

# 17

## Purely Incremental Methods: Load Control

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Incremental methods calculate the nonlinear response through the numerical integration of a *rate* form of the equilibrium equations as the stage control parameter  $\lambda$  is varied. In the nomenclature introduced in Chapter 16, we can characterize these as *predictor-only* methods: no corrective iterations to recover equilibrium are performed. They are also known as step-by-step, initial-value or marching methods in the engineering literature. The qualifier “purely” distinguishes these incremental methods from those that make use of the pseudo-force concept, and which are covered in Chapter 19.

The present Chapter emphasizes purely incremental methods in which the first-order rate equations are integrated by a forward Euler scheme. Furthermore, for simplicity we focus on the simplest increment control strategy: load control, in which  $\lambda$  is treated as an independent variable. This restriction allows subjects such as stability and accuracy to be discussed in a straightforward manner. An arclength-parametrized version, which allows the introduction of more robust increment control techniques and the automatic traversal of limit points, is presented in the following Chapter.

### §17.1. Governing Differential Equation

Recall the first-order rate equation  $\dot{\mathbf{r}} = \mathbf{K}\dot{\mathbf{u}} - \mathbf{q}\dot{\lambda} = \mathbf{0}$  specialized to  $t \equiv \lambda$ :

$$\mathbf{r}' = \mathbf{K}\mathbf{u}' - \mathbf{q} = \mathbf{0}, \quad (17.1)$$

where primes denote differentiation with respect to  $\lambda$ . If the stiffness matrix is nonsingular, this equation uniquely relates the differential of  $\mathbf{u}$  to that of  $\lambda$ :

$$\mathbf{u}' = \frac{d\mathbf{u}}{d\lambda} = \mathbf{K}^{-1}\mathbf{q} = \mathbf{v}, \quad (17.2)$$

where as usual  $\mathbf{v}$  denotes the incremental velocity vector. Purely incremental methods with  $\lambda$  as independent variable are based on the numerical integration of (17.2) to generate an approximate response  $\mathbf{u} = \mathbf{u}(\lambda)$  given the initial condition

$$\mathbf{u} = \mathbf{u}_0 \quad \text{at} \quad \lambda = 0. \quad (17.3)$$

**Remark 17.1.** The exact integral of (17.2) with the initial conditions (17.3) is

$$\mathbf{r}(\mathbf{u}, \lambda) = \mathbf{r}_0 \quad (17.4)$$

where  $\mathbf{r}_0 = \mathbf{r}(\mathbf{u}_0, 0)$ . Thus an initial equilibrium error does not decay even if the integration were carried out exactly. This is the source of the *drifting error* that afflicts purely incremental methods. The error committed at each step moves the equilibrium point to a neighboring curve in the incremental flow (see Figure 16.2). Consequently the solution may “drift away” quickly when the incremental flow paths “flare out” from the equilibrium path.

### §17.2. Forward Euler Integration

In the remaining subsections of this Chapter we consider that the incrementation process is controlled directly by varying the stage parameter  $\lambda$ , which thus assumes the role of independent variable. This is tantamount to using the  $\lambda$ -control increment discussed in Chapter 16. This restriction is removed in the next Chapter.

The simplest incrementation scheme is obtained by using the forward Euler integrator

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta\lambda \mathbf{u}'_n, \quad (17.5)$$

where  $n$  is the incremental step index,  $\mathbf{u}_n \stackrel{\text{def}}{=} \mathbf{u}(\lambda_n)$  and

$$\Delta\lambda_n = \lambda_{n+1} - \lambda_n, \quad (17.6)$$

is the stage parameter stepsize. Treating (17.2) with this integrator yields the scheme

$$\begin{array}{l} \Delta\mathbf{u}_n = \mathbf{K}_n^{-1} \mathbf{q}_n \Delta\lambda_n = \mathbf{v}_n \Delta\lambda_n, \\ \mathbf{u}_{n+1} = \mathbf{u}_n + \Delta\mathbf{u}_n. \end{array} \quad (17.7)$$

In the actual computer implementation of (17.7) the linear system  $\mathbf{K}_n \mathbf{v}_n = \mathbf{q}_n$  is preprocessed by assembling and factoring  $\mathbf{K}_n$ . The right hand side  $\mathbf{q}_n$  is solved for to get  $\mathbf{v}_n$ . This is multiplied by  $\Delta\lambda_n$ , which is either prescribed or (better) adjusted by the stepsize-control techniques discussed below.

