14

The CR Formulation: Space Bar

TABLE OF CONTENTS

§14.1. Introduction

The expressions provided in the foregoing two Chapters and in Appendix R address the formation of incremental equations for an arbitrary CR element. The main restriction is the assumption of small local deformations, which permit the use of the linearized equations for the deformational energy.

Closed form of the incremental expressions become intractable for elements of arbitrary geometry, such as curved shells or beams, because **c** and **R** are complicated functions of the element displacement field, which is turn determined by **u**. Fortunately the CR approach is often used with elements of *simple geometry* in which rotational freedoms, if any, may be ignored in defining the corotated configuration. Under those conditions one can work out the base-to-deformed transformation arrays directly from *geometric arguments*. The transformations may be then systematically applied to existing linear elements through a modular interface, as illustrated in Figure 12.7.

In particular, for a *simplex element* (a constant strain element without rotational freedoms) it is possible to work out all transformation from the *intrinsic geometry* of a line segment, triangle or tetrahedron moving in 3D space. (By “intrinsic” is meant changes in edge lengths, face areas and volume dimensions.) For the line segment modeling a space bar the formulas are worked out and collected in the next sections. These apply to several types of finite elements, such as bars and cables. The results may be specialized to two dimensions, if desired, by setting the third coordinate to zero.

It is important that the kinematic analysis be *exact* so that arbitrary rigid body motions can be accomodated. Restrictions on local deformations in the motion from $\mathcal{C}_R$ to $\mathcal{C}_D$ can then be made when considering specific elements, particularly those endowed with rotational DOFs.

Several of the following results are new. Their closed form derivation was made possible because of the use of *Mathematica* to synthesize abtruse algebraic expressions containing symbolic terms.

§14.2. Line Segment Moving in 3D

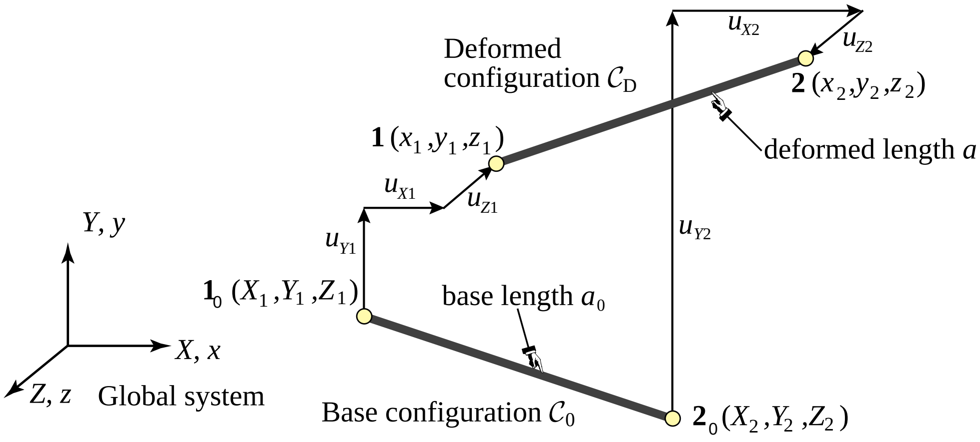


FIGURE 14.1. A line segment moving in 3D space.

Consider the line segment shown in Figure 14.1, defined by the end nodes 1-2. The segment moves in three-dimensional space. The global axes will be denoted by $\{X, Y, Z\}$ instead of $\{X_1, X_2, X_3\}$ so

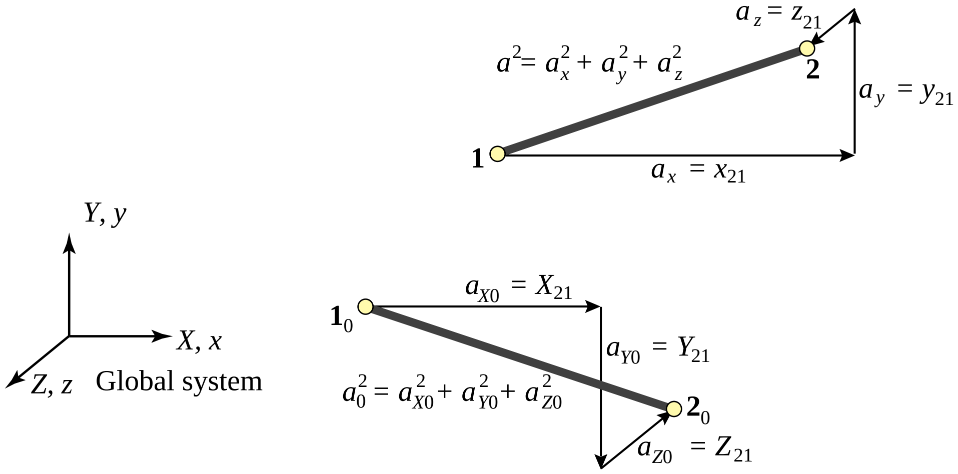


FIGURE 14.2. Line segment components.

that $\{X_n, Y_n, Z_n\}$ can be used for the coordinates of node n. Global axes $\{x, y, z\}$, which are used for the deformed configuration, coalesce with $\{X, Y, Z\}$,

The base line configuration $\mathcal{C}_0$ is specified by the coordinates (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) of the line end nodes. The line moves to the deformed (current) configuration $\mathcal{C}_D$ of length a defined by coordinates $\{x_1 = X_1 + u_{X1}, y_1 = Y_1 + u_{Y1}, z_1 = Z_1 + u_{Z1}\}$ and $\{x_2 = X_2 + u_{X2}, y_2 = Y_2 + u_{Y2}, z_2 = Z_2 + u_{Z2}\}$, where u_{X1} through u_{Z2} are the node displacements. Node coordinate and displacement differences are abbreviated by $X_{21} = X_2 - X_1$, $x_{21} = x_2 - x_1$, $u_{X21} = u_{X2} - u_{X1}$, etc. As illustrated in Figure 14.2, the line lengths are given by

$$a_0^2 = a_{X0}^2 + a_{Y0}^2 + a_{Z0}^2 = X_{21}^2 + Y_{21}^2 + Z_{21}^2, \qquad a^2 = a_x^2 + a_y^2 + a_z^2 = x_{21}^2 + y_{21}^2 + z_{21}^2. \tag{14.1}$$

For further use define the following vectors

$$\mathbf{a}_0 = \begin{bmatrix} a_{X0} \\ a_{Y0} \\ a_{Z0} \end{bmatrix}, \quad \hat{\mathbf{a}}_0 = \frac{1}{a_0} \begin{bmatrix} a_{X0} \\ a_{Y0} \\ a_{Z0} \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_x \end{bmatrix}, \quad \hat{\mathbf{a}} = \frac{1}{a} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad \mathbf{u}_{21} = \mathbf{a} - \mathbf{a}_0 = \begin{bmatrix} u_{X21} \\ u_{Y21} \\ u_{Z21} \end{bmatrix}, \tag{14.2}$$

