24

Bifurcation: Linearized Prebuckling I

TABLE OF CONTENTS

§24.1. Introduction

This Chapter starts a systematic study of the *stability of elastic structures*. We shall postpone the more rigorously mathematical definition of stability (or lack thereof) until later because the concept is essentially dynamic in nature. For the moment the following physically intuitive concept should suffice:

> "A structure is stable at an equilibrium position if it returns to that position upon being disturbed by an extraneous action"

Note that this informal definition is dynamic in nature, because the words "returns" and "upon" convey a sense of history. But it does not imply that the inertial and damping effects of true dynamics are involved.

A structure that is initially stable may lose stability as it moves to another equilibrium position when the control parameter(s) change. Under certain conditions, that transition is associated with the occurrence of a *critical point*. These have been classified into limit points and bifurcation points in Chapter 5.

For the slender structures that occur in aerospace, civil and mechanical engineering, bifurcation points are more practically important than limit points. Consequently, attention will be initially directed to the phenomena of bifurcation or branching of equilibrium states, a set of phenomena also informally known as *buckling*. The analysis of what happens to the structure after it crosses a bifurcation point is called *post-buckling analysis*.

The study of bifurcation and post-buckling while carrying out a full nonlinear analysis is a mathematically demanding subject. But in important cases the loss of stability of a geometrically nonlinear structure by bifurcation can be assessed by solving *linear algebraic eigenvalue problems* or "eigenproblems" for short. This eigenanalysis provides the *magnitude* of the loads (or, more generally, of the control parameters) at which buckling is expected to occur. The analysis yields no information on post-buckling behavior. Information on the buckling load levels is often sufficient, however, for design purposes.

The present Chapter covers the source of such eigenproblems for *conservatively loaded* elastic structures. Chapters 26 through 28 discuss stability in the context of full nonlinear analysis. The two final Chapters (29–30) extend these concepts to structures under nonconservative loading.

Following a brief review of the stability assessment criteria the singular-stiffness test is described. Attention is then focused on the particular form of this test that is most used in engineering practice: the linearized prebuckling (LPB) analysis. The associated buckling eigenproblem is formulated. The application of LPB on a simple problem is worked out using the bar element developed in the previous three sections. The assumptions underlying LPB and its range of applicability are discussed in the next Chapter.

§24.2. Loss of Stability Criteria

For *elastic*, geometrically-nonlinear structures under *static* loading we can distinguish the following techniques for stability assessment.

$$\text{Loading}\begin{cases}\text{Conservative}\begin{cases}\text{Static criterion (Euler method): singular stiffness}\\ \text{Dynamic criterion: zero frequency}\end{cases}\\ \text{Nonconservative}\left\{\text{Dynamic criterion}\begin{cases}\text{zero frequency (divergence)}\\ \text{frequency coalescence (flutter)}\end{cases}\right.\end{cases}$$

§24.2.1. Static criterion

The *static criterion* is also known as *Euler's method*, since Euler introduced it in his famous investigations of the elastica published in 1744. Other names for it are *energy method* and *method of adjacent states*. To apply this criterion we look at *admissible static perturbations of an equilibrium position*[1]. These perturbations generate adjacent states or configurations, which are not generally in equilibrium.

Stability is assessed by comparing the potential energy of these adjacent configurations with that of the equilibrium position. If all adjacent states have a higher potential energy, the equilibrium is stable. If at least one state has a lower (equal) potential energy the equilibrium is unstable (neutrally stable). This comparison can be expressed in terms of the second variation of the potential energy and hence can be reduced to the assessment of the *positive definite character of the tangent stiffness matrix*.

Although stability is a dynamic phenomenon, no true-dynamics concepts such as mass or damping are involved in the application of the static criterion, which is a key reason for its popularity. But the reasoning behind it makes it strictly applicable only to *conservatively loaded* systems, because a load potential function is assumed to exist.

§24.2.2. Dynamic criterion

The dynamic criterion looks at *dynamic perturbations* of the static equilibrium position. In informal terms, "give the structure a (little) kick and see how it moves." More precisely, we consider *small oscillations* about the equilibrium position, and pose an eigenproblem that determines characteristic exponents and associated eigenmodes. The characteristic exponents are generally complex numbers. If all characteristic exponents have no positive real components the equilibrium is dynamically stable, and unstable otherwise.

