ASEN 5107 Spring 2007 Nonlinear FEM Quiz 3 (Last Exam)

Solutions

QUESTION 1

Integrated draft and lift forces: $P_d = p_d\,L = p_{d0}L\sin^2\theta$, $P_t = p_t\,L = p_{t0}L\cos^2\theta$.

$$\mathbf{f}_L = \tfrac{1}{2}\begin{bmatrix} -P_d\sin\theta + P_t\cos\theta \\ P_d\cos\theta + P_t\sin\theta \\ -P_d\sin\theta + P_t\cos\theta \\ P_d\sin\theta + P_t\cos\theta \end{bmatrix} = \tfrac{1}{2}L\begin{bmatrix} -p_{d0}\sin^3\theta + p_{t0}\cos^3\theta \\ p_{d0}\sin^2\theta\cos\theta + p_{t0}\sin\theta\cos^2\theta \\ -p_{d0}\sin^3\theta + p_{t0}\cos^3\theta \\ p_{d0}\sin^2\theta\cos\theta + p_{t0}\sin\theta\cos^2\theta \end{bmatrix} \tag{Q3.1}$$

To abbreviate, call $c = \cos\theta$ and $s = \sin\theta$. The necessary partial derivatives in terms of angles are: $\partial L/\partial u_{X1} = -c$, $\partial L/\partial u_{Y1} = -s$, $\partial L/\partial u_{X2} = c$, $\partial L/\partial u_{Y2} = s$, $\partial\theta/\partial u_{X1} = s/L$, $\partial\theta/\partial u_{Y1} = -c/L$, $\partial\theta/\partial u_{X2} = -s/L$, $\partial\theta/\partial u_{Y2} = c/L$. Using these relations one obtains

$$\mathbf{K}_L = -\frac{\partial\mathbf{f}_L}{\partial\mathbf{u}} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \tag{Q3.2}$$

in which

$$\begin{aligned} K_{11} = K_{31} &= \tfrac{1}{2}p_{t0}c^4 + \tfrac{3}{2}p_{d0}c^2s^2 + p_{t0}cs^3 \\ K_{12} = K_{32} &= -p_{t0}c^3s - \tfrac{3}{2}p_{d0}c^2s^2 - p_{d0}s^4 \\ K_{13} = K_{33} &= -\tfrac{1}{2}p_{t0}c^4 - \tfrac{3}{2}p_{t0}c^2s^2 - p_{d0}s^4 \\ K_{14} = K_{34} &= p_{t0}c^4 - \tfrac{3}{2}p_{d0}c^2s^2 - \tfrac{1}{2}p_{d0}s^4 \\ K_{21} = K_{41} &= p_{d0}c^3s + p_{t0}cs^3 + \tfrac{1}{2}p_{d0}s^4 \\ K_{22} = K_{42} &= \tfrac{1}{2}p_{t0}c^4 + p_{d0}c^3s + \tfrac{1}{2}p_{d0}s^4 \\ K_{23} = K_{43} &= \tfrac{1}{2}p_{t0}c^2s^2 - p_{t0}cs^3 - \tfrac{1}{2}p_{d0}s^4 \\ K_{24} = K_{24} &= -\tfrac{1}{2}p_{t0}c^4 - p_{d0}cs^3 - \tfrac{1}{2}p_{d0}s^4 \end{aligned} \tag{Q3.3}$$

For the given numerical values,

$$\mathbf{K}_L = \begin{bmatrix} 31.9512 & -55.5584 & -31.9512 & 55.5584 \\ 9.8816 & 17.1288 & -9.8816 & -17.1288 \\ 31.9512 & -55.5584 & -31.9512 & 55.5584 \\ 9.8816 & 17.1288 & -9.8816 & -17.1288 \end{bmatrix} \tag{Q3.4}$$

QUESTION 2

The solution does not change with respect to (a), which justifies the exclusion of the geometric stiffness.