Here $\hat{\mathbf{a}}_0 = \mathbf{a}_0/a_0$ and $\hat{\mathbf{a}} = \mathbf{a}/a$ denote the direction cosine vectors of the base and current line segment, respectively. This "hat convention" will be used to identify direction vectors normalized to unit length. It is important not to confuse the 3-vector $\mathbf{u}_{21}$ with the 6-vector of node displacements

$$\mathbf{u} = [\, u_{X1} \quad u_{Y1} \quad u_{Z1} \quad u_{X2} \quad u_{Y2} \quad u_{Z2} \,]^T \tag{14.3}$$

In fact, $\partial \mathbf{u}_{21}/\partial \mathbf{u}$ is the 3×6 matrix $[\, -\mathbf{I} \quad \mathbf{I} \,]$, where $\mathbf{I}$ is the 3×3 identity matrix.

§14.2.1. Line Segment Derivatives

Suppose that the node displacements are functions of two variables, generically denoted by Φ and Ψ: $u_{X1} = u_{X1}(\Phi, \Psi)$, etc. The partial derivative of a with respect to Φ is

$$\frac{\partial a}{\partial \Phi} = \frac{a_x}{a} \frac{\partial u_{X21}}{\partial \Phi} + \frac{a_y}{a} \frac{\partial u_{Y21}}{\partial \Phi} + \frac{a_z}{a} \frac{\partial u_{Z21}}{\partial \Phi} = \hat{\mathbf{a}}^T \, \mathbf{u}_{21,\Phi}, \qquad \text{in which} \quad \mathbf{u}_{21,\Phi} = \begin{bmatrix} \dfrac{\partial u_{X21}}{\partial \Phi} \\ \dfrac{\partial u_{Y21}}{\partial \Phi} \\ \dfrac{\partial u_{Z21}}{\partial \Phi} \end{bmatrix}. \tag{14.4}$$

Note that the base configuration is only "remembered" through the displacements since the initial length a_0 does not appear explicitly.

To obtain the second derivative we take the partial of (14.4) with respect to the generic variable Ψ, which yields $\hat{\mathbf{a}}^T\,\partial^2\mathbf{u}_{21}/(\partial\Phi\partial\Psi)+(\partial\hat{\mathbf{a}}^T/\partial\Psi)\ (\partial\mathbf{u}_{21}/\partial\Phi)$. While the first term is easy, the second one requires the derivatives of $\hat{\mathbf{a}}$. Since this involves the variation of a fixed-length (unit) vector, it can be expected to involve the orthogonal projector associated with $\mathbf{a}$. The final result can be presented in the compact matrix form

$$\frac{\partial^2 a}{\partial\Phi\partial\Psi}=\mathbf{u}^T_{21,\Phi}\,\mathbf{H}\,\mathbf{u}_{21,\Psi}+\hat{\mathbf{a}}^T\,\mathbf{u}_{21,\Phi\Psi},\tag{14.5}$$

in which

$$\mathbf{u}_{21,\Phi\Psi}=\begin{bmatrix}\dfrac{\partial^2 u_{X21}}{\partial\Phi\partial\Psi}\\ \dfrac{\partial^2 u_{Y21}}{\partial\Phi\partial\Psi}\\ \dfrac{\partial^2 u_{Z21}}{\partial\Phi\partial\Psi}\end{bmatrix},\quad \mathbf{H}=\frac{1}{a^3}\begin{bmatrix}a_y^2+a_z^2 & -a_xa_y & -a_xa_z\\ -a_ya_x & a_z^2+a_x^2 & -a_ya_z\\ -a_za_x & -a_za_y & a_x^2+a_y^2\end{bmatrix}=\frac{1}{a}(\mathbf{I}-\hat{\mathbf{a}}\hat{\mathbf{a}}^T)=\frac{1}{a}\mathbf{P}_a.\tag{14.6}$$

Here $\mathbf{I}$ denotes the identity matrix of order 3 whereas $\mathbf{P}_a=\mathbf{I}-\hat{\mathbf{a}}\hat{\mathbf{a}}^T$ is the orthogonal projector associated with the direction $\mathbf{a}$. (To show this, square $\mathbf{P}_a$ and verify that $\mathbf{P}_a^2=\mathbf{P}_a$.)

The results (14.4) and (14.5) can be specialized to various choices. For example, if the displacements are viewed as functions of real time (or a time-like parameter) t, we have $\Phi\equiv\Psi\equiv t$, and

$$\dot{a}=\hat{\mathbf{a}}^T\,\dot{\mathbf{u}}_{21},\qquad \ddot{a}=\hat{\mathbf{a}}^T\,\ddot{\mathbf{u}}_{21}+\dot{\mathbf{u}}_{21}^T\,\mathbf{H}\,\dot{\mathbf{u}}_{21}\tag{14.7}$$

where a superposed dot denotes derivative with respect to t. Of more interest for element derivation is to set Φ and Ψ in turn to the six entries of the node displacement vector $\mathbf{u}$ arranged as (14.3). In this case the term $\mathbf{u}_{21,\Phi\Psi}$ in (14.5) vanishes, and we obtain

$$\frac{\partial a}{\partial\mathbf{u}}=\frac{1}{a}\,[\,-a_x\ \ -a_y\ \ -a_z\ \ a_x\ \ a_y\ \ a_z\,]^T=[\,-\hat{\mathbf{a}}^T\ \ \hat{\mathbf{a}}^T\,]^T\ \overset{\text{def}}{=}\ \mathbf{h},\qquad \frac{\partial^2 a}{\partial\mathbf{u}\,\partial\mathbf{u}}=\begin{bmatrix}\mathbf{H} & -\mathbf{H}\\ -\mathbf{H} & \mathbf{H}\end{bmatrix}\overset{\text{def}}{=}\mathbf{G}.\tag{14.8}$$

Vector $\mathbf{h}$ and matrix $\mathbf{G}$ will appear in the derivation of the space bar element in §14..3. In addition the following definition and relations are useful there:

$$\mathbf{J}\overset{\text{def}}{=}\begin{bmatrix}\mathbf{I} & -\mathbf{I}\\ -\mathbf{I} & \mathbf{I}\end{bmatrix}=a\mathbf{G}+\mathbf{h}\,\mathbf{h}^T,\quad \mathbf{G}=\frac{1}{a}\left(\mathbf{J}-\mathbf{h}\,\mathbf{h}^T\right).\tag{14.9}$$