These exponents change as the control parameter λ is varied. For sufficiently small values the structure is stable. Loss of stability occurs when a characteristic exponent enters the right-hand complex plane. If that happens, the associated mode viewed as a displacement pattern will amplify exponentially in the course of time. A deeper study of the stable-to-unstable transition mechanism

[1] "Admissible" in the sense of the Principle of Virtual Work: variations of the state parameters that are consistent with the essential boundary conditions (kinematic constraints)

reveals two types of instability phenomena, which are associated with the physically-oriented terms terms *divergence* and *flutter*.

Divergence occurs when the characteristic exponent enters the right-hand plane through the origin, and it can therefore be correlated with the zero frequency test and the singular stiffness test.

The dynamic criterion is applicable to both conservative and nonconservative systems. This wider range of application is counterbalanced by the need of incorporating additional information (mass and possibly damping) into the problem. Furthermore, unsymmetric eigenproblems arise in the nonconservative case, and these are the source of many computational difficulties.

§24.3. The Tangent Stiffness Test

The stability of conservative systems can be assessed by looking at the spectrum[2] of the tangent stiffness matrix **K**. Let μ_i denote the i^{th} eigenvalue of **K**. The set of μ_i's are the solution of the algebraic eigenproblem

$$\mathbf{K}\mathbf{z}_i = \mu_i \mathbf{z}_i. \tag{24.1}$$

Since **K** is *real symmetric*[3] all of its eigenvalues are *real*. Thus we can administer the following test:

(I)	If all $\mu_i > 0$	the equilibrium position is strongly stable
(II)	If all $\mu_i \geq 0$	the equilibrium position is neutrally stable
(III)	If some $\mu_i < 0$	the equilibrium position is unstable

In engineering applications one is especially interested in the behavior of the structure as the stage control parameter λ is varied, and so

$$\mathbf{K} = \mathbf{K}(\lambda). \tag{24.2}$$

Given this dependence, a key information is the *transition* from stability to instability at the value of λ closest to stage start, which is usually $\lambda = 0$. This is called the *critical value* of λ, which we shall denote as λ_{cr}.

If the entries of **K** depend continuously on λ the eigenvalues of **K** also depend continuously[4] on λ, although the dependence is not necessarily continuously differentiable. It follows that transition from strong stability — case (I) — to instability — case (III) — has to go through case (II), *i.e.* a zero eigenvalue. Thus a *necessary* condition is that **K** be singular, that is

$$\det \mathbf{K}(\lambda_{cr}) = 0, \tag{24.3}$$

or, equivalently,

$$\mathbf{K}(\mathbf{u}_{cr}, \lambda_{cr})\mathbf{z} = \mathbf{0}, \tag{24.4}$$

where $\mathbf{z} \neq 0$ is the buckling mode introduced in Chapters 4–5, where it was called a null eigenvector. Equation (24.3) or (24.4) is the expression of the *static test* for finding a stability boundary.

---

[2] The spectrum of a matrix is the set of its eigenvalues.

[3] Because $\mathbf{K} = \partial^2\Pi/\partial\mathbf{u}\partial\mathbf{u}$ is the Hessian of the total potential energy Π.

[4] Continuous dependence of eigenvalues on the entries is guaranteed by the perturbation theory for symmetric and Hermitian matrices. This continuous dependence does not hold, however, for eigenvectors.

Remark 24.1. Equation (24.3) is a nonlinear eigenvalue problem because: (a) $\mathbf{K}$ has to be evaluated at an equilibrium position, and (b) $\mathbf{K}$ is a nonlinear function of $\mathbf{u}$, which in turn is a nonlinear function of λ as defined by the equilibrium path. It follows that in general a complete response analysis has to be conducted to solve (24.3). Such techniques were called "indirect methods" in the context of critical point location methods in Chapter 23. This involves evaluating $\mathbf{K}$ at each computed equilibrium position, and then finding the spectrum of $\mathbf{K}$. An analysis of this nature is obviously computationally expensive. One way of reducing part of the cost is noted in the following remark.

