelectric dampers for space structures.

Those devices can be often idealized as lumped objects, modeled as point forces or moments, and said to produce localized damping. The distinction appears primarily at the modeling level, since all motion-damper devices ultimately work as a result of some kind of internal energy conversion at the molecular level.

Localized damping devices may be in turn classified into passive or active, with the latter responding to motion feedback. But this would take us to far into control systems, which are not part of this course.

§22.3. Modeling Structural Damping

The foregoing summary should make clear that damping is a ubiquitous but complicated business. In structures containing joints, for example, Coulomb (dry friction) damping often dominates; this model is partly nonlinear because the damping force depends on the sign of the velocity. Fluid damping tends to be highly nonlinear if the interacting flow is turbulent, since in that case the drag is nonlinear in the relative velocity between solid and fluid. Another modeling complication is that friction may depend on fabrication or construction details that are not easy to predict; e.g., bolted versus welded connections.

Balancing those complications is the fact that damping in most structures, especially metallic ones, is light in the sense that the damping factor $\xi$ introduced in Lecture 17 is much smaller than one. In addition, the presence of damping is usually beneficial to safety in the sense that resonance effects are mitigated. This gives structural engineers some leeway to simplify the dynamic analysis, by proceeding as follows.

- A simple damping model, such as linear viscous damping, can be assumed without much concern.
- Mode superposition is applicable because the EOM is linear. Moreover the frequencies and mode shapes for the undamped system can be reused if additional assumptions, such as Rayleigh damping, or modal damping, are made.

It should be stressed that such simplifications are not recommended if precise modeling of damping effects is important to safety and performance. This occurs in the following scenarios:

- Damping is crucial to function or operation. Think, for instance, of a shock absorber. Fortunately damper devices can be modeled more accurately than, say, dry friction, using manufacturer data.
- Damping may destabilize the system by feeding energy instead of removing it. This can happen in active control systems and aeroelasticity.

The last two scenarios are beyond the scope of this course. In this lecture we will focus on linear viscous damping. Moreover, damping levels will usually assumed to be light in the sense that the damping factor $\xi \ll 1$. 

22–4
§22.4 MATRIX EQUATIONS OF MOTION

Consider again the two-DOF mass-spring-dashpot example system of Lecture 19. This is reproduced in Figure 22.1 for convenience. The physical-coordinate EOM in detailed matrix notation are

\[
\begin{bmatrix}
    m_1 & 0 \\
    0 & m_2
\end{bmatrix}
\begin{bmatrix}
    \ddot{u}_1 \\
    \ddot{u}_2
\end{bmatrix}
+ \begin{bmatrix}
    c_1 + c_2 & -c_2 \\
    -c_2 & c_2
\end{bmatrix}
\begin{bmatrix}
    \dot{u}_1 \\
    \dot{u}_2
\end{bmatrix}
+ \begin{bmatrix}
    k_1 + k_2 & -k_2 \\
    -k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2
\end{bmatrix}
= \begin{bmatrix}
    p_1 \\
    p_2
\end{bmatrix}.
\]  

(22.1)

In compact notation,

\[
M \dddot{u} + C \dot{u} + K u = p.
\]

(22.2)

where \(M\), \(C\) and \(K\) are the mass, damping and stiffness matrix, respectively, \(p, u, \dot{u}\) and \(\ddot{u}\) are the force, displacement, velocity and acceleration vectors, respectively. The latter four are function of time: \(u = u(t)\), etc, but the time argument will be often omitted for brevity.

In the sequel it will be assumed that \(M\), \(C\) and \(K\) are symmetric. Furthermore \(M\) is positive definite (PD) whereas \(K\) is nonnegative definite (NND).

§22.4.1. Equations of Motion Using Undamped Modes

This technique attempts to reuse the modal analysis methods introduced in Lectures 19–21. Suppose that damping is removed whence \(C = 0\). Get the natural frequencies and mode shapes of the unforced, undamped system governed by \(M \dddot{u} + K u = 0\), by solving the eigenproblem \(K U_i = \)
\[ \omega_i^2 \mathbf{M} \mathbf{U}_i. \] Normalize the vibration mode shapes \( \mathbf{U}_i \Rightarrow \phi_i \) so that they are orthonormal with respect to the mass matrix:

\[ \phi_i^T \mathbf{M} \phi_i = \delta_{ij}, \quad (22.3) \]

in which \( \delta_{ij} \) denotes the Kronecker delta: \( \delta_{ij} = 1 \) if \( i = j \), else zero. Let \( \Phi \) be the modal matrix constructed with the orthonormalized mode shapes \( \phi_i \), and denote by \( \eta \) the array of modal amplitudes \( \eta_i \), also called generalized coordinates. As before, assume mode superposition, so that physical DOF are linked to modal amplitudes via

\[ \mathbf{u} = \Phi \eta. \quad (22.4) \]

