5

Critical Points and Related Properties

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This Chapter provides additional material on properties of the one-parameter force residual equations. It begins with a study of critical points, which are classified into limit and bifurcation points. Limit point "sensors" and turning points are briefly described. The section conclude with some mathematical derivations for conservative systems, which will be of use in Chapters dealing with incremental solution methods.

§5.1. Classification of Critical Points

Throughout this Chapter it is assumed that the structural system is conservative[1] and consequently **K** is *symmetric*.

Response points at which **K** becomes singular are of great interest in the applications because of their intimate connection to structural *stability*. These are called *critical points*, and also *nonregular* or *singular* points. At these points the velocity vector **v** is not uniquely determined by **q** from (4.8). Physically this means that the structural behavior cannot be controlled by the parameter λ.

It is convenient to distinguish the following types of critical points:

1. *Isolated limit points*, at which the tangent (4.17) to the equilibrium path is unique but normal to the λ axis so **v** becomes infinitely large.

2. *Multiple limit points*, at which there the tangent lies in the null space of **K** and is not unique but still normal to the λ axis.

3. *Isolated bifurcation points*, also called *branch points* or *branching points*, from which two equilibrium path branches emanate and so there is no unique tangent. The rank deficiency of **K** is one.

4. *Multiple bifurcation points*, from which more than two equilibrium path branches emanate. The rank deficiency of **K** is two or greater.

A critical point that is both a limit and a bifurcation point is classified as a multiple bifurcation point.

Figures 5.1, 5.2 and 5.3 illustrate isolated limit points (identified as $L_1,\ L_2,\ \ldots$) and bifurcation points (identified as $B_1,\ B_2,\ \ldots$).

To classify critical points we proceed as follows. Let **z** be a null right eigenvector of **K** at a critical point, that is,

$$\mathbf{K}\mathbf{z} = \mathbf{0}. \tag{5.1}$$

Since **K** is assumed symmetric, $\mathbf{z}^T\mathbf{K} = \mathbf{0}$; that is, **z** is also a left null eigenvector. The parametric differential equation of the equilibrium path is $\mathbf{K}\dot{\mathbf{u}} = \mathbf{q}\,\dot{\lambda}$, which multiplied through by dt becomes

$$\boxed{\mathbf{K}\,d\mathbf{u} = \mathbf{q}\,d\lambda.} \tag{5.2}$$

Premultiply both sides of (5.2) by $\mathbf{z}^T$ and use $\mathbf{z}^T\mathbf{K} = \mathbf{0}$ to get

$$\mathbf{z}^T\mathbf{q}\ d\lambda = 0. \tag{5.3}$$

[1] A property elaborated upon in Chapter 6. This property is important in that a symmetric **K** is guaranteed to have a full set of eigenvectors. Furthermore left and right eigenvectors coalesce.

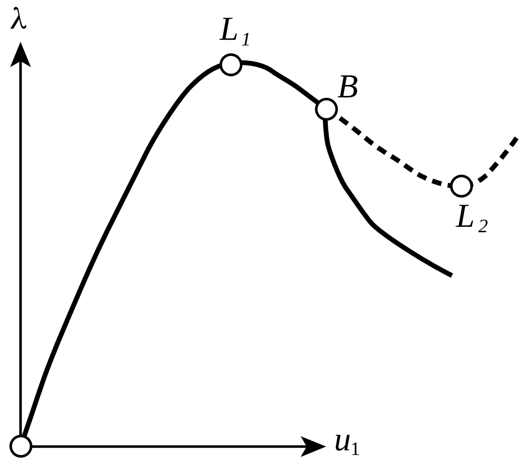

Figure 5.1. Limit points L_1 and L_2 and bifurcation point (B) for a two degree of freedom system (u_1, u_2) shown on the u_1 versus λ plane. Limit point ("snap through" behavior) occurs before bifurcation. Full lines represent physically "preferred" paths.

Two cases may be considered. If

$$\boxed{\mathbf{z}^T\mathbf{q} \neq 0,} \tag{5.4}$$

then $d\lambda$ must vanish, and we have a limit point. The point is isolated if $\mathbf{z}$ is the only null eigenvector and multiple otherwise.