QUESTION 3

Two element flutter load: $P_{cr} = 14.35EI/L^2$. Results were obtained with the *Mathematica* script listed in Figures Q3.2. Results for the two-element discretization are given in Figure Q3.3.

```
ClearAll[λ,EA,EI,L];  L=1; EA=EI=1; rho=1; Ne=2; Le=L/Ne;
KeM={{ EA/Le,          0,          0,-EA/Le,         0,         0},
     {     0, 12*EI/Le^3, 6*EI/Le^2,    0,-12*EI/Le^3, 6*EI/Le^2},
     {     0,  6*EI/Le^2, 4*EI/Le  ,    0, -6*EI/Le^2, 2*EI/Le  },
     {-EA/Le,          0,          0, EA/Le,         0,         0},
     {     0,-12*EI/Le^3,-6*EI/Le^2,    0, 12*EI/Le^3,-6*EI/Le^2},
     {     0,  6*EI/Le^2, 2*EI/Le  ,    0, -6*EI/Le^2, 4*EI/Le  }};
KeG=-{{    0,          0,          0,    0,          0,         0},
     {     0,         36,       3*Le,    0,        -36,      3*Le},
     {     0,       3*Le,     4*Le^2,    0,      -3*Le,     -Le^2},
     {     0,          0,          0,    0,          0,         0},
     {     0,        -36,      -3*Le,    0,         36,     -3*Le},
     {     0,       3*Le,      -Le^2,    0,      -3*Le,    4*Le^2}}*λ/(30*Le);
KeL={{     0,         0,         0,    0,         0,        0},
     {     0,         0,         0,    0,         0,        0},
     {     0,         0,         0,    0,         0,        0},
     {     0,         0,         0,    0,         0,        0},
     {     0,         0,         0,    0,         0,        1},
     {     0,         0,         0,    0,         0,        0}}*λ;
n=3*Ne; K=M=Table[0,{n},{n}];  Mnod=rho*Le/2;
For [e=1, e<=Ne, e++,
     If [e==1,eft={0,0,0,3*e-2,3*e-1,3*e}];
     If [e>1, eft={3*e-5,3*e-4,3*e-3,3*e-2,3*e-1,3*e}];
     If [e<Ne, Ke=KeM+KeG, Ke=KeM+KeG+KeL];
     For [i=1,i<=6,i++, ii=eft[[i]];If[ii==0,Continue[]];
         If [i==1||i==2||i==4||i==5, M[[ii,ii]]+=Mnod];
         For [j=1,j<=6,j++, jj=eft[[j]];If[jj==0,Continue[]];
         K[[ii,jj]]+=Ke[[i,j]] ]]
    ];
Print["K=",K//MatrixForm]; Print["M=",M//MatrixForm];
K11={{K[[2,2]],K[[2,5]]},{K[[5,2]],K[[5,5]]}};
K12={{K[[2,3]],K[[2,6]]},{K[[5,3]],K[[5,6]]}};
K21={{K[[3,2]],K[[3,5]]},{K[[6,2]],K[[6,5]]}};
K22={{K[[3,3]],K[[3,6]]},{K[[6,3]],K[[6,6]]}};
M11={{M[[2,2]],M[[2,5]]},{M[[5,2]],M[[5,5]]}};
M12={{M[[2,3]],M[[2,6]]},{M[[5,3]],M[[5,6]]}};
M21={{M[[3,2]],M[[6,2]]},{M[[3,5]],M[[6,5]]}};
M22={{M[[3,3]],M[[3,6]]},{M[[6,3]],M[[6,6]]}};
Kc=K11-K12.Inverse[K22].K21; Mc=M11;
d=Simplify[Det[pp*Mc+Kc]]; Print["d=",d];
solpp=Simplify[Solve[d==0,pp]];
{pp1,pp2}=pp/.solpp; Print["p^2 roots=",{pp1,pp2}];
solλ=Simplify[Solve[Numerator[pp1]==0,λ]]; Print["divergence roots
pp1=",Chop[N[solλ,6]]];
solλ=Simplify[Solve[Numerator[pp2]==0,λ]]; Print["divergence roots
pp2=",Chop[N[solλ,6]]];
solλ=Simplify[Solve[pp1==pp2,λ]]; Print["flutter roots=",Chop[N[solλ,6]]];
```

Figure Q3.2. *Mathematica* script to solve Question 3.