Following the same scheme as in the previous two Lectures, the transformed EOM in generalized coordinates are

\[ \Phi^T \mathbf{M} \ddot{\Phi} \Phi \ddot{\eta} + \Phi^T \mathbf{C} \Phi \dot{\Phi} \dot{\eta} + \Phi^T \mathbf{K} \Phi \Phi \eta = \Phi^T \mathbf{p}(t). \quad (22.5) \]

Define the generalized mass, damping, stiffness and forces as

\[ \mathbf{M}_g = \Phi^T \mathbf{M} \Phi, \quad \mathbf{C}_g = \Phi^T \mathbf{C} \Phi, \quad \mathbf{K}_g = \Phi^T \mathbf{K} \Phi, \quad \mathbf{f} = \Phi^T \mathbf{p}(t). \quad (22.6) \]

Of these, the generalized mass matrix \( \mathbf{M}_g \) and the generalized stiffness matrix \( \mathbf{K}_g \) were introduced in Lecture 20. If \( \Phi \) is built by stacking mass-orthonormalized vibration modes as columns, it was shown there that

\[ \mathbf{M}_g = \mathbf{I}, \quad \mathbf{K}_g = \text{diag}[\omega_i^2]. \quad (22.7) \]

That is, \( \mathbf{M}_g \) reduces to the identity matrix while \( \mathbf{K}_g \) becomes a diagonal matrix with squared frequencies stacked along its diagonal. The generalized forces \( \mathbf{f}(t) \), also called modal forces, were introduced in Lecture 21. The new term in (22.5) is the generalized damping matrix \( \mathbf{C}_g = \Phi^T \mathbf{C} \Phi \), also called the modal damping matrix in the literature.

Substituting (22.7) into (22.6) we arrive at the modal EOM for the damped system:

\[ \ddot{\eta}(t) + \mathbf{C}_g \dot{\eta}(t) + \text{diag}(\omega_i^2) \eta(t) = \mathbf{f}(t). \quad (22.8) \]

Here we ran into a major difficulty: matrix \( \mathbf{C}_g \) generally will not be diagonal. If that happens, the modal EOM (22.8) will not decouple. We seem to have taken a promising path, but hit a dead end.

§22.4.2. Three Ways Out

There are three ways out\(^1\) of the dead end:

- **Diagonalization.** Keep working with (22.8), but make \( \mathbf{C}_g \) diagonal through some artifice.
- **Complex Eigensystem.** Set up and solve a different (augmented) eigenproblem that diagonalizes two matrices that comprise \( \mathbf{M}, \mathbf{C} \) and \( \mathbf{K} \) as submatrices. The name comes from the fact that it generally leads to frequencies and mode shapes that are complex.

\(^1\) In the textbook of Craig-Kurdila the first two approaches listed above are called mode superposition through real modes of the undamped system, and mode superposition through complex modes of the damped system, respectively. See their Section 10.1.10 for details. Those mouthfuls are abbreviated above to more easily remembered labels.
§22.5 Diagonalization by Modal Damping

- **Direct Time Integration or DTI.** Integrate directly (22.2) numerically in time without passing to modal coordinates.

Each approach has strengths and weaknesses. (Obviously, else we would mention just one.)

Diagonalization allows straightforward reuse of undamped frequencies and mode shapes, which are fairly easy to obtain with standard eigensolution software. The uncoupled modal equations often have physical interpretation and can be validated with experiments. Only real arithmetic is necessary.

The downside is that we don’t solve the original EOM (22.2) directly, so some form of approximation is inevitable. This is counteracted by the fact that structural damping is often difficult to quantify since it can come from many sources, as discussed earlier. Thus the approximation in solving (22.2) may be tolerable in view of modeling uncertainties. This is particularly true if damping is *light*.

There are certain problems, however, in which diagonalization cannot adequately represent damping effects within engineering accuracy. Three of them are:

1. Structures with localized damper devices (e.g., shock absorbers, piezoelectric dampers)
2. Structure-media interaction (e.g., building foundations, tunnels, aeroelasticity, marine structures, surface ships)
3. Active control systems

In those situations one of the two remaining approaches must be taken.