**Remark 17.2.** As discussed in Chapter 16, (17.7) is also the usual predictor for incremental-corrective methods.

### §17.3. More Accurate Integration

To increase accuracy, more refined integration formulas have been proposed. An attractive second-order choice is the explicit midpoint rule (also called Heun's rule by some authors):

$$\begin{array}{l} \mathbf{u}_{n+1/2} = \mathbf{u}_n + \frac{1}{2} \mathbf{K}_n^{-1} \mathbf{q}_n \Delta\lambda_n, \\ \mathbf{K}_{n+1/2} \stackrel{\text{def}}{=} \mathbf{K}(\mathbf{u}_{n+1/2}), \quad \mathbf{q}_{n+1/2} \stackrel{\text{def}}{=} \mathbf{q}(\mathbf{u}_{n+1/2}), \\ \mathbf{u}_{n+1} = \mathbf{u}_n + \mathbf{K}_{n+1/2}^{-1} \mathbf{q}_{n+1/2} \Delta\lambda_n. \end{array} \quad (17.8)$$

This scheme was used in the author's thesis<sup>1</sup> to treat problems with combined geometric and material nonlinearities. The midpoint rule has attractive features for flow-plasticity studies, since local elastic unloading can be detected during the first "trial" step and  $\mathbf{K}_{n+1/2}$  adjusted accordingly. The same feature can be used to advantage in bifurcation analysis if a stiffness-determinant change is detected between  $\mathbf{K}_n$  and  $\mathbf{K}_{n+1/2}$ . But note that the stiffness matrix has to be formed and factored *twice* per incremental step.

<sup>1</sup> C. A. Felippa, Refined Finite Element Analysis of Linear and Nonlinear Two-dimensional Structures, Ph.D. thesis, Dept. of Civil Engineering, University of California, Berkeley (1966)

Natural extensions of (17.8) are third and fourth-order Runge-Kutta (RK) formulas, which require three and four stiffness evaluations and factorizations per step, respectively. These more refined methods, however, are rarely used in structural mechanics for the amount of work per step is considerable. Remark 17.8, however, indicates a possible niche for the classical fourth-order RK in nonconservative problems.

#### §17.4. Numerical Stability of Forward Euler

Can the integration process (17.7) become numerically unstable? The subject is rarely mentioned in the finite element literature. For simplicity we begin with the one-degree-of-freedom counterpart  $u' = v$  of  $\mathbf{u}' = \mathbf{v}$ . The right-hand side  $v = \mathbf{K}^{-1}q$  is Taylor-series expanded in  $\Delta u = u - u_n$  about  $u_n$  as

$$u' = u_n + \mu \Delta u + O(\Delta u^2) \quad \text{with} \quad \mu = \frac{\partial v}{\partial u}. \quad (17.9)$$

For the linearized stability analysis only the homogeneous part of (17.9) is retained, which yields the *model equation*

$$u' = \mu u. \quad (17.10)$$

Consider the case in which  $\mu$  is *negative real* and  $h = \Delta\lambda > 0$ . Then the solution  $u = u(\lambda)$  of the model equation is exponentially decreasing as  $\lambda$  increases. The forward Euler integration is absolutely stable<sup>2</sup> if

$$|1 + h\mu| \leq 1, \quad \text{or} \quad h \leq -2/\mu. \quad (17.11)$$

If  $h$  exceeds this value, the computed solution exhibits oscillatory instability. If  $\mu$  is *positive real* the solution of the model equation grows exponentially as  $\lambda$  increases and the forward Euler integration is “relatively stable” for all  $h > 0$ .