§14.2.2. Derivatives of Length Functions

Often the derivatives of a function $F(a)$ are needed, for example for bar strain measures other than engineering strains. The function $F(a)$ is assumed twice differentiable respect to a. The a-derivative abbreviations are

$$F_{,a}=\frac{\partial F(a)}{\partial a},\qquad F_{,aa}=\frac{\partial^2 F(a)}{\partial a^2},\tag{14.10}$$

(Partials are used since F could be a function of other variables in addition to a.) Using the previous results and the chain rule we obtain

$$\begin{aligned}\frac{\partial F(a)}{\partial \Phi} &= \frac{\partial F(a)}{\partial a}\frac{\partial a}{\partial \Phi} = F_{,a}\,\hat{\mathbf{a}}^T\,\mathbf{u}_{21,\Phi},\\ \frac{\partial^2 F(a)}{\partial \Phi \partial \Psi} &= \frac{\partial F_{,a}}{\partial a}\,\frac{\partial a}{\partial \Psi}\,\hat{\mathbf{a}}^T\,\mathbf{u}_{21,\Phi} + F_{,a}\,\frac{\partial^2 a}{\partial \Phi \partial \Psi} = \mathbf{u}^T_{21,\Phi}\big(F_{,aa}\hat{\mathbf{a}}\hat{\mathbf{a}}^T + F_{,a}\,\mathbf{H}\big)\mathbf{u}_{21,\Psi} + F_{,a}\,\hat{\mathbf{a}}^T\,\mathbf{u}_{21,\Phi\Psi}\end{aligned} \tag{14.11}$$

For $F(a) = a$ the previous results are recovered. The case of powers of a: $F(a) = a^n$ has immediate applications. For $n = 2$, with Φ and Ψ specialized first to t and then to $\mathbf{u}$, we obtain the squared-length derivatives:

$$\dot{\overline{a^2}} = 2a\,\hat{\mathbf{a}}\,\dot{\mathbf{u}}_{21}, \qquad \ddot{\overline{a^2}} = 2\dot{\mathbf{u}}^T\dot{\mathbf{u}}_{21} + 2a\ddot{\mathbf{u}}_{21} \tag{14.12}$$

$$\frac{\partial a^2}{\partial \mathbf{u}} = 2a\,[\,-\hat{\mathbf{a}}^T \quad \hat{\mathbf{a}}^T\,]^T = 2a\,\mathbf{h}, \qquad \frac{\partial^2 a^2}{\partial \mathbf{u}\,\partial \mathbf{u}} = 2\begin{bmatrix} \mathbf{I} & -\mathbf{I}\\ -\mathbf{I} & \mathbf{I}\end{bmatrix} = 2\mathbf{J}. \tag{14.13}$$

The simplicity of the Hessian of a^2 will be a cogent argument for the use of the Green measure of strain in bars. For $n = -1$, with Φ and Ψ specialized as above, we obtain the inverse-length derivatives:

$$\dot{\overline{1/a}} = -(1/a^2)\,\hat{\mathbf{a}}\,\dot{\mathbf{u}}, \qquad \ddot{\overline{1/a}} = (1/a^3)\,\dot{\mathbf{u}}^T_{21}\,(3\hat{\mathbf{a}}\,\hat{\mathbf{a}}^T - \mathbf{I})\,\dot{\mathbf{u}}_{21} - (1/a^2)\,\ddot{\mathbf{u}}_{21} \tag{14.14}$$

$$\frac{\partial(1/a)}{\partial \mathbf{u}} = -(1/a^2)\;[\,-\hat{\mathbf{a}} \quad \hat{\mathbf{a}}\,], \qquad \frac{\partial^2(1/a)}{\partial \mathbf{u}\,\partial \mathbf{u}} = (1/a^3)\begin{bmatrix} 3\hat{\mathbf{a}}\,\hat{\mathbf{a}}^T - \mathbf{I} & -3\hat{\mathbf{a}}\,\hat{\mathbf{a}}^T + \mathbf{I}\\ -3\hat{\mathbf{a}}\,\hat{\mathbf{a}}^T + \mathbf{I} & 3\hat{\mathbf{a}}\,\hat{\mathbf{a}}^T - \mathbf{I}\end{bmatrix} \tag{14.15}$$

The foregoing equations may be specialized to two-dimensional motions of a segment moving in the $\{X, Y\}$ plane by setting the Z component to zero and then removing that component from vectors and matrices.

All of the results given so far are *geometrically exact* and pose no limit on how much the segment stretches or contracts. Restrictions in the form of small deformations will appear when the formulas are applied to a bar element in §14.3.

§14.2.3. Mathematica Implementation and FD Verification

The foregoing formulas for the first and second derivatives of an arbitrary function $F(a)$, given in (14.11), have been implemented in *Mathematica* in the form of two modules. Results are numerically verified by finite differences.

Scripts for the computation of first partial derivatives of $F(a)$ with respect to Φ are shown in Figure 14.3. This is done by module `LengthFunctionFirstDerivatives`. The results when Φ is identified with the node displacement vector are verified with central finite differences with module `LengthFunctionFirstDerivativesByFD`. The driver code is shown at the bottom of the figure.

The computations are exercised for $F(a)$ set to a, a^2, $1/a$, $1/a^2$, $\sqrt{a}$ and $\log(a)$ in the loop shown at the bottom of Figure 14.3. The coordinate and displacement values used for numerical verification are