Remark 24.2. If $\mathbf{K}$ is known at a given λ, an explicit solution of the eigenproblem (24.1) is not necessary for assessing stability. It is sufficient to factor $\mathbf{K}$ as

$$\mathbf{K} = \mathbf{L}\mathbf{D}\mathbf{L}^T \tag{24.5}$$

where $\mathbf{L}$ is unit lower triangular and $\mathbf{D}$ is diagonal. The number of negative eigenvalues of $\mathbf{K}$ is equal to the number of negative diagonal elements ("pivots") of $\mathbf{D}$. Matrix factorization is considerably cheaper than carrying out a complete eigenanalysis because sparseness can be exploited more effectively.

Remark 24.3. The condition (24.3) is not *sufficient* for concluding that a system that is stable for $\lambda < \lambda_{cr}$ will go unstable as λ exceeds λ_{cr}. A counterexample is provided by the stable-symmetric bifurcation point discussed in later Chapters. The Euler column furnishes a classical example. At such points (24.3) holds implying neutral stability but the system does not lose stability as the bifurcation state is traversed. Nonetheless the displacements may become so large that the structure is practically rendered useless.

§24.4. Linearized Prebuckling

We investigate now the first critical state of an elastic system if *the change in geometry prior to it can be neglected*. We shall see that in this case the nonlinear equilibrium equations can be partly linearized, a process that leads to the classical stability eigenproblem or *buckling eigenproblem*.

The eigenstability analysis procedure that neglects prebuckling displacements is known as *linearized prebuckling* (LPB). The modeling assumptions that are tacitly or explicitly made in LPB are discussed in some detail in the next Chapter, as well as the practical limitations that emanate from these assumptions. In the present Chapter we discuss the formulation of the LPB eigenproblem and illustrate these techniques on a simple problem using the bar elements developed in previous Chapters.

§24.5. The LPB Eigensystem

The two key results from the LPB assumptions (which are studied in the next Chapter) can be summarized as follows. Recall from Chapters 8–10 that the tangent stiffness matrix can be decomposed as the sum of material and geometric stiffness matrices:

$$\mathbf{K} = \mathbf{K}_M + \mathbf{K}_G. \tag{24.6}$$

Then the LPB leads to the following simplifications:

(1) The material stiffness is the stiffness evaluated at the reference configuration:

$$\mathbf{K}_M = \mathbf{K}_0. \tag{24.7}$$

(2) The geometric stiffness is linearly dependent on the control parameter λ:

$$\mathbf{K}_G = \lambda \mathbf{K}_1. \tag{24.8}$$

where $\mathbf{K}_1$ is constant and also evaluated at the reference configuration.

Now the stability test (24.3) requires that $\mathbf{K}$ be singular, which leads to the stability eigenproblem

$$\mathbf{K}\mathbf{z} = (\mathbf{K}_0 + \lambda \mathbf{K}_1)\, \mathbf{z} = \mathbf{0}. \tag{24.9}$$

In the following Chapter we shall prove that under certain restrictions the critical states determined from this eigenproblem are *bifurcation points* and not limit points. That is, they satisfy the orthogonality test

$$\mathbf{z}^T \mathbf{q} = 0. \tag{24.10}$$

The eigenproblem (24.9) befits the generalized symmetric algebraic eigenproblem

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{B}\mathbf{x}, \tag{24.11}$$

where both matrices $\mathbf{A} \equiv \mathbf{K}_0$ and $\mathbf{B} \equiv -\mathbf{K}_1$ are real symmetric, and $\mathbf{x} \equiv \mathbf{z}$ are the buckling mode eigenvectors. If (as usual) the material stiffness $\mathbf{K}_0$ is positive definite, eigensystem theory says that all eigenvalues of (24.11) are real. We cannot in general make statements, however, about the *sign* of these eigenvalues. That will depend on the physics of the problem as well as on the sign conventions chosen for the control parameter(s).