Now if  $\lambda$  *decreases* so that  $h = \Delta\lambda < 0$  the roles are reversed (cf. Remark 17.3). The stability condition is  $h \geq -2/\mu$  if  $\mu > 0$ . If  $\lambda$  is a load parameter, loading and unloading sequences may be viewed as equally likely; consequently a safe stability constraint is

$$\boxed{|\Delta\lambda| \leq \frac{2}{|\mu|}}. \quad (17.12)$$

For the general system (17.2), let  $\mu_i$  ( $i = 1, 2, \dots, N$ ,  $N$  being the number of degrees of freedom) be the eigenvalues of the so-called *amplification matrix*

$$\mathbf{A} = \frac{\partial \mathbf{v}}{\partial \mathbf{u}} = \frac{\partial(\mathbf{K}^{-1}\mathbf{q})}{\partial \mathbf{u}}. \quad (17.13)$$

It is shown in Remark 5.8 that this matrix, although generally unsymmetric, has *real* eigenvalues if the problem is conservative, *i.e.*  $\mathbf{K}$  is the Hessian of a potential  $\Pi(\mathbf{u}, \lambda)$  for fixed  $\lambda$ , and  $\mathbf{K}$  is positive

<sup>2</sup> C. W. Gear, *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice-Hall, Englewood Cliffs, N.J. (1971)

L. Lapidus and J. H. Seinfeld, *Numerical Solution of Ordinary Differential Equations*, Academic Press, New York (1971).

definite. The eigenvalues are given by the eigensystems (5.22) or (5.23). Under such conditions a safe increment is given by (17.12), where now

$$\mu = \mu_{max} = \max_i |\mu_i|, \quad i = 1, \dots, N. \quad (17.14)$$

If  $\mathbf{K}$  does not depend on  $\lambda$  (as in the linear case) all  $\mu_i$  vanish and the increment is unrestricted. If  $\mathbf{K}$  depends mildly upon  $\lambda$ , eigensystems (5.21)–(5.22) show that the largest  $|\mu_i|$  are associated with the smallest eigenvalues of  $\mathbf{K}$ , *i.e.* the fundamental stiffness modes (see also Remark 17.4).

Of course the actual calculation of all  $\mu_i$  at each step would be a formidable computational task. But the following finite-difference “path” estimate is easily obtained:

$$\mu \approx \frac{\|\mathbf{v}_{n+1} - \mathbf{v}_n\|}{\|\mathbf{u}_{n+1} - \mathbf{u}_n\|} = \frac{\|\mathbf{v}_{n+1} - \mathbf{v}_n\|}{\|\mathbf{v}_n\|} \frac{1}{\Delta\lambda_n} = \frac{a_n}{\Delta\lambda_n}, \quad (17.15)$$

where  $a_n = \|\mathbf{v}_{n+1} - \mathbf{v}_n\|/\|\mathbf{v}_n\|$  and  $\|\cdot\|$  denotes the 2-norm or Euclidean norm of a vector:  $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}}$ . ( $a_n$  may be viewed as a kind of “incremental acceleration”.) Unfortunately this quantity is not available until the  $n^{\text{th}}$  step is completed, and to get a practical estimate we replace it by the previous step estimate:

$$\mu \approx \frac{a_{n-1}}{\Delta\lambda_{n-1}}, \quad a_{n-1} = \frac{\|\mathbf{v}_n - \mathbf{v}_{n-1}\|}{\|\mathbf{v}_{n-1}\|}. \quad (17.16)$$

Insertion into (17.12) yields the stability condition

$$\boxed{|\Delta\lambda_n| \leq \frac{2|\Delta\lambda_{n-1}|}{a_{n-1}}}. \quad (17.17)$$

If  $\mathbf{A}$  can have complex eigenvalues, however, this simple rule does not apply (see Remark 17.8).

