15

The CR Description: C1 Plane Beam

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§15.1. Introduction

In this Chapter we use the CR description to construct a geometrically nonlinear, 2-node Bernoulli-Euler plane beam. Unlike Chapter 9 we will do a C^1 (Hermitian) beam from the start, since with CR it is as easy to do C^1 or C^0, and the former has a much better geometric stiffness matrix.

§15.2. CR Beam Kinematics

The CR formulation of the beam motion is quite similar to that of the bar element in many respects, and much of the development can be reused. Only the major differences will be noted here.

§15.2.1. Coordinate Systems

As in Chapter 9, we consider a plane, straight, prismatic beam element with two nodes. The element is initially aligned with the global X axis in the initial configuration $\mathcal{C}_0$, with the origin O_0 located at the element midpoint. This configuration is assumed to be straight and undeformed although it may be under initial uniform axial stress with resultant N^0. The bar properties include the elastic modules, E, the cross section area A_0 and the moment of inertia I_0 about the neutral axis. The length in C_0 is L_0.

The motion on the $\{X, Y\}$ plane carries it to the current configuration $\mathcal{C}$. The corotated configuration $\mathcal{C}_R$ is selected as depicted in Figure 15.1:

1. The longitudinal axis passes through the current position of the end nodes. This defines the local axis x^e. The origin of $\{x^e, y^e\}$ is placed halfway between the nodes. This forms an angle ψ with X.
2. The $\mathcal{C}_R$ nodes are placed at an equal distance from the $\mathcal{C}$ nodes. Hence the corotated axes $\{x^e_R, y^e_R\}$, including origin, coincide with $\{x^e, y^e\}$.

The new ingredient is the rotation angle θ about Z or z. With $\mathcal{C}_R$ chosen as indicated, the deformation part of these rotations is easily extracted: $\bar{\theta} = \theta - \psi$.

Other possibilities for selecting $\mathcal{C}_R$ are possible. The foregoing choice has the advantage of being compatible with that of the bar element discussed in the previous Chapter.

§15.2.2. Degrees of Freedom

The beam element has six degrees of freedom, which are placed in the vectors

$$\mathbf{u} = \begin{bmatrix} u_{X1} \\ u_{Y1} \\ \theta_1 \\ u_{X2} \\ u_{Y2} \\ \theta_2 \end{bmatrix}, \qquad \bar{\mathbf{u}}^e = \begin{bmatrix} \bar{u}^e_{X1} \\ u^e_{Y1} \\ \bar{\theta}^e_1 \\ \bar{u}^e_{X2} \\ \bar{u}^e_{Y2} \\ \bar{\theta}^e_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}d \\ 0 \\ \bar{\theta}_1 \\ \frac{1}{2}d \\ 0 \\ \bar{\theta}_2 \end{bmatrix}. \tag{15.1}$$

See Figure 15.2 for a picture of the global displacements and Figure 15.3 for the deformation displacements.