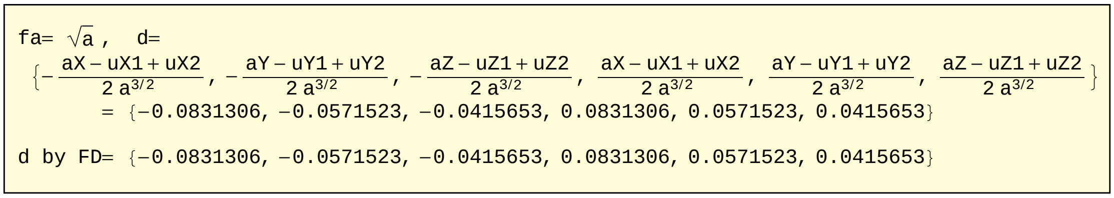
$$\begin{aligned}&(a_X, a_Y, a_Z) = (11, 10, 2), \quad (u_{X1}, u_{Y1}, u_{Z1}) = (3, -5, -5), \quad (u_{X2}, u_{Y2}, u_{Z2}) = (8, -4, 1),\\ &(a_x, a_y, a_z) = (a_X, a_Y, a_Z) + (u_{X2}, u_{Y2}, u_{Z2}) - (u_{X1}, u_{Y1}, u_{Z1}) = (16, 11, 8),\\ &a_0 = \sqrt{a_X^2 + a_Y^2 + a_Z^2} = \sqrt{11^2 + 10^2 + 2^2} = 15, \quad a = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{16^2 + 11^2 + 8^2} = 21.\end{aligned} \tag{14.16}$$

```
  LengthFunctionFirstDerivative[{fa_,a_},aXYZ_,u21_,Φ_]:=Module[
    {nu=Length[u21],nΦ=Length[Φ],axyz,u21Φ}, axyz=aXYZ+u21;
     u21Φ=Table[D[u21[[i]],Φ[[j]]],{i,1,nu},{j,1,nΦ}];
     Return[(D[fa,a]*axyz/a).u21Φ] ];

  LengthFunctionFirstDerivativeByFD[{fa_,a_},aXYZ_,u21_,δ_]:=
     Module[{ax,ay,az,axyz,axyzp,axyzm,axp,ayp,azp,axm,aym,azm,
     i,inc,dfada,da,d=Table[0,{6}]}, {ax,ay,az}=axyz=aXYZ+u21;
     dfada=D[fa,a]/.a->Sqrt[ax^2+ay^2+az^2];
     inc=δ*{{-1,0,0},{0,-1,0},{0,0,-1},{1,0,0},{0,1,0},{0,0,1}};
     For [i=1,i<=6,i++,  axyz=aXYZ+u21;
         axyzp=axyz+inc[[i]];  axyzm=axyz-inc[[i]];
         {axp,ayp,azp}=axyzp;  {axm,aym,azm}=axyzm;
         da=Sqrt[axp^2+ayp^2+azp^2]-Sqrt[axm^2+aym^2+azm^2];
         d[[i]]=dfada*da/(2*δ);
         ];  Return[d]];

  ClearAll[a,axyz,a,aX,aY,aZ,uX1,uY1,uZ1,uX2,uY2,uZ2,Φ,δ]; δ=1/100;
  rep={aX->11,aY->10,aZ->2,uX1->3,uY1->-5,uZ1->-5,uX2->8,uY2->-4,
       uZ2->1,a0->15,a->21};
  aXYZ={aX,aY,aZ}; u21={uX2-uX1,uY2-uY1,uZ2-uZ1};
  axyz=aXYZ+u21; {ax,ay,az}=axyz;
  Print["aXYZ=",aXYZ/.rep," u21=",u21/.rep," axyz=",axyz/.rep,
   " a0=",Sqrt[aX^2+aY^2+aZ^2]/.rep," a=",Sqrt[ax^2+ay^2+az^2]/.rep];
  u={uX1,uY1,uZ1,uX2,uY2,uZ2};
  For [if=1,if<=6,if++,  fa={a,a^2,1/a,1/a^2,Sqrt[a],Log[a]}[[if]];
      d=Simplify[LengthFunctionFirstDerivative[{fa,a},aXYZ,u21,u]];
      Print["fa= ",fa,",  d=",d,"\n       = ",N[d/.rep]];
      dFD=LengthFunctionFirstDerivativeByFD[{fa,a},aXYZ/.rep,u21/.rep,δ];
      Print["d by FD= ", N[dFD/.rep]];
      ];
```

FIGURE 14.3. Mathematica implementation of first-derivative computations in (14.11) for $F(a)$ and numerical check by central finite differences.



fa= $\sqrt{a}$,   d=

$$\left\{-\frac{aX-uX1+uX2}{2\,a^{3/2}},\ -\frac{aY-uY1+uY2}{2\,a^{3/2}},\ -\frac{aZ-uZ1+uZ2}{2\,a^{3/2}},\ \frac{aX-uX1+uX2}{2\,a^{3/2}},\ \frac{aY-uY1+uY2}{2\,a^{3/2}},\ \frac{aZ-uZ1+uZ2}{2\,a^{3/2}}\right\}$$

```
      = {-0.0831306, -0.0571523, -0.0415653, 0.0831306, 0.0571523, 0.0415653}

d by FD= {-0.0831306, -0.0571523, -0.0415653, 0.0831306, 0.0571523, 0.0415653}
```

FIGURE 14.4. Partial results from script of Figure 14.3 for $F(a) = \sqrt{a}$ and numerical values (14.16). Displacement increment $\delta = 0.01$ used for verification by central finite differences.

Figure 14.4 shows results for the case $F(a) = \sqrt{a}$. Numerical results correspond to the data (14.16). It can be seen that the finite difference values, obtained with a displacement increment of $\delta = 0.01$, agree to all places shown with the analytical results.

Scripts for the computation of second partial derivative of $F(a)$ with respect to Φ and Ψ are shown in Figure 14.5. This is done by module `LengthFunctionSecondDerivatives`. The results when both Φ and Ψ are identified with the node displacement vector are verified through finite differences with module `LengthFunctionSecondDerivativesByFD`. The driver code is shown at the bottom of the figure. The computations are exercised for $F(a)$ set to a, a^2, $1/a$, $1/a^2$, $\sqrt{a}$ and $\log(a)$ in the loop shown at the bottom of Figure 14.5.