$$
K=\begin{pmatrix}
4 & 0 & 0 & -2 & 0 & 0\\
0 & 192-\frac{24\lambda}{5} & 0 & 0 & -96+\frac{12\lambda}{5} & 24-\frac{\lambda}{10}\\
0 & 0 & 16-\frac{2\lambda}{15} & 0 & -24+\frac{\lambda}{10} & 4+\frac{\lambda}{60}\\
-2 & 0 & 0 & 2 & 0 & 0\\
0 & -96+\frac{12\lambda}{5} & -24+\frac{\lambda}{10} & 0 & 96-\frac{12\lambda}{5} & -24+\frac{11\lambda}{10}\\
0 & 24-\frac{\lambda}{10} & 4+\frac{\lambda}{60} & 0 & -24+\frac{\lambda}{10} & 8-\frac{\lambda}{15}
\end{pmatrix}
$$

$$
M=\begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{4} & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1}{4} & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
d=\frac{72\,(3840+64\lambda+\lambda^2)^2+pp^2\,(403200-8160\lambda+31\lambda^2)-48\,pp\,(-2304000+119040\lambda-1816\lambda^2+7\lambda^3)}{8\,(403200-8160\lambda+31\lambda^2)}
$$

p^2 roots=

$$
\Big\{-\frac{1}{403200-8160\lambda+31\lambda^2}\Big(6\Big(9216000-476160\lambda+7264\lambda^2-28\lambda^3+\sqrt{(73043804160000-8932294656000\lambda+358229606400\lambda^2-7375257600\lambda^3+79983104\lambda^4-398400\lambda^5+722\lambda^6)}\Big)\Big),
$$
$$
\frac{1}{403200-8160\lambda+31\lambda^2}\Big(6\Big(-9216000+476160\lambda-7264\lambda^2+28\lambda^3+\sqrt{(73043804160000-8932294656000\lambda+358229606400\lambda^2-7375257600\lambda^3+79983104\lambda^4-398400\lambda^5+722\lambda^6)}\Big)\Big)\Big\}
$$

divergence roots pp1 $=\{\{\lambda\Rightarrow 197.305\}\}$

divergence roots pp2 $=\{\{\lambda\Rightarrow 65.9204\},\{\lambda\Rightarrow -32.-53.066\,i\},\{\lambda\Rightarrow -32.+53.066\,i\}\}$

flutter roots $=\{\{\lambda\Rightarrow 14.3477\},\{\lambda\Rightarrow 64.6765\},\{\lambda\Rightarrow 36.7169-36.683\,i\},\{\lambda\Rightarrow 36.7169+36.683\,i\},\{\lambda\Rightarrow 199.671-24.5675\,i\},\{\lambda\Rightarrow 199.671+24.5675\,i\}\}$

Figure Q3.3. Results of running the script of Figure Q3.2 for the 2-element discretization.

QUESTION 4

Solution using *Mathematica*:

```
ClearAll[a,b,rho,CL,kS,mb];
sub={CL->1,rho->1,a->3/4,b->1,kS->1,mb->1};
KM=(kS/4)*{{1,1},{1,1}}+alpha*kS*{{1,-1},{-1,1}};
KL=-(1/2)*rho*CL*v^2*{{(a/b),-(a/b)},{(1-a/b),-(1-a/b)}};
K=KM+KL; M=(mb/6)*{{2,1},{1,2}};
d=Simplify[Det[K+pp*M]]; Print["d=",d];
sol=Solve[d==0,pp];sol=Simplify[sol];
Print["p^2 roots=",sol//InputForm];
pp1=pp/.sol[[1]]; pp2=pp/.sol[[2]];
Print["pp1=",pp1]; Print["pp2=",pp2];
solf=Solve[pp1==pp2,v];
sold=Solve[pp2==0,v];
vf=Simplify[v/.solf[[2]]];  (* pick +v *)
vd=Simplify[v/.sold[[2]]];  (* pick +v *)
Print["vf=",vf]; Print["vd=",vd];
numvals={CL->1,rho->1,a->3/4,b->1,kS->1};
Plot[vd/.numvals,{alpha,0.0834,1},PlotLabel->"Divergence speed"];
```