**Remark 17.3.** The fact that  $h = \Delta\lambda$  can be either positive or negative is a distinguishing feature of incremental static analysis. In the numerical integration of actual dynamical systems, the time increment  $h = \Delta\tau$  is never negative; therefore stability results from conventional ODE theory should be used with caution.

**Remark 17.4.** Having the largest  $\mu_i$  associated with the smallest eigenvalues of  $\mathbf{K}$  represents another noteworthy difference with dynamic problems. In the latter, the stability limit of explicit integrators such as (17.7) is determined by the *highest frequencies* of the system. As discussed in Chapters dealing with dynamic relaxation methods, the difference is due to the fact that the matrix multiplying the highest derivative is the mass (stiffness) in a dynamic (static) problem.

**Remark 17.5.** If the midpoint rule (17.8) is used, the stability limit remains the same for real  $\mu_i$ .

**Remark 17.6.** Replacing  $\|\mathbf{v}_{n-1}\|$  by  $\|\mathbf{v}_n\|$  in (17.16) is inconsequential, as it is only a gross estimate. In fact, choosing the smallest of the two norms would be the more conservative policy. A more serious objection is the choice of the 2-norm unless the problem is well scaled and all degrees of freedom have common physical dimension. Otherwise a diagonal scaling matrix may have to be introduced as discussed in Chapter 4; for example

$$a_{n-1}^2 = \frac{|(\mathbf{v}_n - \mathbf{v}_{n+1})^T \mathbf{S}^2 (\mathbf{v}_n - \mathbf{v}_{n+1})|}{\min(|\mathbf{v}_n^T \mathbf{S}^2 \mathbf{v}_n|, |\mathbf{v}_{n-1}^T \mathbf{S}^2 \mathbf{v}_{n-1}|)}. \quad (17.18)$$

**Remark 17.7.** As  $\mathbf{K}$  approaches singularity,  $\|\mathbf{v}\| \rightarrow \infty$  and the stable  $\Delta\lambda$  approaches zero. This is an indication of the problems encountered by this type of incremental method at critical points. “Flat” limit points can be traversed using the step-controlled parametric formulation discussed in the next Chapter. At those points  $\Delta\lambda$  changes sign. But as the limit point becomes progressively sharper, numerical difficulties increase. Bifurcation points, which in some sense may be viewed as infinitely sharp limit points, are difficult to traverse without resort to either perturbation or special techniques that necessarily involve buckling mode estimation, as discussed later.

**Remark 17.8.** If the problem is not derivable from a potential or  $\mathbf{K}$  is not positive definite,  $\mathbf{A}$  may have complex eigenvalues such as  $\mu_i = \alpha_i + j\beta_i$ ,  $j = \sqrt{-1}$ . Let  $h = \Delta\lambda$ . If  $h\alpha_i < 0$ , the appropriate stability condition for forward Euler reads

$$(1 + h\alpha_i)^2 + h^2\beta_i^2 \leq 1. \quad (17.19)$$

If the imaginary component  $\beta_i$  dominates, the stable stepsize may be sharply reduced over that of the potential case, and if  $\alpha_i = 0$  there is no stable  $h$ . It is not difficult to construct “load follower” problems that yield almost imaginary  $\mu_i$ . The morale is that *purely incremental methods should be used with caution in nonconservative problems*. For this class of problems, third- and fourth-order Runge-Kutta methods do enjoy a substantial stability edge; see, for example, the stability charts on page 120 of Lapidus and Seinfeld (referenced cited in footnote 2).