The complex eigensystem method is mathematically irrefutable and can solve (22.2) without additional approximations. No assumptions as to light versus heavy damping are needed. But it involves a substantial amount of preparatory work since the EOM must be transformed to the so-called *state space* form. For a large number of DOF, solving complex eigensystems is unwieldy. Physical interpretation of complex frequencies and modes is less immediate and may require substantial expertise in math as well as engineering experience. But in the three scenarios listed above it should be preferred to diagonalization.

Direct time integration (DTI) has the advantage of being completely general. In fact it can handle not only the linear EOM (22.2), but also *nonlinear systems*, as well as non-viscous damping mechanisms. No transformations to mode coordinates is required, and no complex arithmetic emerges. The main disadvantage is that it requires substantial expertise in computational handling of ODE, which is a hairy topic onto itself. Since DTI can only handle fully numerically-specified models, the approach is not particularly useful during preliminary design stages when many free parameters float around.

Because the last two approaches lie outside the scope of an introductory course (they are usually taught at the graduate level), our choice is easy: diagonalization it is.

### §22.5. Diagonalization by Modal Damping

In this approach the generalized damping matrix $C_g = \Phi^T C \Phi$ is assumed to be diagonal from the start by using modal damping factors

$$C_g = \text{diag} \left[ 2 \xi_i \omega_i \right] M_i \quad i = 1, 2, \ldots n.$$  \hspace{1cm} (22.9)

in which $n$ is the number of DOF and $\xi_i$ denotes the damping factor for the $i^{th}$ natural mode. As a result, the modal EOM (22.8) *decouple*, reducing to $n$ canonical second-order equations in the
modal amplitudes

\[ \ddot{\eta}_i + 2\xi_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i = f_i(t), \quad i = 1, 2, \ldots, n. \] (22.10)

These equations can be solved using the methods described in Lecture 21. The solutions can be superposed via the mode decomposition assumption \( \mathbf{u}(t) = \Phi \eta(t) \) to get the physical response.

The method is straightforward. Two technical difficulties remain. First, how are modal damping factors picked? Second, what is the error incurred by the decoupling assumption?

### 22.5.1. Damping Factor Guessing

One time honored approach is educated guessing. This is of course the only possibility if (a) little is known about the damping level and sources and (b) there are no experimental results; for example the structure only exists on paper. The structural engineer then makes recourse to experience with similar systems. With an air of authority she says: “No problema. Let’s assign 1% to modes 1 through 5, 2% to modes 6 through 20, and 4% to all higher ones.” Done.

There is some method in the madness. First, damping factors of well constructed structures are typically small compared to unity: 1 to 5% is typical. Second, damping generally increases with frequency, the reason being that more hysteresis cycles take place within a fixed time interval. Third, the effect of light damping is not significant on the response unless the force (or base motion) happens to excite a resonance; so the difference between, say, 1% and 2% may be well within modeling uncertainty.

Note that this procedure directly constructs a diagonal \( \mathbf{C}_g \) rather than \( \mathbf{C} \). If it is desirable to obtain \( \mathbf{C} \) \emph{a posteriori}, one may use the following relations, the second of which assumes \( \mathbf{M}_g = \mathbf{I} \):

\[
\mathbf{C} = \Phi^{-T} \mathbf{C}_g \Phi^{-1} = \mathbf{M} \Phi \mathbf{C}_g \Phi^T \mathbf{M} = \sum_{i=1}^{n} 2\xi_i \phi_i (\mathbf{M} \phi_i) (\mathbf{M} \phi_i)^T.
\] (22.11)

Taking account of mode orthogonality gives the useful relation

\[
\phi_i^T \mathbf{C} \phi_i = 2\xi_i \omega_i.
\] (22.12)

### 22.5.2. Energy Equivalent Damping Factor

This technique is applicable if \( \mathbf{C} \) is available. It can be shown that the power dissipated by viscous damping in an unforced system governed by the physical-coordinates EOM (22.6) with \( \mathbf{p} = \mathbf{0} \) is

\[
D = \frac{1}{2} \mathbf{u} \mathbf{C} \mathbf{u}.
\] (22.13)

The kinetic and potential energies of the system are

\[
T = \frac{1}{2} \dot{\mathbf{u}} \mathbf{M} \dot{\mathbf{u}}, \quad V = \frac{1}{2} \mathbf{u} \mathbf{K} \mathbf{u},
\] (22.14)

Now suppose that the structure is moving in one of the undamped modes, say \( \phi_i \). The damping factor can be chosen so that the energy dissipated in that mode matches that of the full discrete system. This leads to the Rayleigh Quotient rule. (Topic to be developed further, since it is not in C-K.)
§22.6. Diagonalization by Rayleigh Damping

