

## ASEN 5107: Nonlinear Finite Element Methods, Spring 2007. Second Midterm Quiz

Begin answering each Question on a new page. Attach this exam as cover with name on it

**Important: Work individually. No consultation with others is permitted**

Take home Th April 5, 2007. Due on or before Th April 12 at class time. If returning exam before April 12 leave in instructor's mailbox at ECAE 196 or at the mailbox in ECOT 635 (the Aero department mail office). Please place in clearly labeled envelope to avoid loss. Do not leave outside or under door.

For this exam, the two-bar arch used as example in Chapter 8 is now built as a rigid-jointed frame, with two identical straight, prismatic, plane *beam members* of length  $L_0$  clamped at supports 1 and 3, as shown in Figure 1. The arch is subjected to a vertical load  $f_Y = \lambda P$  applied to the crown (node 2). The reference load  $P$  is set to  $-1$ . Therefore  $\lambda > 0$  means that the load acts downward. Axes  $X$  and  $Y$  are placed as indicated.

The arch span is  $S$ . The initial height is  $H$ , whence  $L_0^2 = (S/2)^2 + H^2$ . The properties of both beams are  $E$ ,  $A_0$  and  $I_0$ . The structure is modeled with two  $C^0$  (Timoshenko) TL beam elements of the kind derived in Chapter 9. The shear rigidity  $GA_0$  is replaced by  $12EI_0/L_0^2$ , which is MacNeal's simplified RBF. To characterize the arch geometry set  $H = \frac{1}{2}\beta S$  and  $\beta = \tan \alpha$ , where  $\alpha \geq 0$  is the rise angle. Furthermore, take  $I_0 = \rho A_0 L_0^2$ , in which  $\rho \geq 0$  is a dimensionless parameter that characterizes the beam bending rigidities. All initial stresses in  $C_0$  are zero.

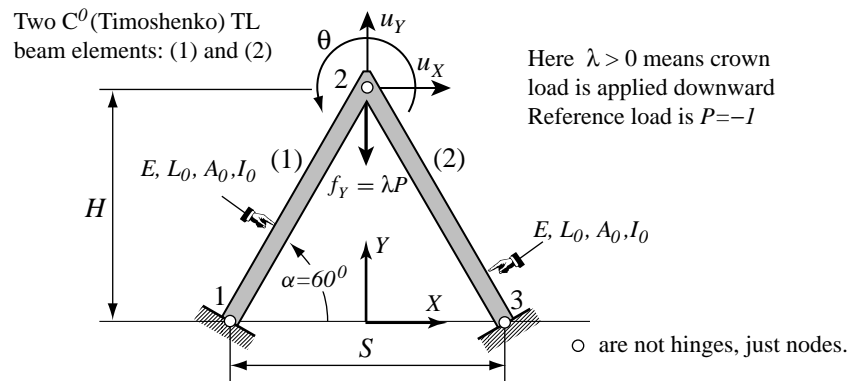


Figure 1. Clamped framework arch structure for exam. Each member is modeled with one  $C^0$  TL beam element.

Because all freedoms at nodes 1 and 3 are suppressed, the structure has only three DOFs at node 2:  $u_{X2}$ ,  $u_{Y2}$  and  $\theta_2$ . These are renamed  $u_X$ ,  $u_Y$  and  $\theta$ , respectively, for simplicity. The fundamental equilibrium path is  $u_X = \theta = 0$ , which is called the *symmetric* deformation of the arch. The dimensionless deflections  $\mu_X$  and  $\mu_Y$  are defined as  $\mu_X = u_X/(S/2)$  and  $\mu_Y = u_Y/(S/2)$ . To get rid of all remaining dimensioned quantities set  $E = 1$ ,  $A_0 = 1$  and  $S = 2$ .