### §17.5. Accuracy Monitoring

If the response is twice differentiable, the local truncation error of the Euler integrator (17.5) at a regular point is easily obtained from the truncated Taylor expansion about  $(\mathbf{u}_n, \lambda_n)$  as<sup>3</sup>

$$\mathbf{e} = \frac{1}{2}(\Delta\lambda)^2 \mathbf{u}''_{\xi} = \frac{1}{2}(\Delta\lambda)^2 \mathbf{v}'_{\xi}, \quad (17.20)$$

where the symbol  $\mathbf{v}'_{\xi} = \mathbf{u}''_{\xi}$  denotes the second derivative  $\partial^2 \mathbf{u} / \partial \lambda^2 = \partial \mathbf{v} / \partial \lambda$  evaluated at points  $\xi_i \in [\lambda_n, \lambda_{n+1}]$ , which generally differ from component to component. To assess the magnitude  $e = \|\mathbf{e}\|$  of this error we need an estimate of the norm of  $\mathbf{v}'$ . An obvious finite-difference estimate for this quantity is  $(\mathbf{v}_{n+1} - \mathbf{v}_n) / \Delta\lambda_n$  but — as in the stability analysis —  $\Delta\lambda_n$  and  $\mathbf{v}_{n+1}$  are not available until the step is completed. For a practical estimation of  $e$  we are forced to use previous step values:

$$e = \|\mathbf{e}\| \approx \frac{1}{2}(\Delta\lambda_n)^2 \frac{\|\mathbf{v}_n - \mathbf{v}_{n-1}\|}{\Delta\lambda_{n-1}}. \quad (17.21)$$

For accuracy monitoring a convenient dimensionless measure is the ratio  $\epsilon$  of  $e$  to the increment length  $\|\Delta\mathbf{u}_n\| = \|\Delta\lambda_n \mathbf{v}_n\|$ :

$$\epsilon = \frac{1}{2} \left| \frac{\Delta\lambda_n}{\Delta\lambda_{n-1}} \right| a_{n-1} \quad \text{with} \quad a_{n-1} = \frac{\|\mathbf{v}_n - \mathbf{v}_{n-1}\|}{\|\mathbf{v}_n\|}. \quad (17.22)$$

Here we again denote by  $a$  a ratio similar to the one in (17.16) but with  $\|\mathbf{v}_n\|$  in the denominator, which is an inconsequential change. To strive for uniform local accuracy the basic idea is to specify  $\epsilon$  (say  $\epsilon \approx 0.1$ ) and adjust the stepsize so that

$$\boxed{|\Delta\lambda_n| \leq \frac{2\epsilon |\Delta\lambda_{n-1}|}{a_{n-1}}}. \quad (17.23)$$

<sup>3</sup> P. Henrici, *Error Propagation for Difference Methods*, Wiley, New York (1963).

This increment size must also be subjected to other bounds provided by rules discussed later. Similar accuracy monitoring techniques can be devised for more complicated integration schemes such as the midpoint rule.

**Remark 17.9.** Comparing (17.23) with (17.17) furnishes a simple rule: *choosing  $\epsilon < 1$  takes care of stability* if the roots of  $\mathbf{A}$  are real. In any case the similarity between the stability and accuracy control rules is striking.

**Remark 17.10.** For poorly scaled problems the use of a weighted norm, as in (17.18), is advisable.

**Remark 17.11.** Given bounds on  $e$  at each step, bounds on the accumulated drift error can be obtained but they are usually so pessimistic as to be useless, unless some detailed problem information is available; see *e.g.* Gear (*loc.cit.* in footnote 2). The only reliable way to assess global accuracy is to rerun the problem with several values of  $\epsilon$ , for example  $\epsilon = 0.2, 0.1$  and  $0.05$ .

**Remark 17.12.** This truncation error monitoring technique works in static nonlinear analysis because  $e$  is controlled by physically relevant low-frequency modes. It fails on direct time integration of dynamical problems — see *e.g.* Park<sup>4</sup> — because  $e$  is then controlled by physically irrelevant high-frequency modes. In numerical analysis parlance, problems in structural dynamics are said to be *stiff*.

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<sup>4</sup> K. C. Park and C. A. Felippa, Direct Time Integration Methods in Nonlinear Structural Dynamics, *Comp. Meth. Appl. Mech. Engrg.*, **17/18**, pp. 277–313 (1979)

K. C. Park, Time Integration of Structural Dynamics: A Survey, Ch. 4.2 in *Pressure Vessels and Piping Design Technology — A Decade of Progress* ASME, New York (1982).