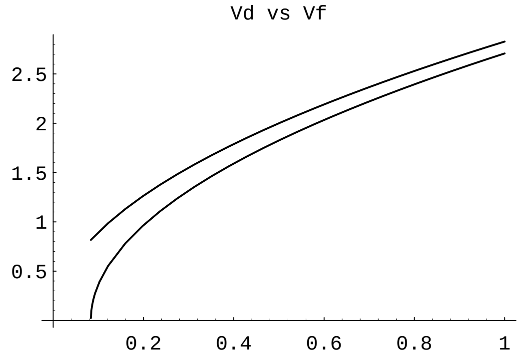

Figure Q3.4. Critical speeds v_d and v_f plotted as functions of α for $\alpha > 1/12$. Note, however, that flutter does not occur at $v = v_f$ because $p_1^2 \neq p_2^2$ for $v_d > v > v_f$, whence the frequencies do not coalesce.

```
Plot[vf/.numvals,{alpha,0.0834,1},PlotLabel->"Flutter speed"];
Plot[{vd/.numvals,vf/.numvals},{alpha,0.0834,1},PlotLabel->"Vd vs Vf"];
```

```
                                                        2               2
  (kS + mb pp) (12 alpha b kS + b mb pp - 6 a CL rho v  + 3 b CL rho v )
d=---------------------------------------------------------------------
                                   12 b
p^2 roots={{pp -> -(kS/mb)},
{pp -> (-3*(4*alpha*b*kS - 2*a*CL*rho*v^2 + b*CL*rho*v^2))/(b*mb)}}
      kS
pp1=-(--)
      mb
                                   2             2
    -3 (4 alpha b kS - 2 a CL rho v  + b CL rho v )
pp2=---------------------------------------------
                         b mb

   Sqrt[-1 + 12 alpha] Sqrt[b] Sqrt[kS]
vf=------------------------------------
    Sqrt[6 a - 3 b] Sqrt[CL] Sqrt[rho]
    2 Sqrt[alpha] Sqrt[b] Sqrt[kS]
vd=--------------------------------
   Sqrt[2 a - b] Sqrt[CL] Sqrt[rho]
```

For the given numerical values, $v_d = 2\sqrt{2}\sqrt{\alpha}$ and $v_f = 2\sqrt{2/3}\sqrt{12\alpha - 1}$. If $\alpha > 1/12 = 0.0833$, v_f is real and $v_f < v_d$ so in theory "flutter" occurs first. [If $\alpha \leq 1/12$ instability occurs for any speed because p_2^2 is positive real.] A plot of v_f and v_d versus α is shown in Figure Q3.4. Note, however, that the p^2 roots obtained above

$$p_1^2 = -\frac{k_S}{m_b}, \qquad p_2^2 = \frac{-3(4\alpha b k_S) - 3(b - 2a)C_L \rho v^2}{b m_b} \tag{Q3.5}$$

remain distinct for $v > v_f$. Hence flutter, in the sense of coalescence of two frequencies, *does not happen*. Dynamic instability always occurs by divergence.

BONUS QUESTION 1

Four element idealization gives $P_{cr} = 18.383\, EI/L^2$ as minimum flutter load.

BONUS QUESTION 2

For small deflections, $f_Y = -\lambda p \sin\theta \approx -\lambda p u_Y/L$ and $f_X = \lambda p \cos\theta \approx -\lambda p,\ f_Z = 0.$ The only nonzero entry in the load stiffness matrix is $K_{L22} = -f_Y/\partial u_Y = \lambda p/L.$ The tangent stiffness is

$$\mathbf{K} = \mathbf{K}_M + \mathbf{K}_G + \mathbf{K}_L = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} - \frac{\lambda p}{30L} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 36 & -3L \\ 0 & -3L & 4L^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\lambda p}{L} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{Q3.6}$$

Since $\mathbf{K}$ is symmetric the system is conservative and cannot lose stability by flutter. The singular-stiffness static criterion $\det(\mathbf{K}) = 0$ gives $P_{cr} = 12EI/L^2$, which is the same result as that of a cantilevered column treated by a single Euler-Bernoulli beam element.

BONUS QUESTION 3

A one DOF system cannot flutter because the linearized perturbation system can have only one root: p_1^2. Two distinct roots, p_1^2 and p_2^2, must coalesce for flutter to occur.