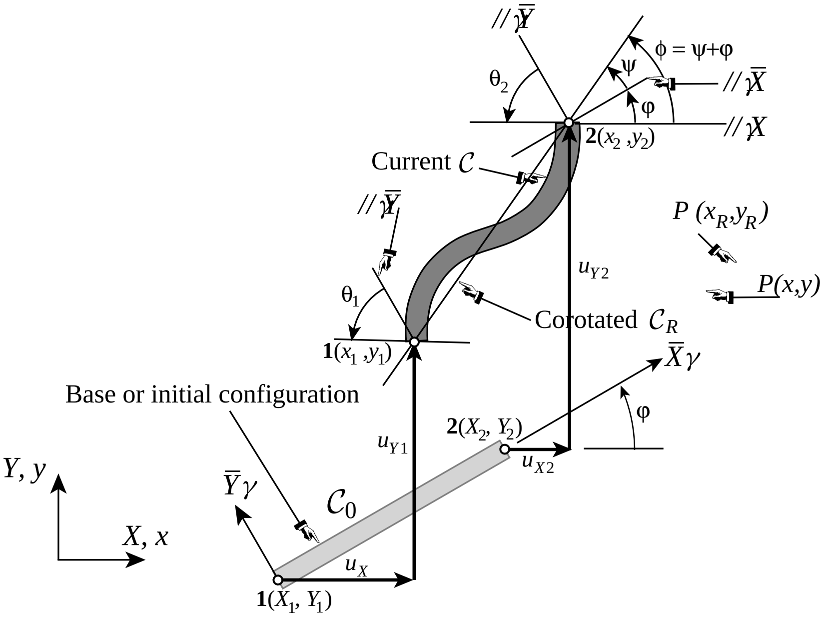

Figure 15.1. Kinematics of corotational C^1 plane beam element.

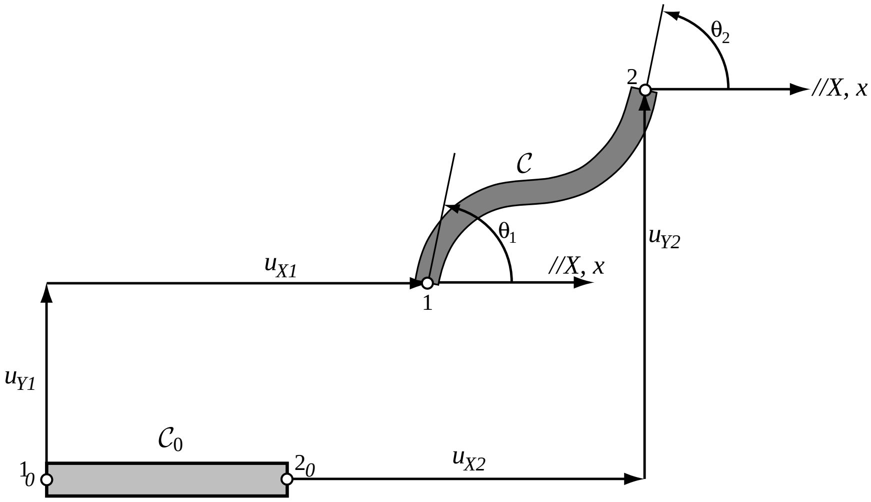

Figure 15.2. Global element displacements.

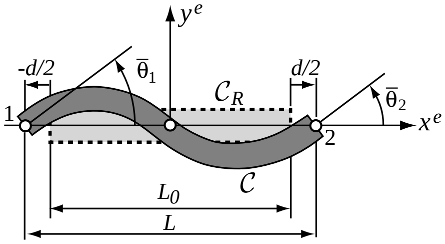

Figure 15.3. Deformational displacements in element system.

Proceeding as in the general formulation specialized to the 2D case, we can obtain the following relation:

$$\bar{\mathbf{u}}^e = \begin{bmatrix} \bar{u}^e_{x1} \\ \bar{u}^e_{y1} \\ \bar{\theta}^e_1 \\ \bar{u}^e_{x2} \\ \bar{u}^e_{y2} \\ \bar{\theta}^e_2 \end{bmatrix} = \begin{bmatrix} c_\psi & s_\psi & 0 & 0 & 0 & 0 \\ -s_\psi & c_\psi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_\psi & s_\psi & 0 \\ 0 & 0 & 0 & -s_\psi & c_\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{X1} - u_{X0} \\ u_{Y1} - u_{Y0} \\ \theta_1 \\ u_{X2} - u_{X0} \\ u_{Y2} - u_{Y0} \\ \theta_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} L_0 (1 - c_\psi) \\ \frac{1}{2} L_0 s_\psi \\ -\psi \\ \frac{1}{2} L_0 (c_\psi - 1) \\ -\frac{1}{2} L_0 s_\psi \\ -\psi \end{bmatrix} \tag{15.2}$$

Here c_ψ and s_ψ and the angle ψ are implicitly defined by the displacements through the trigonometric relations

$$s_\psi = \sin\psi = \frac{L_y}{L}, \quad c_\psi = \cos\psi = \frac{L_x}{L}, \quad \psi = \arctan\frac{L_y}{L_x} \tag{15.3}$$

where $L_x = L_0 + u_{X2} - u_{X1}$, $L_y = u_{Y2} - u_{Y1}$, and

$$L = \sqrt{L_x^2 + L_y^2} \tag{15.4}$$

is the bar length in the current configuration, ignoring the bending deformation.