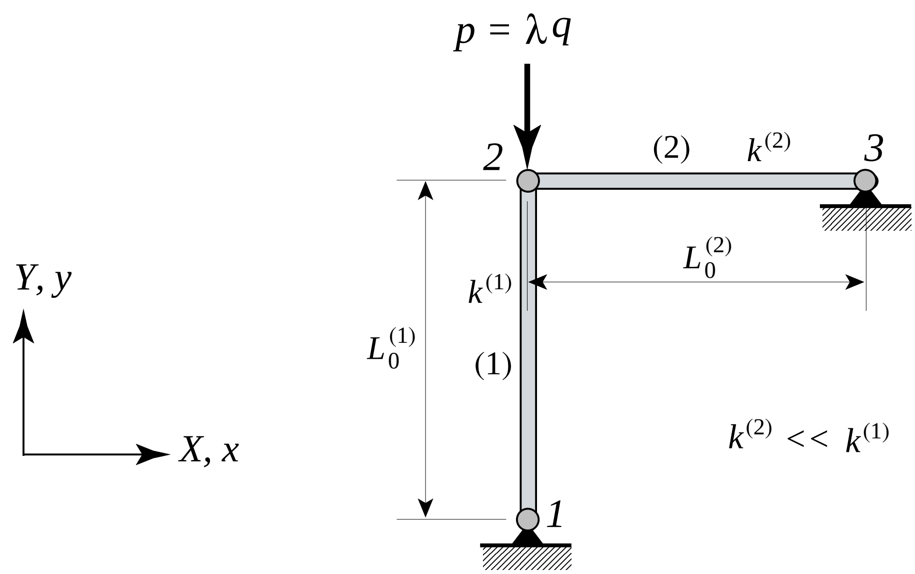

Figure 24.1. LPB example involving two bar elements displacing on the $X \equiv x$, $Y \equiv y$ plane.

§24.6. Solving the Stability Eigenproblem

In production FEM codes, the stability eigenproblem (24.9) is generally treated with special solution techniques that take full advantage of the sparsity of both $\mathbf{K}_0$ and $\mathbf{K}_1$, such as subspace iteration or Lanczos methods.

For small systems an expedient solution method consists of reducing it to canonical form by premultiplying both sides by the inverse of $\mathbf{K}_0$. This is possible if $\mathbf{K}_0$ is nonsingular, which means that the $\lambda = 0$ configuration is not a critical one. Calling $\mathbf{A} = \mathbf{K}_0^{-1}\mathbf{K}_1$ and $\mu = -1/\lambda$ one gets

$$\mathbf{A}\mathbf{z}_i = \mu_i \mathbf{z}_i \tag{24.12}$$

This is a standard algebraic eiegnproblem, which can be solved by standard library routines for the eigenvalues μ_i and eigenvectors $\mathbf{z}_i$. For example, `EigenSystem` in *Mathematica* or `Eig` in *Matlab*. The μ_i farthest away from zero gives the $\lambda_i = -1/\mu_i$ closest to zero.

One disadvantage of this reduction is that $\mathbf{A}$ is unsymmetric even if $\mathbf{K}_0$ and $\mathbf{K}_1$ are. There are more complicated reduction methods that preserve symmetry. These may be studied in standard numerical analysis textbooks covering linear algebra; for example Golub and Van Loan.

§24.7. LPB Analysis Example

To illustrate the application of LPB to a very simple example, the 2-bar assembly shown in Figure 24.1 is chosen. The bars can only displace on the x, y plane, thus the problem is two dimensional. The equivalent-spring stiffness of the bars is denoted by

$$k^{(1)} = \frac{E A_0^{(1)}}{L_0^{(1)}}, \qquad k^{(2)} = \frac{E A_0^{(2)}}{L_0^{(2)}}, \tag{24.13}$$

in which $A_0^{(e)}$ and $L_0^{(e)}$ denote the cross sectional areas and lengths, respectively, of the e^{th} bar in the reference configuration, and E is the elastic modulus common to both bars.

The figure shows the reference configuration $\mathcal{C}_0$ for the two bars. That configuration is taken when the applied load is zero, that is, $\lambda = 0$. We shall assume that the stiffness of bar 1 is much greater than that of bar 2, *i.e.*, $k^{(1)} >> k^{(2)}$ and is such that the vertical displacement u_{Y2} of node 2 under the load is very small compared to the dimensions of the structure.