This diagonalization procedure is widely used for civil structures, especially for seismic response calculations. The viscous damping matrix \( C \) is directly defined as a linear combination of the mass and stiffness matrix:

\[
C = a_0 M + a_1 K.
\]

in which \( a_0 \) and \( a_1 \) are selected constants (with appropriate physical dimensions). The method is also called proportional damping in the literature. Applying the modal matrix congruential transformation to this \( C \) results in

\[
C_g = \Phi^T C \Phi = \text{diag} \{ C_i \} = \text{diag} \{ a_0 + a_1 \omega_i^2 \} = \text{diag} \{ 2 \xi_i \omega_i \},
\]

whence the effective modal damping factor is

\[
\xi_i = \frac{1}{2} \left( \frac{a_0}{\omega_i} + a_1 \omega_i \right).
\]

Choosing the damping factor for two modes of different frequencies and solving (22.17) for \( a_0 \) and \( a_1 \) yields \( C \) from (22.15). In practice the stiffness proportional term is more physically relevant (because, as previously noted, damping usually increases with frequency). Thus structural engineers tend to adjust the choice of those frequencies so that \( \xi \) is roughly minimized for the lowest frequency mode.

§22.7. Remainder of Lecture

The reminder of this lecture just presents the spring-mass-dashpot example system of previous lectures. For various damping levels, this is solved exactly (as a coupled system, by direct numerical integration) and also modally, with a particular damping-matrix-diagonalization method. The results are compared for various damping levels.

§22.8. Damped matrix EOM

Consider again the two-DOF mass-spring-dashpot example system of Lecture 19. This is reproduced in Figure 22.1 for convenience. The physical-coordinate EOM in detailed matrix notation are

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\begin{bmatrix}
  \ddot{u}_1 \\
  \ddot{u}_2
\end{bmatrix} +
\begin{bmatrix}
  c_1 + c_2 & -c_2 \\
  -c_2 & c_2
\end{bmatrix}
\begin{bmatrix}
  \dot{u}_1 \\
  \dot{u}_2
\end{bmatrix} +
\begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} =
\begin{bmatrix}
  p_1 \\
  p_2
\end{bmatrix}.
\]

(22.18)

Suppose \( m_1 = 2, m_2 = 1, k_1 = 6, k_2 = 3, c_2 = c_1 = c, p_1 = 0, \) and \( p_2 = F_2 \cos \Omega t, \) in which \( c \) (a dashpot coefficient), \( \Omega \) (excitation frequency of harmonic force), and \( F_2 \) (harmonic force amplitude) are for the moment kept as free parameters. Then

\[
\begin{bmatrix}
  2 & 0 \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  \ddot{u}_1 \\
  \ddot{u}_2
\end{bmatrix} +
\begin{bmatrix}
  2c & -c \\
  -c & c
\end{bmatrix}
\begin{bmatrix}
  \dot{u}_1 \\
  \dot{u}_2
\end{bmatrix} +
\begin{bmatrix}
  9 & -3 \\
  -3 & 3
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  F_2 \cos \Omega t
\end{bmatrix}.
\]

(22.19)
Lecture 22: MODAL ANALYSIS OF MDOF FORCED DAMPED SYSTEMS

\[ F = m \ddot{u}, \quad F = k u, \quad F = c \dot{u} \]

(a) \hspace{1cm} (b)

\[ p_1(t), \quad F_{d1} = c_1 \dot{u}_1, \quad F_{d1} = c_1 \dot{u}_1 \]

\[ p_2(t), \quad F_{d2} = c_2 (\dot{u}_2 - \dot{u}_1), \quad F_{d2} = c_2 (\dot{u}_2 - \dot{u}_1) \]

\[ u_1 = u(t), \quad u_2 = u(t) \]

\[ k_1, \quad c_1, \quad m_1 \]

\[ k_2, \quad c_2, \quad m_2 \]

\[ x \]

\[ \Phi = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{6} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0.4082 & 0.8165 \\ 0.5773 & -0.5773 \end{bmatrix} \]

\[ \omega_1^2 = \frac{3}{2}, \quad \omega_2^2 = 6 \]

\[ M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix}, \quad K = \begin{bmatrix} 9 & -3 \\ -3 & 3 \end{bmatrix}, \quad p = \begin{bmatrix} 0 \\ F_2 \cos \Omega t \end{bmatrix} \]  

\[ \Phi \]

\[ \Phi = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{6} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0.4082 & 0.8165 \\ 0.5773 & -0.5773 \end{bmatrix} \]