We note the following relations

$$\begin{aligned} &\frac{\partial L}{\partial u_{X2}} = -\frac{\partial L}{\partial u_{X1}} = c_\psi, \quad \frac{\partial L}{\partial u_{Y2}} = -\frac{\partial L}{\partial u_{Y1}} = s_\psi, \quad \frac{\partial L}{\partial \theta_1} = \frac{\partial L}{\partial \theta_2} = 0, \\ &\frac{\partial c_\psi}{\partial u_{X2}} = -\frac{\partial c_\psi}{\partial u_{X1}} = \frac{s_\psi^2}{L}, \quad \frac{\partial c_\psi}{\partial u_{Y2}} = -\frac{\partial c_\psi}{\partial u_{Y1}} = -\frac{s_\psi c_\psi}{L}, \quad \frac{\partial c_\psi}{\partial \theta_1} = \frac{\partial c_\psi}{\partial \theta_2} = 0, \\ &\frac{\partial s_\psi}{\partial u_{X2}} = -\frac{\partial s_\psi}{\partial u_{X1}} = -\frac{s_\psi c_\psi}{L}, \quad \frac{\partial s_\psi}{\partial u_{Y2}} = -\frac{\partial s_\psi}{\partial u_{Y1}} = \frac{c_\psi^2}{L}, \quad \frac{\partial s_\psi}{\partial \theta_1} = \frac{\partial s_\psi}{\partial \theta_2} = 0, \\ &\frac{\partial \psi}{\partial u_{X2}} = -\frac{\partial \psi}{\partial u_{X1}} = \frac{s_\psi}{L}, \quad \frac{\partial \psi}{\partial u_{Y2}} = -\frac{\partial \psi}{\partial u_{Y1}} = -\frac{c_\psi}{L}, \quad \frac{\partial \psi}{\partial \theta_1} = \frac{\partial \psi}{\partial \theta_2} = 0. \end{aligned} \tag{15.5}$$

which are useful in the calculations that follow.

§15.2.3. Partial Derivatives

The first and second partial derivatives of the deformations d, $\bar{\theta}_1$ and $\bar{\theta}_2$ with respect to the node displacements are necessary for the computations of internal forces and stiffness matrices.

Using (15.5) and *Mathematica*, one obtains for the first derivatives:

$$\begin{bmatrix} \delta\bar{u}^e_{x1} \\ \delta\bar{u}^e_{y1} \\ \delta\bar{\theta}^e_1 \\ \delta\bar{u}^e_{x2} \\ \delta\bar{u}^e_{y2} \\ \delta\bar{\theta}^e_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}c_\psi & \frac{1}{2}s_\psi & 0 & -\frac{1}{2}c_\psi & -\frac{1}{2}s_\psi & 0 \\ -s_\psi c_\psi L_0/L & c_\psi^2 L_0/L & 0 & s_\psi c_\psi L_0/L & -c_\psi^2 L_0/L & 0 \\ -s_\psi/L & c_\psi/L & 1 & s_\psi/L & -c_\psi/L & 0 \\ -\frac{1}{2}c_\psi & -\frac{1}{2}s_\psi & 0 & \frac{1}{2}c_\psi & \frac{1}{2}s_\psi & 0 \\ s_\psi c_\psi L_0/L & -c_\psi^2 L_0/L & 0 & -s_\psi c_\psi L_0/L & c_\psi^2 L_0/L & 0 \\ -s_\psi/L & c_\psi/L & 0 & s_\psi/L & -c_\psi/L & 1 \end{bmatrix} \begin{bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta\theta_1 \\ \delta u_{X2} \\ \delta u_{Y2} \\ \delta\theta_2 \end{bmatrix} \tag{15.6}$$