Now let the load $p = \lambda q$ be gradually applied by increasing λ. The structure assumes a deformed current configuration in equilibrium, that is, a target configuration $\mathcal{C}$. According to the LPB basic assumption, the displacements prior to the buckling load level characterized by λ_{cr} are *negligible*. Therefore $\mathcal{C} \equiv \mathcal{C}_0$ as long as $|\lambda| < |\lambda_{cr}|$.

The *linear* finite element equations for the example problem are as follows. For element (1):

$$k^{(1)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{Y1} \\ u_{X2} \\ u_{Y2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\lambda q \end{bmatrix}. \tag{24.14}$$

For element (2):

$$k^{(2)} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{X2} \\ u_{Y2} \\ u_{X3} \\ u_{Y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{24.15}$$

Assembling and applying the boundary conditions $u_{X1} = u_{Y1} = u_{X3} = u_{Y3} = 0$ we get

$$\begin{bmatrix} k^{(2)} & 0 \\ 0 & k^{(1)} \end{bmatrix} \begin{bmatrix} u_{X2} \\ u_{Y2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda q \end{bmatrix}. \tag{24.16}$$

The linear solution is

$$u_{X2} = 0, \qquad u_{Y2} = -\frac{\lambda q}{k^{(1)}} = -\frac{\lambda q L_0^{(1)}}{E A_0^{(1)}}. \tag{24.17}$$

The axial linear strain and Cauchy (true) stress developed in element (1) are

$$\epsilon^{(1)} = \frac{u_{Y2}}{L_0^{(1)}} = \lambda \frac{q}{E A_0^{(1)}}\ , \quad \sigma^{(1)} = E\epsilon^{(1)} = -\lambda \frac{q}{A_0^{(1)}}. \tag{24.18}$$

According to the assumptions stated above the change in geometry prior to buckling is neglected. Consequently

$$e^{(1)} \approx \epsilon^{(1)} \qquad s^{(1)} \approx \sigma^{(1)}$$

The axial strain and stress of element (2) are zero.

The simplified nonlinear finite element equations are, for element (1)

$$\left\{ k^{(1)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + \frac{N^{(1)}}{L_0^{(1)}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} u_{X1} \\ u_{Y1} \\ u_{X2} \\ u_{Y2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\lambda q \end{bmatrix}. \tag{24.19}$$

where $N^{(1)} = A_0^{(1)} s^{(1)} = -p = -\lambda q$ denotes the axial force in bar element (1).

For element (2) we have the same linear matrix equations as before because its geometric stiffness vanishes. Assembling and applying displacement boundary conditions we get the equations

$$\begin{bmatrix} k^{(2)} - \dfrac{\lambda q}{L_0^{(2)}} & 0 \\ 0 & k^{(1)} - \dfrac{\lambda q}{L_0^{(1)}} \end{bmatrix} \begin{bmatrix} u_{X2} \\ u_{Y2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda q \end{bmatrix}. \tag{24.20}$$

One now regards **K** in (24.20) as unaffected by the displacements u_{X2} and u_{Y2}, which is consistent with the assumption that the change of geometry prior to buckling is neglected. This having being done, setting the determinant of **K** to zero yields the buckling eigenproblem:

$$\det \begin{bmatrix} k^{(2)} - \dfrac{\lambda q}{L_0^{(1)}} & 0 \\ 0 & k^{(1)} - \dfrac{\lambda q}{L_0^{(1)}} \end{bmatrix} = 0. \tag{24.21}$$

This matrix is singular if either diagonal element vanishes, which yields the two eigenvalues

$$\lambda_{cr1} = k^{(1)} L_0^{(1)}/q, \qquad \lambda_{cr2} = k^{(2)} L_0^{(1)}/q, \tag{24.22}$$

as critical values of the load parameter. Since $k^{(2)} << k^{(1)}$ the lowest critical load will be

$$p_{cr} = \lambda_{cr2} q = k^{(2)} L_0^{(1)}. \tag{24.23}$$

This is the *buckling load* obtained under the LPB assumptions.