For Questions 1–3 below set  $\rho = 1/24$ ,  $\alpha = 60^\circ$  and  $\beta = \tan 60^\circ = \sqrt{3}$ . The residual  $\mathbf{r}$ , internal force  $\mathbf{p}$  and the tangent stiffness matrix  $\mathbf{K}$  as functions of  $\{\lambda, \mu_X, \mu_Y, \theta\}$  were obtained from the program developed in Chapter 9. They are provided in the Addendum and posted on a Notebook since finite element derivations are not part of this exam.

The questions below may be done by hand calculator or using any computer language you like (Matlab, Mathematica, MathCad, Fortran, C, etc). If you use a computer please attach the important source code in addition to results.

### QUESTION 1 (Analytical/numerical, 30 points)

Determine the critical point defined by  $\lambda_{cr} > 0$  and  $\mu_{Ycr}$  closest to  $\lambda = 0$  on the fundamental equilibrium path  $\mu_X = \theta = 0$ , for parameters  $\rho = 1/24$  and  $\alpha = 60^\circ$ . Check (by exact arithmetic) that it is a bifurcation.

### QUESTION 2 (Numerical, 40 points)

Compute numerically the response  $\mathbf{u}(\lambda)$  for  $\rho = 1/24$  and  $\alpha = 60^\circ$ , using a purely incremental Forward Euler or Midpoint Rule (pick one) with Arclength Control (FE/AC). Start at the reference state  $\lambda = \mu_X = \mu_Y = \theta = 0$  and step  $\ell$  to get to the vicinity of the bifurcation point determined above. Trigger the structure to buckle by injecting artificial imperfections. (Without imperfections the solution process will continue forever on the fundamental path, because the solution method is purely incremental.) As basic results provide response plots similar to those of Figure 2.

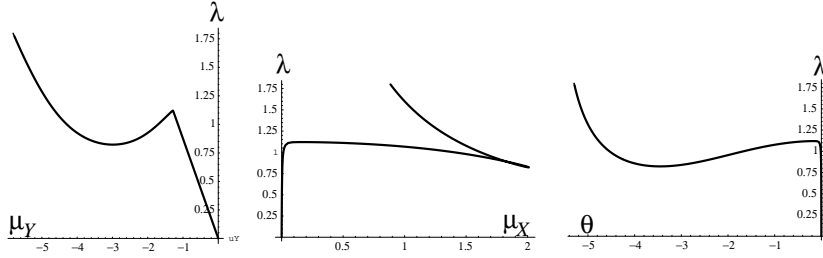


Figure 2. Sketch of response of  $\mu_Y$ ,  $\mu_X$  and  $\theta$  versus  $\lambda$  for the arch structure; given to suggest how to present results.

Note: the fundamental path (symmetric) response  $\mu_Y = \mu_Y(\lambda)$  is *exactly linear* because of the beam element derivation assumptions and the fact that each member is modeled with only one element. See leftmost plot in Figure 2. To make the program take the secondary path you will need to disturb the arch. Put a tiny horizontal load or a small moment on the crown, and play with the load imperfection and the increment  $\ell$  until you get some decent looking response plots.

### QUESTION 3 (Numerical, 30 points)

If Question 2 works OK, you will likely notice that the secondary path is very sensitive to increment length and imperfections, in the sense that it may change significantly if those solution parameters are modified. Away from bifurcation that sensitivity can be eliminated by a corrective process that eliminates the drift error. Do so by following the FE/AC predictor with a Conventional Newton process. Rerun with different  $\ell$  until you see the same response (if that happens) and report your findings.

### BONUS QUESTION CHOICE 1 (up to 5 pts, total may not exceed 100) Choose only one Bonus Question

Provide a contour plot that displays  $\lambda_{cr}$  curves as a function of  $0 \leq \alpha \leq 75^\circ$  and  $0 \leq \rho \leq 1/5$  on the horizontal and vertical axes, respectively, for the arch structure with variable  $\alpha$  and  $\rho$ . Notes: (i)  $\mathbf{K}$  for arbitrary  $\rho$  and  $\beta = \tan \alpha$  is available on the posted Notebook; (ii) Try this one only if you are a programming wizard, comfortable with graphics.