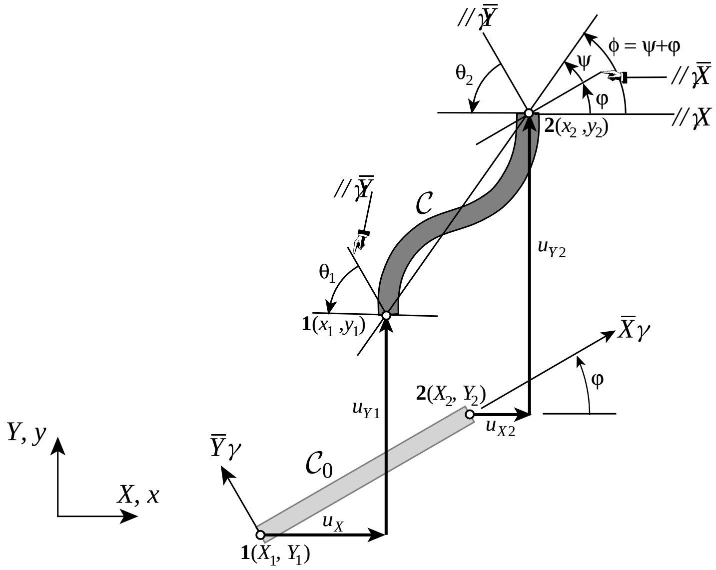

Figure 15.4. Beam element with arbitrarily oriented initial configuration $\mathcal{C}_0$, forming an angle φ with X. Corotated configuration not shown to reduce clutter.

Since $-u^e_{x1} = u^e_{x2} = \frac{1}{2}d$, $\bar{\theta}^e_1 = \bar{\theta}_1$, and $\bar{\theta}^e_2 = \bar{\theta}_2$ we get

$$\begin{bmatrix} \delta d \\ \delta\bar{\theta}_1 \\ \delta\bar{\theta}_2 \end{bmatrix} = \begin{bmatrix} -c_\psi & -s_\psi & 0 & c_\psi & s_\psi & 0 \\ -s_\psi/L & c_\psi/L & 1 & s_\psi/L & -c_\psi/L & 0 \\ -s_\psi/L & c_\psi/L & 0 & s_\psi/L & -c_\psi/L & 1 \end{bmatrix} \begin{bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta\theta_1 \\ \delta u_{X2} \\ \delta u_{Y2} \\ \delta\theta_2 \end{bmatrix} \tag{15.7}$$

The second derivatives of deformation variables are

$$\frac{\partial^2 d}{\partial \mathbf{u}\,\partial \mathbf{u}} = \frac{1}{L} \begin{bmatrix} s^2_\psi & -s_\psi c_\psi & 0 & -s^2_\psi & s_\psi c_\psi & 0 \\ -s_\psi c_\psi & c^2_\psi & 0 & s_\psi c_\psi & -c^2_\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2_\psi & s_\psi c_\psi & 0 & s^2_\psi & -s_\psi c_\psi & 0 \\ s_\psi c_\psi & -c^2_\psi & 0 & -s_\psi c_\psi & c^2_\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15.8}$$

$$\frac{\partial^2 \bar{\theta}_1}{\partial \mathbf{u}\,\partial \mathbf{u}} = \frac{1}{L^2} \begin{bmatrix} -2s_\psi c_\psi & c^2_\psi - s^2_\psi & 0 & 2s_\psi c_\psi & s^2_\psi - c^2_\psi & 0 \\ c^2_\psi - s^2_\psi & 2s_\psi c_\psi & 0 & s^2_\psi - c^2_\psi & -2s_\psi c_\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2s_\psi c_\psi & s^2_\psi - c^2_\psi & 0 & -2s_\psi c_\psi & c^2_\psi - s^2_\psi & 0 \\ s^2_\psi - c^2_\psi & -2s_\psi c_\psi & 0 & c^2_\psi - s^2_\psi & 2s_\psi c_\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15.9}$$