§24.8. Summary of LPB Steps

The foregoing example illustrates the key steps of LPB analysis. These are summarized below for completeness.

1. Assemble the linear stiffness $\mathbf{K}_0$ and solve the *linear static problem*

$$\mathbf{K}_0\mathbf{u} = \mathbf{q}_0\lambda, \tag{24.24}$$

for $\lambda = 1$ and obtain the internal force (stress) distribution. Note: In statically determinate structures, such as Exercise 24.4, the internal forces and stresses may be obtained directly from equilibrium. However $\mathbf{K}_0$ is still necessary for step 3.

2. Form the reference geometric stiffness $\mathbf{K}_1$ for that internal force distribution. The geometric stiffness is $\mathbf{K}_G = \lambda\mathbf{K}_1$.

3. Solve the stability eigenproblem

$$(\mathbf{K}_0 + \lambda\mathbf{K}_1)\,\mathbf{z}_i = \mathbf{0}, \qquad \text{or} \quad \mathbf{K}_0\mathbf{z}_i = -\lambda_i\mathbf{K}_1\,\mathbf{z}_i, \tag{24.25}$$

The eigenvalue λ_i closest to zero is the critical load multiplier, and the associated eigenvector $\mathbf{z}_i$ gives the corresponding buckling mode.

Homework Exercises for Chapter 24

Bifurcation: Linearized Prebuckling I

EXERCISE 24.1 [A:15] Find the buckling mode (null eigenvector) $\mathbf{z}$ (normalized to unit length) for the example problem of §24.6, and verify the orthogonality condition $\mathbf{z}^T\mathbf{q} = 0$.

EXERCISE 24.2 [A:15] Find the buckling load for the two-bar problem if bar (2) forms an angle $0 \leq \varphi < 90^\circ$ with the x axis. Assume still that the system displaces only on the $x - y$ plane and that $k^{(2)} << k^{(1)}$ so that the stress in bar (2) can be neglected in forming the geometric stiffness. Determine $\mathbf{z}$ and verify orthogonality.

EXERCISE 24.3 [A:20] Suppose the load of the two-bar example problem of §24.6 depends on the vertical displacement of point 2 as $p = -\lambda c u_{Y2}^2$, where c is a constant with dimensions of stress. Show that even if prebuckling deformations are neglected, the singular stiffness test leads to a nonlinear eigenvalue problem.

EXERCISE 24.4 [A:25] The "Euler column" shown in Figure E24.1 is modelled by *one* 2-node Euler-Bernoulli beam column element along its length:

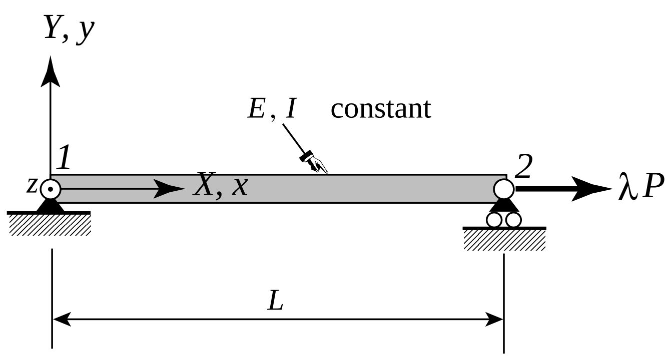

Figure E24.1. One-element model of Euler column

The state parameters are the nodal displacements degrees of freedom arranged as

$$\mathbf{u} = \begin{bmatrix} u_{X1} \\ u_{Y1} \\ \theta_{z1} \\ u_{X2} \\ u_{Y2} \\ \theta_{z2} \end{bmatrix} \tag{E24.1}$$

where θ_{z1} and θ_{z2} are (to first order) the end rotations, positive counterclockwise about z.

The linear material matrix in the reference (undeformed) configuration is

$$\mathbf{K}_0 = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ & & & \frac{EA}{L} & 0 & 0 \\ & & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ symm & & & & & \frac{4EI}{L} \end{bmatrix} \tag{E24.2}$$

in which E is the elastic modulus, L the element length, A the cross section area, and I the moment of inertia of the cross section about the z neutral axis.