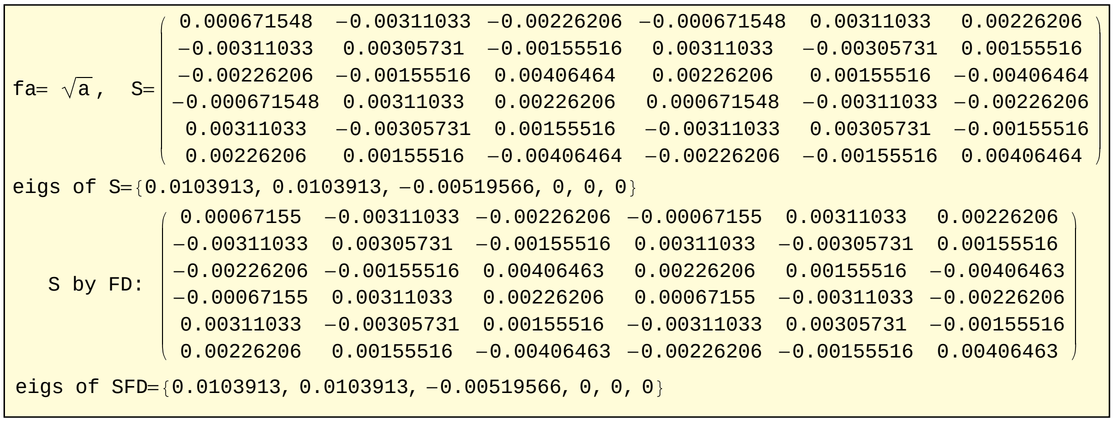
Figure 14.6 shows numerical results for $F(a) = \sqrt{a}$ and the data (14.16). Symbolic results are omitted to reduce clutter. It can be seen that the finite difference values, obtained with a displacement increment of $\delta = 0.01$, agree to at least 5 places with the analytical results.

```
LengthFunctionSecondDerivative[{fa_,a_},aXYZ_,u21_,{Φ_,Ψ_}]:=Module[
  {nc=Length[aXYZ],nu=Length[u21],nΦ=Length[Φ],nΨ=Length[Ψ],
   u21Φ,u21Ψ,J,H,P,S,Q,dfda,d2fdada}, axyz=aXYZ+u21; S=Q=Table[0,{6},{6}];
   u21Φ=Table[D[u21[[i]],Φ[[j]]],{i,1,nu},{j,1,nΦ}];
   u21Ψ=Table[D[u21[[i]],Ψ[[j]]],{i,1,nu},{j,1,nΨ}];
   J=Table[axyz[[i]]*axyz[[j]]/a^2,{i,1,nc},{j,1,nc}];
   H=(IdentityMatrix[nc]-J)/a; Q=Table[0,{nΦ},{nΨ}];
   For [i=1,i<=nc,i++, Q=Q+(axyz[[i]]/a)*
        Table[D[u21[[i]],Φ[[j]],Ψ[[k]]],{j,1,nΦ},{k,1,nΨ}] ];
   dfda=D[fa,a]; d2fdada=D[dfda,a];
   S=Transpose[u21Φ].(d2fdada*J+dfda*H).u21Ψ;
   Return[{S,Q}];
 ];

LengthFunctionSecondDerivativeByFD[{fa_,a_},aXYZ_,u21_,δ_]:=
   Module[{aX,aY,aZ,uX21,uY21,uZ21,ax,ay,az,
           i,inc,du,S=Table[0,{6},{6}]},
   inc=δ*{{-1,0,0},{0,-1,0},{0,0,-1},{1,0,0},{0,1,0},{0,0,1}};
   For [i=1,i<=6,i++,  du=inc[[i]];
       dp=LengthFunctionFirstDerivativeByFD[{fa,a},aXYZ,u21+du,δ];
       dm=LengthFunctionFirstDerivativeByFD[{fa,a},aXYZ,u21-du,δ];
       S[[i]]=(dp-dm)/(2*δ);
       ];  Return[S]];

ClearAll[a,n,axyz,a,aX,aY,aZ,uX1,uY1,uZ1,uX2,uY2,uZ2,Φ,δ]; δ=1/100;
rep={aX->11,aY->10,aZ->2,uX1->3,uY1->-5,uZ1->-5,uX2->8,uY2->-4,uZ2->1,a0-
>15,a->21};
aXYZ={aX,aY,aZ}; u21={uX2-uX1,uY2-uY1,uZ2-uZ1};
axyz=aXYZ+u21; {ax,ay,az}=axyz;
Print["aXYZ=",aXYZ/.rep," u21=",u21/.rep," axyz=",axyz/.rep,
 " a0=",Sqrt[aX^2+aY^2+aZ^2]/.rep," a=",Sqrt[ax^2+ay^2+az^2]/.rep];
u={uX1,uY1,uZ1,uX2,uY2,uZ2};
For [if=1,if<=6,if++,  fa={a,a^2,1/a,1/a^2,Sqrt[a],Log[a]}[[if]];
    {S,Q}=Simplify[LengthFunctionSecondDerivative[{fa,a},aXYZ,u21,{u,u}]];
    Print["fa= ",fa,",  S=",S//MatrixForm];
    Print["fa= ",fa,",  S=",N[Simplify[S/.rep]]//MatrixForm];
    Print["eigs of S=", Chop[Eigenvalues[N[S/.rep]]]];
    SFD=LengthFunctionSecondDerivativeByFD[{fa,a},aXYZ/.rep,u21/.rep,δ];
    Print["S by FD: ",Chop[N[SFD],.000001]//MatrixForm];
    Print["eigs of SFD=", Chop[Eigenvalues[N[SFD]],.00001]];
    ];
```

FIGURE 14.5. Mathematica implementation of second-derivative computations in (14.11) for $F(a)$ and check by central finite differences.



```
                ( 0.000671548  -0.00311033 -0.00226206 -0.000671548  0.00311033   0.00226206 )
                ( -0.00311033   0.00305731 -0.00155516  0.00311033  -0.00305731   0.00155516 )
                ( -0.00226206  -0.00155516  0.00406464  0.00226206   0.00155516  -0.00406464 )
fa= √a ,   S=   ( -0.000671548  0.00311033  0.00226206  0.000671548 -0.00311033  -0.00226206 )
                (  0.00311033  -0.00305731  0.00155516 -0.00311033   0.00305731  -0.00155516 )
                (  0.00226206   0.00155516 -0.00406464 -0.00226206  -0.00155516   0.00406464 )
eigs of S={0.0103913, 0.0103913, -0.00519566, 0, 0, 0}
                ( 0.00067155  -0.00311033 -0.00226206 -0.00067155  0.00311033   0.00226206 )
                ( -0.00311033  0.00305731 -0.00155516  0.00311033 -0.00305731   0.00155516 )
                ( -0.00226206 -0.00155516  0.00406463  0.00226206  0.00155516  -0.00406463 )
    S by FD:    ( -0.00067155  0.00311033  0.00226206  0.00067155 -0.00311033  -0.00226206 )
                (  0.00311033 -0.00305731  0.00155516 -0.00311033  0.00305731  -0.00155516 )
                (  0.00226206  0.00155516 -0.00406463 -0.00226206 -0.00155516   0.00406463 )
eigs of SFD={0.0103913, 0.0103913, -0.00519566, 0, 0, 0}
```

FIGURE 14.6. Partial results from script of Figure 14.5 for $F(a) = \sqrt{a}$ and numerical values (14.16). Symbolic results not shown to reduce clutter. Displacement increment $\delta = 0.01$ used for verification by central finite differences.