\[ \Phi = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{6} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0.4082 & 0.8165 \\ 0.5773 & -0.5773 \end{bmatrix} \]

\[ M_g = \Phi^T M \Phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad C_g = \Phi^T C \Phi = \begin{bmatrix} \frac{c}{3} & -\frac{c}{3\sqrt{2}} \\ -\frac{c}{3\sqrt{2}} & \frac{5c}{2} \end{bmatrix} \]

\[ K_g = \Phi^T K \Phi = \begin{bmatrix} 3/2 & 0 \\ 0 & 6 \end{bmatrix} = \text{diag}[3/2, 6], \quad f(t) = \Phi^T p(t) = F_2 \cos \Omega t \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \]

\[ \ddot{\eta}(t) + C_g \dot{\eta}(t) + \text{diag}[3/2, 6] \eta(t) = f(t). \]  

Since \( C_g \) is not diagonal if \( c > 0 \), the modal EOM (22.23) are coupled through that matrix.
22.9. Diagonalization By Energy Balance, aka RQ Diagonalization

We will investigate the following “Rayleigh quotient” (RQ) diagonalization technique\(^\dagger\)

\[
C_g^{RQ} = \text{diag}[C_1^{RQ}, C_2^{RQ}] = \begin{bmatrix} C_1^{RQ} & 0 \\ 0 & C_2^{RQ} \end{bmatrix}, \quad C_1^{RQ} = \frac{\phi_1^T C \phi_1}{\phi_1^T \phi_1} = \frac{2c}{5}, \quad C_2^{RQ} = \frac{\phi_2^T C \phi_2}{\phi_2^T \phi_2} = \frac{5c}{2}.
\]

(22.24)

This relation is obtained by equating the energy dissipated by damping over one cycle. Replacing \(C_g\) with \(C_g^{RQ}\) gives the “RQ modal EOM”

\[
\ddot{\eta}(t) + \text{diag}\left[\frac{2c}{5}, \frac{5c}{2}\right] \dot{\eta}(t) + \text{diag}[3/2, 6] \eta(t) = f(t).
\]

(22.25)

These now decouple to

\[
\ddot{\eta}_1(t) + \frac{2c}{5} \dot{\eta}_1(t) + \frac{3}{2} \eta_1(t) = \frac{2}{\sqrt{6}} F_2 \cos \Omega t, \quad \ddot{\eta}_2(t) + \frac{5c}{2} \dot{\eta}_2(t) + 6 \eta_2(t) = -\frac{1}{\sqrt{3}} F_2 \cos \Omega t.
\]

(22.26)

The effective modal damping factors are

\[
\xi_1^{RQ} = \frac{2c}{5(2\omega_1)} = 0.1633 c, \quad \xi_2^{RQ} = \frac{5c}{2(2\omega_2)} = 0.5103 c.
\]

(22.27)

The IC required for solving the ODE (22.26) are, from Lecture 20, \(\eta(0) = \Phi^T M u(0)\) and \(\dot{\eta}(0) = \Phi^T M \dot{u}(0)\), and replacing numbers:

\[
\eta_0 = \begin{bmatrix} \eta_{10} \\ \eta_{20} \end{bmatrix} = \begin{bmatrix} 1 \sqrt{6} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 \sqrt{6} \\ 0 \sqrt{3} \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 2(u_1(0) + u_3(0)) \\ 2(u_1(0) - u_3(0)) \end{bmatrix},
\]

\[
\dot{\eta}_0 = \begin{bmatrix} \dot{\eta}_{10} \\ \dot{\eta}_{20} \end{bmatrix} = \begin{bmatrix} 1 \sqrt{6} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 \sqrt{6} \\ 0 \sqrt{3} \end{bmatrix} \begin{bmatrix} \dot{u}_1(0) \\ \dot{u}_2(0) \end{bmatrix} = \begin{bmatrix} 2(\dot{u}_1(0) + \dot{u}_3(0)) \\ 2(\dot{u}_1(0) - \dot{u}_3(0)) \end{bmatrix}.
\]

(22.28)

Once the solutions \(\eta_1^{RQ}(t)\) and \(\eta_2^{RQ}(t)\) are obtained they can be combined via the modal matrix as

\[
u^{RQ}(t) = \Phi \eta^{RQ}
\]

to produce the response in physical coordinates.

22.10. Comparison of Exact Versus Diagonalized Responses

This will be demonstrated in class using Mathematica from a laptop if there is time, else it will be skipped. The damping level will be varied as well as the IC and forcing function.

\[^\dagger\] Do not confuse with the Rayleigh damping, also known as proportional damping, approach. Same name, different matrices.