The exactly-integrated geometric stiffness at the reference configuration is[1]

$$\mathbf{K}_G = \lambda\mathbf{K}_1, \quad \mathbf{K}_1 = \frac{P}{30L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 36 & 3L & 0 & -36 & 3L \\ & & 4L^2 & 0 & -3L & -L^2 \\ & & & 0 & 0 & 0 \\ & & & & 36 & -3L \\ symm & & & & & 4L^2 \end{bmatrix} \tag{E24.3}$$

where N is the axial force in the element (here obviously equal to the applied force λP because the structure is statically determinate.)

For this problem:

(a) Check that $\mathbf{K}_0$ and $\mathbf{K}_1$ satisfy translational infinitesimal rigid body motion conditions $u_X \equiv 1$ and $u_y \equiv 1$ if the six degrees of freedom are left unconstrained. (Convert those modes to node displacements, then premultiply by the stiffness matrices.)

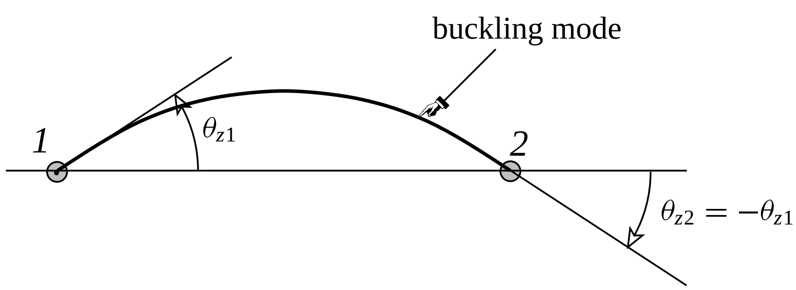

Figure E24.2. Symmetric buckling of one-FE model of Euler column.

(b) Set up the linearized prebuckling eigenproblem

$$(\mathbf{K}_0 + \lambda\mathbf{K}_1)\mathbf{z} = \mathbf{0} \tag{E24.4}$$

Apply the support end conditions to remove u_{X1}, u_{Y1} and u_{Y2} as degrees of freedom.

[1] See e.g. Przemieniecki's *Theory of Matrix Structural Analysis*, *loc. cit.*

(c) Justify that freedom u_{X2} can be isolated from the eigenproblem, and proceed to drop it to reduce the eigenproblem to 2×2.

(d) Reduce the 2×2 eigenproblem to a scalar one for the symmetric buckling mode sketched in Figure E24.2. by setting $\theta_{z1} = -\theta_{z2}$ (see Figure), and get the first critical load parameter λ_1.

(e) Repeat (e) for the antisymmetric buckling mode sketched in Figure E24.3 by setting $\theta_{z1} = \theta_{z2}$ (see Figure) and obtain the second critical load parameter λ_2.

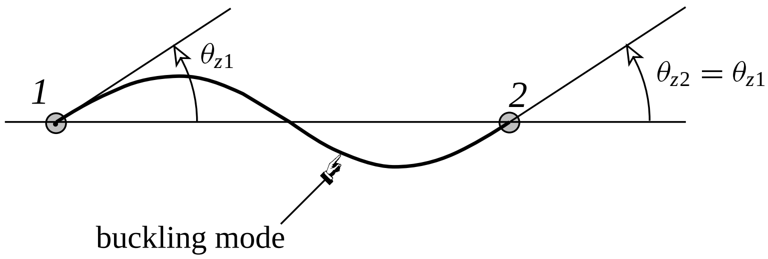

Figure E24.3. Antisymmetric buckling of one-FE model of Euler column.

(f) Compare the results of (d)–(e) to the exact critical load values

$$P_1^E = -\pi^2 \frac{EI}{L^2}, \qquad P_2^E = -4\pi^2 \frac{EI}{L^2}, \tag{E24.5}$$

The first one was determined by Euler in 1744 and therefore is called the *Euler critical load*. For P_1^E the FEM result should be within 25%, which is good for one element.