$$\frac{\partial^2 \bar{\theta}_2}{\partial \mathbf{u}\,\partial \mathbf{u}} = \frac{1}{L^2}\begin{bmatrix} -2s_\psi c_\psi & c_\psi^2 - s_\psi^2 & 0 & 2s_\psi c_\psi & s_\psi^2 - c_\psi^2 & 0 \\ c_\psi^2 - s_\psi^2 & 2s_\psi c_\psi & 0 & s_\psi^2 - c_\psi^2 & -2s_\psi c_\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2s_\psi c_\psi & s_\psi^2 - c_\psi^2 & 0 & -2s_\psi c_\psi & c_\psi^2 - s_\psi^2 & 0 \\ s_\psi^2 - c_\psi^2 & -2s_\psi c_\psi & 0 & c_\psi^2 - s_\psi^2 & 2s_\psi c_\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15.10}$$

§15.2.4. Arbitrary Initial Configuration

The foregoing relations can be generalized to the case of a initial configuration $\mathcal{C}_0$ not aligned with the X axis as shown in Figure 15.4. Given the node coordinates and displacements shown in the figure, it is easily shown (Section §9.4) that $\cos\varphi = X_{21}/L_0$, $\sin\varphi = Y_{21}/L_0$, $\cos\phi = \cos(\psi + \varphi) = x_{21}/L$, $\sin\phi = \sin(\psi + \varphi) = y_{21}/L$, $\cos\psi = (X_{21}x_{21} + Y_{21}y_{21})/(LL_0)$ and $\sin\psi = (X_{21}y_{21} - Y_{21}x_{21})/(LL_0)$.

The preceding transformation rules remain correct if ψ is replaced by $\phi = \varphi + \psi$, except for the deformation angle computation, which remain $\bar{\theta}_1 = \theta_1 - \psi$ and $\bar{\theta}_2 = \theta_2 - \psi$ because the θs are measured from $\bar{X}$.

The relation between deformational and global displacements become

$$\begin{aligned} d &= L - L_0 = u_{X21}\,c_\phi + u_{Y21}\,s_\phi + L_0(1 - c_\phi) \\ \bar{\theta}_1 &= \theta_1 - \psi \\ \bar{\theta}_2 &= \theta_2 - \psi \end{aligned} \tag{15.11}$$

The first derivatives of d=eformation variables are

$$\begin{bmatrix} \delta d \\ \delta\bar{\theta}_1 \\ \delta\bar{\theta}_2 \end{bmatrix} = \begin{bmatrix} -c_\phi & -s_\phi & 0 & c_\phi & s_\phi & 0 \\ -s_\phi/L & c_\phi/L & 1 & s_\phi/L & -c_\phi/L & 0 \\ -s_\phi/L & c_\phi/L & 0 & s_\phi/L & -c_\phi/L & 1 \end{bmatrix} \begin{bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta\theta_1 \\ \delta u_{X2} \\ \delta u_{Y2} \\ \delta\theta_2 \end{bmatrix} \tag{15.12}$$

The second derivatives of deformation variables are

$$\frac{\partial^2 d}{\partial \mathbf{u}\,\partial \mathbf{u}} = \frac{1}{L}\begin{bmatrix} s_\phi^2 & -s_\phi c_\phi & 0 & -s_\phi^2 & s_\phi c_\phi & 0 \\ -s_\phi c_\phi & c_\phi^2 & 0 & s_\phi c_\phi & -c_\phi^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s_\phi^2 & s_\phi c_\phi & 0 & s_\phi^2 & -s_\phi c_\phi & 0 \\ s_\phi c_\phi & -c_\phi^2 & 0 & -s_\phi c_\phi & c_\phi^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15.13}$$