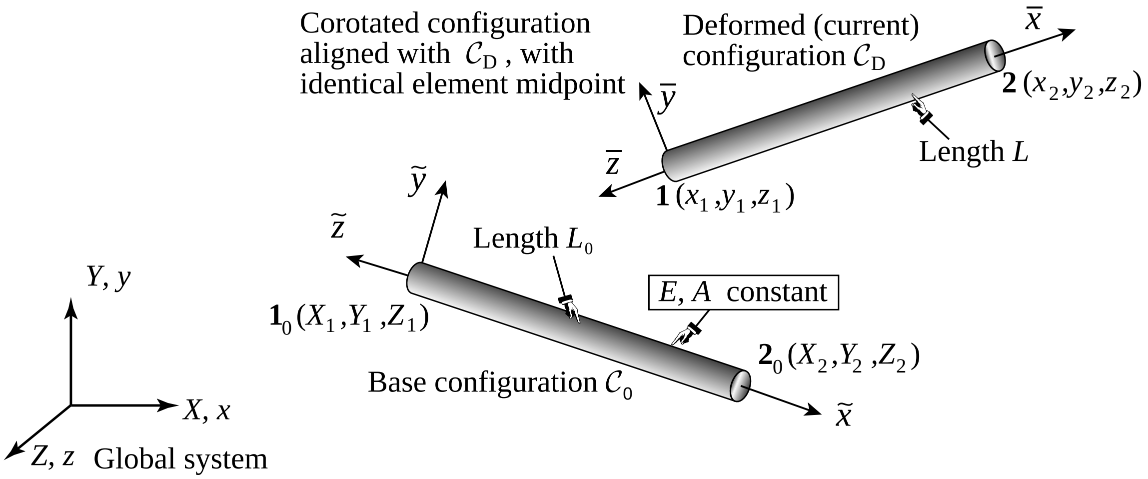


FIGURE 14.7. The 2-node bar element moving in 3D space.

§14.3. The CR Bar Element

The space bar element shown in Figure 14.7 will be presented here as an application of the foregoing results on line segment moving through 3D space. The static EICR formulation is illustrated with a linearly elastic, prismatic, 2-node bar element moving in 3D space, as depicted in that figure. The element has six degrees of freedom collected in the 6-vector (14.3).

The area and length in the base configuration $\mathcal{C}_0$ are A_0 and L_0, respectively. These become A and L in the deformed configuration $\mathcal{C}_D$. The element elongation is called $d = L - L_0$. The elastic modulus is E. In accordance with the usual assumptions of the CR description, bar deformations are assumed to be small. This allows to carry all integrals over the initial volume $A_0 L_0$.

Application of the "best corotational fit criterion" shows that, as may be expected, the corotated configuration $\mathcal{C}_R$ is aligned with, and lies halfway from, the current end nodes. But this result will not be required to develop the element equations.

§14.3.1. Internal Energy, Force and Stiffness

The axial strain and stress measures are denoted by e and s, respectively, with s being the energy conjugate of e. Both are assumed to be constant over the element volume. For the moment the choice of e and s will be left open. The strain and stress in $\mathcal{C}_0$ are 0 and s_0, respectively. In $\mathcal{C}_D$ they become e and $s = s_0 + Ee$. The axial forces in $\mathcal{C}_0$ and $\mathcal{C}_D$ are $N_0 = A_0 s_0$ and $N = A_0\, s = N_0 + E A_0\, e$, respectively.

The energy density in $\mathcal{C}_0$ is taken as zero. It becomes $\mathcal{U} = s_0\, e + \frac{1}{2} E\, e^2$ in $\mathcal{C}_D$, which is constant over the element. The total internal energy in $\mathcal{C}_D$ is

$$U = \int_{V_0} \mathcal{U}\, dV = \mathcal{U}\, V_0 = A_0 L_0 (s_0\, e + \tfrac{1}{2} E\, e^2) = L_0 (N_0\, e + \tfrac{1}{2} E A_0\, e^2). \tag{14.17}$$

The strain measure is taken to be a function $e = e(d, L_0) = u(L, L_0)$ to be chosen later. Since $d = L - L_0$ and L_0 is fixed, $e_{,d} = \partial e/\partial d = \partial e/\partial L = e_{,L}$ and likewise $e_{,dd} = \partial^2 e/(\partial d \partial d) =$

Table 14.1. Strain Measures for Bar Element and their Derivatives

Strain Measure	Symbol	Definition for bar	$\frac{\partial e}{\partial L}$	$\frac{\partial^2 e}{\partial L^2}$
Engineering	e_E	$(L-L_0)/L_0$	$1/L_0$	0
Green-Lagrange	e_G	$(L^2-L_0^2)/(2L_0^2)$	L/L_0^2	$1/L_0^2$
Hencky	e_H	$\log(L/L_0)$	$1/L$	$-1/L^2$
Midpoint	e_M	$2(L-L_0)/(L+L_0)$	$4L_0/(L+L_0)^2$	$-8L_0/(L+L_0)^3$

$\partial^2 e/(\partial L\,\partial L) = e_{,LL}$. The derivatives of e with respect to the nodal displacements DOF (14.3) are obtained by the chain rule: $\partial e/\partial \mathbf{u} = e_{,L}\,(\partial L/\partial \mathbf{u})$, etc. The necessary partials are already worked out for the line segment in §14.2, with the substitutions $a \rightarrow L$, $\mathbf{a} \rightarrow \mathbf{L}$, $\hat{\mathbf{a}} \rightarrow \hat{\mathbf{L}}$, etc. Recall that

$$\frac{\partial L}{\partial \mathbf{u}} = \mathbf{h} = \begin{bmatrix} -\hat{\mathbf{L}} \\ \hat{\mathbf{L}} \end{bmatrix}, \qquad \frac{\partial^2 L}{\partial \mathbf{u}\partial \mathbf{u}} = \mathbf{G} = \begin{bmatrix} \mathbf{H} & -\mathbf{H} \\ -\mathbf{H} & \mathbf{H} \end{bmatrix} = \frac{1}{L}(\mathbf{J} - \mathbf{h}\,\mathbf{h}^T), \tag{14.18}$$

where $\hat{\mathbf{L}}$ is the $\hat{\mathbf{a}}$ given in in (14.2), $\mathbf{H}$ is defined in (14.8) and $\mathbf{J}$ in (14.9), with the replacements indicated above.

The internal force is the gradient of the internal energy with respect to the node displacements:

$$\mathbf{p} = \frac{\partial U}{\partial \mathbf{u}} = \frac{\partial U}{\partial e}\,\frac{\partial e}{\partial \mathbf{u}} = L_0\,(N_0 + E A_0 e)\frac{\partial e}{\partial \mathbf{u}} = L_0\,N\frac{\partial e}{\partial L}\,\frac{\partial L}{\partial \mathbf{u}} = L_0\,N\frac{\partial e}{\partial L}\mathbf{h}. \tag{14.19}$$