(g) Repeat the calculations of λ_1 and λ_2 with the following reduced-integration geometric stiffness matrices[6]

$$\mathbf{K}_1^* = \frac{P}{8L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 9 & 3L & 0 & -9 & 3L \\ & & L^2 & 0 & -3L & -L^2 \\ & & & 0 & 0 & 0 \\ & & & & 9 & -3L \\ symm & & & & & L^2 \end{bmatrix} \tag{E24.6}$$

$$\mathbf{K}_1^{**} = \frac{P}{24L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 24 & 0 & 0 & -24 & 0 \\ & & 2L^2 & 0 & 0 & -2L^2 \\ & & & 0 & 0 & 0 \\ & & & & 24 & 0 \\ symm & & & & & 2L^2 \end{bmatrix} \tag{E24.7}$$

and comment on the relative accuracy obtained against the exact values.

(h) Repeat the calculations of steps (b) through (e) for a *two* equal-element discretization. Verify that the symmetric-mode buckling load is now $-10EI/L^2$, which is (surprisingly) close to Euler's value $P_1^E = -\pi^2 EI/L^2$.

[6] These matrices are obtained by one-point and two-point Gauss integration, respectively, whereas the $\mathbf{K}_1$ of (E24.3) is obtained by either 3-point Gauss or analytical integration.

EXERCISE 24.5 [A:25] This is identical in all respects to Exercise 24.4, except that the 2-node Timoshenko beam model is used, with Mac Neals's RBF correction for the material stiffness matrix. The net result is that $\mathbf{K}_0$ is *identical* to (E24.2) but $\mathbf{K}_1$ is different. In fact $\lambda\mathbf{K}_1$ is given by Equation (9.45) where $V = 0$ and $N = \lambda P$. Note: dont be surprised if the one-element results are poor when compared to the analytical buckling load.

EXERCISE 24.6 [A/C: 20] The column shown in Figure E24.4 consists of 3 rigid bars of equal length L connected by hinges and stabilized by two lateral springs of linear stiffness k. The applied axial load is λP, where $\lambda > 0$ means compression. Compute the two buckling loads $\lambda_1 P$ and $\lambda_2 P$ in terms of k and L, assuming the LPB model of infinitesimal displacements from the initial state. Show that one load corresponds to a symmetric buckling mode and the other to an antisymmetric buckling mode, and find which one is critical.

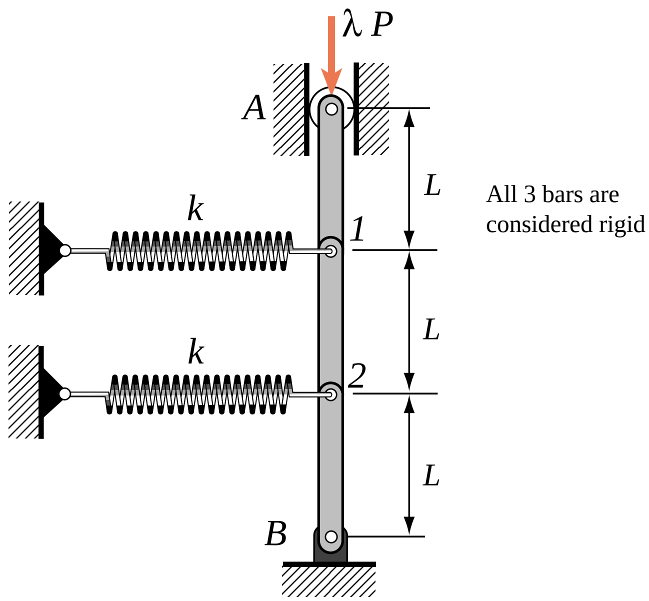

Figure E24.4. Buckling of a segmented-hinged column propped by two springs.

Hint. The two degrees of freedom are the small lateral displacements u_1 and u_2 of hinges 1 and 2, where $u_1 << L, u_2 << L$. Write the total potential energy as

$$\Pi = U - W, \qquad U = \tfrac{1}{2}ku_1^2 + \tfrac{1}{2}ku_2^2, \qquad W = \frac{\lambda P}{2L}\left(u_1^2 + (u_1 - u_2)^2 + u_2^2\right). \tag{E24.8}$$

Explain where the expression of W comes from. Once Π is in hand, it is smooth sailing.