$$\frac{\partial^2 \bar{\theta}_1}{\partial \mathbf{u}\,\partial \mathbf{u}} = \frac{1}{L^2}\begin{bmatrix} -2s_\phi c_\phi & c_\phi^2 - s_\phi^2 & 0 & 2s_\phi c_\phi & s_\phi^2 - c_\phi^2 & 0 \\ c_\phi^2 - s_\phi^2 & 2s_\phi c_\phi & 0 & s_\phi^2 - c_\phi^2 & -2s_\phi c_\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2s_\phi c_\phi & s_\phi^2 - c_\phi^2 & 0 & -2s_\phi c_\phi & c_\phi^2 - s_\phi^2 & 0 \\ s_\phi^2 - c_\phi^2 & -2s_\phi c_\phi & 0 & c_\phi^2 - s_\phi^2 & 2s_\phi c_\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15.14}$$

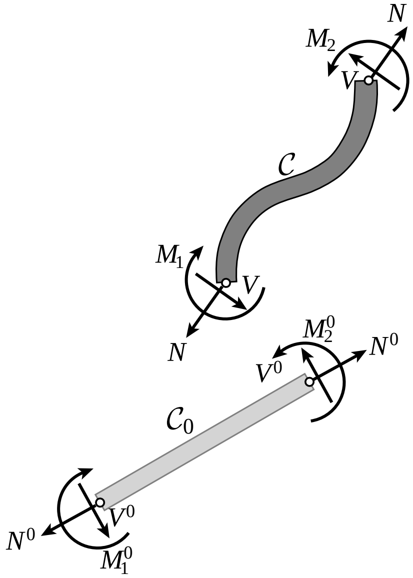

Figure 15.5. Beam stress resultants depicting positive sign conventions. Axial forces N and transverse shear forces V are constant along the length, but the bending moments M vary linearly. Hence two nodal values of M are required.

$$\frac{\partial^2\bar{\theta}_2}{\partial\mathbf{u}\,\partial\mathbf{u}} = \frac{1}{L^2}\begin{bmatrix} -2s_\phi c_\phi & c_\phi^2 - s_\phi^2 & 0 & 2s_\phi c_\phi & s_\phi^2 - c_\phi^2 & 0 \\ c_\phi^2 - s_\phi^2 & 2s_\phi c_\phi & 0 & s_\phi^2 - c_\phi^2 & -2s_\phi c_\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2s_\phi c_\phi & s_\phi^2 - c_\phi^2 & 0 & -2s_\phi c_\phi & c_\phi^2 - s_\phi^2 & 0 \\ s_\phi^2 - c_\phi^2 & -2s_\phi c_\phi & 0 & c_\phi^2 - s_\phi^2 & 2s_\phi c_\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15.15}$$

§15.2.5. Stress Resultants

The stress resultants in the reference configuration (either $\mathcal{C}_0$ or $\mathcal{C}_R$) are N^0, M_1^0 and M_2^0. The initial shear force is $V^0 = (M_1^0 - M_2^0)/L_0$. See Figure 15.5 for sign conventions.

Denote by N, V and M the stress resultants in the current configuration. Whereas N and V are constant along the element, $M = M(x^e)$ varies linearly along the length because this is a Hermitian or model, which relies on cubic transverse displacements. Consequently we will define its variation by the two node values M_1 and M_2. The shear V is recovered from equilibrium as $V = (M_1 - M_2)/L$, which is also constant. The stress resultants can be obtained from the deformations as

$$\begin{aligned} N &= N^0 + \frac{EA_0}{L_0}d, \quad M_1 = M_1^0 - \frac{2EI_0}{L_0}(2\bar{\theta}_1 + \bar{\theta}_2), \\ M_2 &= M_2^0 + \frac{2EI_0}{L_0}(\bar{\theta}_1 + 2\bar{\theta}_2), \quad V = \frac{M_1 - M_2}{L} = V^0\frac{L_0}{L} + \frac{2EI}{LL_0}(\bar{\theta}_1 - \bar{\theta}_2). \end{aligned} \tag{15.16}$$

§15.3. The Deformational Strain Energy

The next step in the CR formulation is to work out the deformational strain energy of the beam. The basic choices are:

1. A linear beam
2. A nonlinear TL beam

The strain energy of the beam for small strains can be written

$$U = U^a + U^b + U^g \tag{15.17}$$

where U^a, U^b and U^g are the energy taken by axial (bar) deformation, bending deformation, and initial-stress geometric effects, respectively. We adopt the following energy expressions:

$$\begin{aligned} U^a &= N^0\,d + \tfrac{1}{2}(N - N^0)d^2 = N^0\,L_0\,e + \tfrac{1}{2}EA_0L_0\,e^2,\\ U^b &= M_2^0\,\bar\theta_2 - M_1^0\,\bar\theta_1 + \tfrac{1}{2}\begin{bmatrix}\bar\theta_1\\ \bar\theta_2\end{bmatrix}^T \frac{EI_0}{L_0}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}\bar\theta_1\\ \bar\theta_2\end{bmatrix},\\ U^g &= \tfrac{1}{2}\begin{bmatrix}\bar\theta_1\\ \bar\theta_2\end{bmatrix}^T \frac{N^0L_0}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\begin{bmatrix}\bar\theta_1\\ \bar\theta_2\end{bmatrix}. \end{aligned} \tag{15.18}$$

The 2×2 matrices appearing in U^b and U^g may be derived from those given in Chapters 5 and 15, respectively, of Przemieniecki's book.[1] This book, howevr, omits the initial stress terms.

§15.4. Internal Force Vector and Tangent Stiffness Matrix

The internal force vector and tangent stiffness matrix of the corrotational element are then obtained by the usual formulas:

$$\mathbf{p} = \frac{\partial U}{\partial \mathbf{u}}, \qquad \mathbf{K} = \frac{\partial \mathbf{p}}{\partial \mathbf{u}} = \mathbf{K}_M + \mathbf{K}_G \tag{15.19}$$

To develop these quantities it is necessary to find the first and second partial derivatives of d, $\bar\theta_1$ and $\bar\theta_2$ in terms of the node displacements.

§15.4.1. Internal Force Vector

Using the partial derivatives compiled above and *Mathematica*, one obtains the following expression for the internal forces.

$$\mathbf{p} = \mathbf{p}^a + \mathbf{p}^b + \mathbf{p}^g \tag{15.20}$$

where

$$\begin{aligned} \mathbf{p}^a &= \frac{\partial U^a}{\partial \mathbf{u}} = N\,[\,-c_\phi \quad -s_\phi \quad 0 \quad c_\phi \quad s_\phi \quad 0\,]^T\\ \mathbf{p}^b &= \frac{\partial U^b}{\partial \mathbf{u}} = [\,Vs_\phi \quad -Vc_\phi \quad -M_1 \quad -Vs_\phi \quad Vc_\phi \quad M_2\,]^T\\ \mathbf{p}^g &= \frac{\partial U^g}{\partial \mathbf{u}} = \frac{N^0L_0}{30}\,[\,-3s_\phi(\bar\theta_1+\bar\theta_2)/L \quad 3c_\phi(\bar\theta_1+\bar\theta_2)/L \quad 4\bar\theta_1-\bar\theta_2\\ &\qquad 3s_\phi(\bar\theta_1+\bar\theta_2)/L \quad -3c_\phi(\bar\theta_1+\bar\theta_2)/L \quad 4\bar\theta_2-\bar\theta_1\,]^T \end{aligned} \tag{15.21}$$

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[1] J. S. Przemieniecki, *Theory of Matrix Structural Analysis*, Dover, New York, 1985.

§15.4.2. Material Stiffness Matrix

Carrying out the computations one obtains the following compact expression for the material stiffness:

$$\mathbf{K}_M = \mathbf{T}^T \mathbf{K}_{M0} \mathbf{T} \tag{15.22}$$

where

$$\mathbf{K}_{M0} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{15.23}$$

is the stiffness matrix of the linear beam element, and **T** is the transformation matrix

$$\mathbf{T} = \begin{bmatrix} c_\phi & s_\phi & 0 & 0 & 0 & 0 \\ -s_\phi & c_\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_\phi & s_\phi & 0 \\ 0 & 0 & 0 & -s_\phi & c_\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{15.24}$$

which introduces the effect of finite rigid body motions.