The tangent stiffness is the Hessian of U:

$$\begin{aligned}
\mathbf{K} &= \frac{\partial^2 U}{\partial \mathbf{u}\partial \mathbf{u}} = \frac{\partial \mathbf{p}}{\partial \mathbf{u}} = L_0\,\frac{\partial N}{\partial \mathbf{u}}\,\frac{\partial e}{\partial L}\,\frac{\partial L}{\partial \mathbf{u}} + L_0\,N\frac{\partial^2 e}{\partial L^2}\,\frac{\partial L}{\partial \mathbf{u}}\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T + L_0\,N\frac{\partial e}{\partial L}\,\frac{\partial^2 L}{\partial \mathbf{u}\partial \mathbf{u}} \\
&= L_0\,E A_0\,\frac{\partial e}{\partial L}\,\frac{\partial e}{\partial L}\,\frac{\partial L}{\partial \mathbf{u}}\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T + L_0\,N\left[\frac{\partial^2 e}{\partial L^2}\,\frac{\partial L}{\partial \mathbf{u}}\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T + \frac{\partial e}{\partial L}\,\frac{\partial^2 L}{\partial \mathbf{u}\partial \mathbf{u}}\bigg)\right] \\
&= E A_0\,L_0\Big(\frac{\partial e}{\partial L}\Big)^2\,\mathbf{h}\,\mathbf{h}^T + N\,L_0\Big(\frac{\partial^2 e}{\partial L^2}\mathbf{h}\,\mathbf{h}^T + \frac{\partial e}{\partial L}\mathbf{G}\Big) \\
&= E A_0\,L_0\Big(\frac{\partial e}{\partial L}\Big)^2\,\mathbf{h}\,\mathbf{h}^T + N\,L_0\left[\frac{1}{L}\,\frac{\partial e}{\partial L}\mathbf{J} + \left(\frac{\partial^2 e}{\partial L^2} - \frac{1}{L}\,\frac{\partial e}{\partial L}\right)\mathbf{h}\mathbf{h}^T\right] = \mathbf{K}_M + \mathbf{K}_G.
\end{aligned} \tag{14.20}$$

Here $\mathbf{K}_M$ and $\mathbf{K}_G$ denote the material and geometric stiffness, a decomposition already encountered in the TL description of Chapters 8–9.

§14.3.2. Matrices for Specific Strain Measures

Some specific strain measures and the values of their partials with respect to L are collected in Table 14.1. The appropriate choice should be replaced in (14.19) and (14.20) to get the final form of the internal force vector and tangent stiffness matrix.

Example 14.1. If the engineering strain $e = e_E = (L - L_0)/L_0$ is used,

$$\mathbf{p} = N\,\mathbf{h}, \quad \mathbf{K}_M = \frac{E A_0}{L_0}\mathbf{h}\,\mathbf{h}^T, \quad \mathbf{K}_G = \frac{N}{L}\mathbf{G}, \quad \mathbf{K} = \mathbf{K}_M + \mathbf{K}_G. \tag{14.21}$$

Example 14.2. If the Green-Lagrange strain $e = e_G = (L^2 - L_0^2)/(2L_0^2)$ is used,

$$\mathbf{p} = N\,\frac{L}{L_0}\mathbf{h}, \quad \mathbf{K}_M = \frac{E A_0 L^2}{L_0^3}\mathbf{h}\,\mathbf{h}^T, \quad \mathbf{K}_G = \frac{N}{L_0}\mathbf{J} = \frac{N}{L_0}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \mathbf{K}_M + \mathbf{K}_G. \tag{14.22}$$

For this choice the geometric stiffness $\mathbf{K}_G$ has the same form as that of the TL bar element derived in Chapter 8. The material matrix, however, is different unless one sets $L = L_0$.

Example 14.3. If the Hencky strain $e = e_H = \log(L/L_0)$ is used,

$$\mathbf{p} = N\,\frac{L_0}{L}\mathbf{h}, \quad \mathbf{K}_M = \frac{E A_0 L_0}{L^2}\mathbf{h}\,\mathbf{h}^T, \quad \mathbf{K}_G = \frac{N L_0}{L^2}(\mathbf{J} - 2\mathbf{h}\,\mathbf{h}^T), \quad \mathbf{K} = \mathbf{K}_M + \mathbf{K}_G. \tag{14.23}$$

```
CRSpaceBar2InternalForce[ncoor0_,Em_,A0_,ue_,N0_,sm_,numer_]:=
  Module[{X1,Y1,Z1,X2,Y2,Z2,LX,LY,LZ,LL0,L0,x1,y1,z1,x2,y2,z2,
          Lx,Ly,Lz,LL,L,Lavg,d,EA,e,dedL,d2edL2,ND,h,p},
  {{X1,Y1,Z1},{X2,Y2,Z2}}=ncoor0; {{uX1,uY1,uZ1},{uX2,uY2,uZ2}}=ue;
  {LX,LY,LZ}={X2-X1,Y2-Y1,Z2-Z1}; LL0=LX^2+LY^2+LZ^2; L0=Sqrt[LL0];
  {{x1,y1,z1},{x2,y2,z2}}=ncoor0+ue;
  {Lx,Ly,Lz}={x2-x1,y2-y1,z2-z1}; LL=Lx^2+Ly^2+Lz^2; L=Sqrt[LL];
  d=L-L0; Lavg=(L+L0)/2; e=Null;
  If [sm=="eE", e=d/L0;        dedL=1/L0;      d2edL2=0];
  If [sm=="eG", e=d*Lavg/L0^2; dedL=L/L0^2;    d2edL2=1/L0^2];
  If [sm=="eH", e=Log[L/L0];   dedL=1/L;       d2edL2=-1/L^2];
  If [sm=="eM", e=d/Lavg;      dedL=L0/Lavg^2; d2edL2=-L0/Lavg^3];
  If [e==Null,Print["CRSpaceBar2Force: Illegal sm arg"]; Return[Null]];
  ND=N0+Em*A0*e;  h={-Lx,-Ly,-Lz,Lx,Ly,Lz}/L;
  p=ND*L0*dedL*h; If [numer,p=N[p]];
  Return[p]];

ClearAll[ ];  ncoor0=N[{{0,0,0},{11,10,2}}];
Em=5000; A0=3; N0=10; numer=True;
For [iε=1, iε<=4, iε++,  ={0,0.0001,0.01,0.1}[[iε]];
     ue=N[{{2,3,-4},{-4,-5,8}}*(1+ε)^2];
     For [ism=1,ism<=4,ism++,  sm={"eE","eG","eH","eM"}[[ism]];
     p=CRSpaceBar2InternalForce[ncoor0,Em,A0,ue,N0,sm,numer];
     Print["sm: ",sm,",  ε=", ε,",  p=",p];
    ]];
```

FIGURE 14.8. Mathematica implementation of internal force vector calculation for 2-node space bar.