### BONUS QUESTION CHOICE 2 (up to 5 pts, total may not exceed 100) Choose only one Bonus Question

Suppose  $\alpha = 0^\circ$  and  $\rho = 0$ . The “arch” degenerates to a pair of bars that looks like Figure Q1.2 of Quiz 1. At the reference state  $\lambda = \mu_X = \mu_Y = \theta = 0$ ,  $\mathbf{K}$  is singular. No surprise there. However, if we set  $\mu_X = \theta = 0$  to impose symmetric deformations, what’s left of the stiffness (namely  $K_{YY}$ ) is exactly zero for any  $\mu_Y$ . But this kind of structure is supposed to stiffen if it displaces laterally. Explain the reason for this deficiency of the FEM model.

### Addendum - Internal Force, Residual and Tangent Stiffness

The following expressions were computed using *Mathematica* for  $E = A_0 = 1$ ,  $S = 2$ ,  $\alpha = 60^\circ$ ,  $\rho = 1/24$  and symbolic  $\lambda$ ,  $\mu_X$ ,  $\mu_Y$  and  $\theta$ . They are directly available in the Quiz 2 Notebook *posted on the course web site*. They are listed here in case a program other than *Mathematica* is chosen.

$$\mathbf{r} = \mathbf{p} - \mathbf{f}, \quad \mathbf{p} = \begin{bmatrix} p_X \\ p_Y \\ p_\theta \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_X \\ f_Y \\ f_\theta \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} K_{XX} & K_{XY} & K_{X\theta} \\ & K_{YY} & K_{Y\theta} \\ \text{symm} & & K_{\theta\theta} \end{bmatrix}, \quad (\text{Q2.1})$$

$$p_X = \frac{6\mu_X - \mu_X c + 8\sqrt{3}s_h - 2\sqrt{3}s - \mu_Y s}{8}, \quad p_Y = \frac{6\sqrt{3} + 6\mu_Y - 8\sqrt{3}c_h + 2\sqrt{3}c + \mu_Y c - \mu_X s}{8},$$

$$p_\theta = \frac{\theta}{6} + \frac{\sqrt{3}\mu_X c_h}{2} - \frac{\sqrt{3}\mu_X c}{4} - \frac{\mu_X \mu_Y c}{8} + 2s_h + \frac{\sqrt{3}\mu_Y s_h}{2} - \frac{s}{2} + \frac{\mu_X^2 s}{16} - \frac{\sqrt{3}\mu_Y s}{4} - \frac{\mu_Y^2 s}{16},$$

$$K_{XX} = \frac{6-c}{8}, \quad K_{XY} = -\frac{s}{8}, \quad K_{X\theta} = \frac{4\sqrt{3}c_h - 2\sqrt{3}c - \mu_Y c + \mu_X s}{8}, \quad (\text{Q2.2})$$

$$K_{YY} = \frac{6+c}{8}, \quad K_{Y\theta} = \frac{4\sqrt{3}s_h - 2\sqrt{3}s - \mu_Y s - \mu_X c}{8},$$

$$K_{\theta\theta} = \frac{1}{6} + c_h + \frac{\sqrt{3}\mu_Y c_h}{4} - \frac{c}{2} + \frac{\mu_X^2 c}{16} - \frac{\sqrt{3}\mu_Y c}{4} - \frac{\mu_Y^2 c}{16} - \frac{\sqrt{3}\mu_X s_h}{4} + \frac{\sqrt{3}\mu_X s}{4} + \frac{\mu_X \mu_Y s}{8}.$$

Here  $c = \cos \theta$ ,  $c_h = \cos(\theta/2)$ ,  $s = \sin \theta$  and  $s_h = \sin(\theta/2)$ . To inject artificial imperfections for Questions 2-3,  $f_X$  or  $f_\theta$  are set to a tiny nonzero value, the sign of which is unimportant.