§15.4.3. Geometric Stiffness Matrix

The expression for the geometric stiffness is a bit more complicated. It can be presented in a compact form as follows:

$$\mathbf{K}_G = \mathbf{T}^T \mathbf{K}_G^N \mathbf{T} + \mathbf{K}_G^V \tag{15.25}$$

where **T** is the transformation matrix (15.24), $\mathbf{K}_G^N$ is the well known geometric stiffness for a Hermitian beam element under axial force:

$$\mathbf{K}_G^N = \frac{N}{30L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3L & 0 & -36 & 3L \\ 0 & 3L & 4L^2 & 0 & -3L & -L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3L & 0 & 36 & -3L \\ 0 & 3L & -L^2 & 0 & -3L & 4L^2 \end{bmatrix} \tag{15.26}$$

and the remaining term introduces the effect of varying moments through the transverse shear force in $\mathcal{C}$:

$$\mathbf{K}_G^V = \frac{V}{L} \begin{bmatrix} \sin 2\phi & -\cos 2\phi & 0 & -\sin 2\phi & \cos 2\phi & 0 \\ -\cos 2\phi & -\sin 2\phi & 0 & \cos 2\phi & \sin 2\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin 2\phi & \cos 2\phi & 0 & \sin 2\phi & -\cos 2\phi & 0 \\ \cos 2\phi & \sin 2\phi & 0 & -\cos 2\phi & -\sin 2\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15.27}$$

in which $\sin 2\phi = 2 s_\phi c_\phi$ and $\cos 2\phi = c_\phi^2 - s_\phi^2$.

Homework Exercises for Chapter 15

The Corotational Description: 2D C1 Beam

EXERCISE 15.1 Complete the derivation of $\mathbf{p}$ for the 2-node C^1 beam element and implement in *Mathematica*, using the same inputs as in Chapter 9 Addendum. (Implemented and posted on Web)

EXERCISE 15.2 Complete the derivation of $\mathbf{K}$ for the 2-node C^1 beam element and implement in *Mathematica*, using the same inputs as in Chapter 9 Addendum. (Implemented and posted on Web)

EXERCISE 15.3 A plane 2-node C^1 beam element has properties $L_0 = 6$, $E = 3000$, $A_0 = 2$, $I_0 = 12$, $N^0 = 5$ in the initial state $\mathcal{C}_0$ along X, with node 1 at (0,0) and node 2 at $(L_0, 0)$. The beam rotates by 45° about the origin so that at the current configuration $\mathcal{C}$ node 1 stays at {0, 0} while node 2 moves to $\{(L_0+d)/\sqrt{2}, (L_0+d)/\sqrt{2}\}$, where $d = L_0/1000$. The rotational freedoms at $\mathcal{C}$ are $\theta_1 = \theta_2 = 45^\circ = \pi/4$ radians. Compute $\mathbf{p}$, $\mathbf{K}_M$ and $\mathbf{K}_G$ at the current configuration, and compare those quantities with those of the C^0 beam element presented in Chapter 9, using RBF for the latter.

Note: A *Mathematica* implementation of this C^1 element has been posted on the Web as a *Mathematica 4.1* Notebook `PlaneBeamC1.nb`. The element checks out when moving about the reference configuration $\mathcal{C}_0$. It gives excellent buckling values for the problem of Exercise 9.3. More tests are needed, however, for an arbitrary configuration to make sure the internal force vector and the tangent stiffness are consistent.

EXERCISE 15.4 Confirm the previous statement by repeating the buckling calculations of Exercise 9.3 using the CR beam element provided in the Mathematica Notebook mentioned above (extract the material and stiffness matrices, ignore the rest). Compare the speed of convergence of the CR and TL element for the cantilever buckling problem.