§14.3.3. Mathematica Implementation

The calculation of the internal force vector for the CR space bar is implemented via module `CRSpaceBar2InternalForce`. This is listed in Figure 14.8, along with test statements described below.

```
sm: eE,  ϵ=0,  p={-3.33333, -1.33333, -9.33333, 3.33333, 1.33333, 9.33333}
sm: eG,  ϵ=0,  p={-3.33333, -1.33333, -9.33333, 3.33333, 1.33333, 9.33333}
sm: eH,  ϵ=0,  p={-3.33333, -1.33333, -9.33333, 3.33333, 1.33333, 9.33333}
sm: eM,  ϵ=0,  p={-3.33333, -1.33333, -9.33333, 3.33333, 1.33333, 9.33333}
sm: eE,  ϵ=0.0001,  p={-3.87431, -1.54886, -10.8525, 3.87431, 1.54886, 10.8525}
sm: eG,  ϵ=0.0001,  p={-3.87476, -1.54904, -10.8538, 3.87476, 1.54904, 10.8538}
sm: eH,  ϵ=0.0001,  p= {-3.87386, -1.54868, -10.8513, 3.87386, 1.54868, 10.8513}
sm: eM,  ϵ=0.0001,  p={-3.87386, -1.54868, -10.8513, 3.87386, 1.54868, 10.8513}
sm: eE,  ϵ=0.01,  p={-56.577, -21.3257, -165.128, 56.577, 21.3257, 165.128}
sm: eG,  ϵ=0.01,  p={-57.5008, -21.6739, -167.824, 57.5008, 21.6739, 167.824}
sm: eH,  ϵ=0.01,  p={-55.6686, -20.9833, -162.477, 55.6686, 20.9833, 162.477}
sm: eM,  ϵ=0.01,  p={-55.6664, -20.9824, -162.47, 55.6664, 20.9824, 162.47}
sm: eE,  ϵ=0.1,  p={-430.732, -36.854, -1902.59, 430.732, 36.854, 1902.59}
sm: eG,  ϵ=0.1,  p={-517.786, -44.3025, -2287.12, 517.786, 44.3025, 2287.12}
sm: eH,  ϵ=0.1,  p={-358.761, -30.6961, -1584.69, 358.761, 30.6961, 1584.69}
sm: eM,  ϵ=0.1,  p={-356.998, -30.5453, -1576.9, 356.998, 30.5453, 1576.9}
```

FIGURE 14.9. Results obtained by running the script of Figure 14.8.

The module is invoked as

$$\texttt{p = CRSpaceBar2InternalForce[ncoor0, Em, A0, ue, N0, sm, numer]} \tag{14.24}$$

The arguments are

`ncoor0`	Node coordinates of element base configuration, arranged as `{{X1,Y1,Z1},{X2,Y2,Z2}}`.
`Em`	Elastic modulus.
`A0`	Cross section area in base configuration.
`ue`	Node displacements from base to deformed configuration, arranged as `{{uX1,uY1,uZ1},{uX2,uY2,uZ2}}`.
`N0`	Axial force in base configuration.
`sm`	A two-character string that specifies which strain measure to use for the bar constitutive equations: `"eE"`, `"eG"`, `"eH"` and `"eM"` for the engineering, Green-Lagrange, Hencky and midpoint measures, respectively. If `sm` is not one of these, an error message is printed and the module returns `Null`.
`numer`	A logical logical flag with the value `True` or `False`. If `True` the computations are carried out in floating-point arithmetic. If `False` symbolic processing is assumed.

The module returns as function value

`p`	The internal force vector arranged as a one-dimensional list: `{pX1,pY1,pZ1,pX2,pY2,pZ2}`.

The module is numerically exercised by the test statements shown at the bottom of Figure 14.8. The test case corresponds to the following data: $E = 5000$, $A_0 = 3$, $N_0 = 10$, base configuration node coordinates $(X_1, Y_1, Z_1) = (0, 0, 0)$ and $(X_2, Y_2, Z_2) = (11, 12, 2)$ and node displacements $(u_{X1}, u_{Y1}, u_{Z1} = (2, 3, -4) * (1 + \epsilon)^2$ and $(u_{X2}, u_{Y2}, u_{Z2}) = (-4, -5, 8) * (1 + epsilon)^2$, in which ϵ is an adjustable parameter. If $\epsilon = 0$ the bar moves in space but does not change length: $L_0 = L = 15$. If $\epsilon > 0$ the bar stretches by approximately ϵL_0.

The four strain measures e_E, e_G, e_H and e_M are exercised by cycling over `sm = "eE"`, `"eG"`, `"eH"`, and `"eM"`, respectively, in the inner test loop. For values of ϵ: 0, 0.0001, 0.01 and 0.1 are tested in the outer test loop.

As can be observed the internal force vectors are identical if $\epsilon = 0$, since for zero strain it does not matter which measure is used. The forces are very close for the different measures if $\epsilon = 0.0001$, which is very small strain (roughly 100 micros) but they differ substantially as ϵ gets large.

The implementation and verification of the tangent stiffness matrix is deferred to Exercises 14.3 and 14.4.

The formulation of the mass and damping matrices for dynamic analysis is not provided in this Chapter.

Homework Exercises for Chapter 14

The CR Formulation: Space Bar

EXERCISE 14.1 [A:15] Verify by hand the formulas given in §14..3.2.

EXERCISE 14.2 [A:15] Verify the expressions given in Examples 14.1–3, and append to these the $\mathbf{p}$, $\mathbf{K}_M$ and $\mathbf{K}_G$ matrices for the choice $e = e_M$.

EXERCISE 14.3 [C:20] Implement the calculation of the tangent stiffness matrix for the space bar element (any language is OK) to return $\mathbf{K}_M$ and $\mathbf{K}_G$. If done in *Mathematica*, the code of Figure 14.8, which is posted on the web site linked to the Chapter 14 Index, may be used as template. It is convenient to compute and return $\mathbf{K}_M$ and $\mathbf{K}_G$ as two separate matrices as function value in *Mathematica*: `Return[{KM,KG}]`.

As numerical test, run the bar used in the test statements of that figure using $\epsilon = 0$ and any strain measure. Print out $\mathbf{K}_M$, $\mathbf{K}_G$, $\mathbf{K} = \mathbf{K}_M + \mathbf{K}_G$ and the eigenvalues of the three matrices. Validation check: the rank of $\mathbf{K}_M$, $\mathbf{K}_G$ and $\mathbf{K}$ should be 1, 3 and 3, respectively (except in the case of engineering strain, in which case the rank should be 1, 2 and 3, respectively.)

EXERCISE 14.4 [C:20] Using the module developed in the previous Exercise, compute and show $\mathbf{K}_M$, $\mathbf{K}_G$ and $\mathbf{K}$ for the bar data used in the example of Figure 14.8. For the inner loop cycle over the 4 strain measure choices used in the internal force test. For the outer test loop, cycle over ϵ equal to 0, 0.01 and -0.01. Compute and show the eigenvalues of the three matrices. Validation check: the rank of $\mathbf{K}_M$, $\mathbf{K}_G$ and $\mathbf{K}$ should be 1, 3 and 3, respectively. For $\epsilon = -0.01$ you may see some negative eigenvalues in $\mathbf{K}_G$ and $\mathbf{K}$; do not be alarmed.