On the other hand, if

$$\boxed{\mathbf{z}^T\mathbf{q} = 0,} \tag{5.5}$$

then we have a bifurcation or branching point. The point is isolated if $\mathbf{z}$ is the only null eigenvector and multiple otherwise. The key physical characteristic of a bifurcation point is *an abrupt transition from one deformation mode to another mode*; the latter having been previously "concealed" by virtue of being orthogonal to the incremental load vector.

Remark 5.1. If $\mathbf{K}$ is not symmetric, several changes must be made in the previous assumptions and derivations. These are explained in the Chapters that deal with nonconservative systems (29-30) and the possible loss of stability by growing dynamic oscillations (flutter).

Remark 5.2. If λ is an applied load multiplier, a limit point such as L_1 in Figure 5.1 is called a *snap-through* point because the structure "snaps" dynamically to another equilibrium position. The term *collapse* applies to critical points beyond which the structure becomes useless.

Remark 5.3. As an isolated limit point is approached, $\mathbf{v}$ tends to become parallel to $\mathbf{z}$ whereas its magnitude goes to ∞; that is

$$\frac{\mathbf{v}}{|\mathbf{v}|} \to \mathbf{z}. \tag{5.6}$$

Consequently the normalized $\mathbf{v}$ may be a good eigenvector estimate if $\mathbf{K}$ has been factored near the limit point. (This is nothing more that a statement of the well known inverse iteration process for finding eigenvectors.)

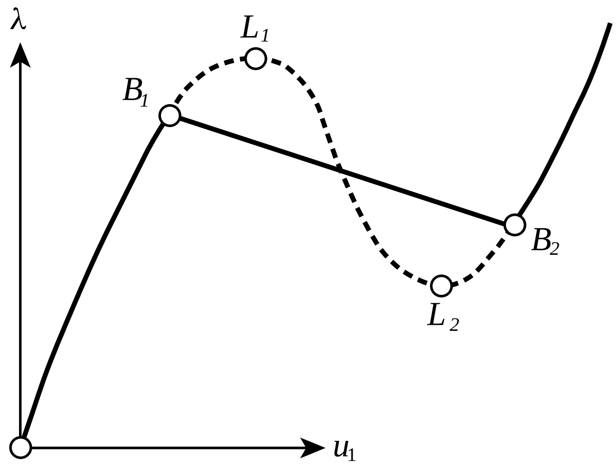

Figure 5.2. Similar to Figure 5.1, but here the bifurcation point B_1 implying "buckling" behavior occurs before the limit point L_1, which is physically unreachable. A more realistic three-dimensional view of this case is shown in Figure 5.3.

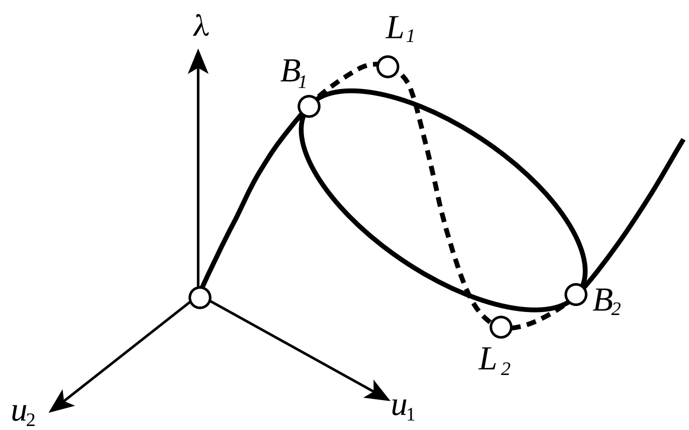

Figure 5.3. The equilibrium path of Figure 5.2 shown in the 3D space (u_1, u_2, λ). This is the type of response exhibited by a uniformly pressurized deep arch, for which u_1 and u_2 are amplitudes of the symmetric and antisymmetric deformation mode, respectively, and λ is a pressure multiplier.

Remark 5.4. The set of control parameters for which $\det \mathbf{K} = 0$ while $\mathbf{r}(\mathbf{u}, \lambda) = \mathbf{0}$ is sometimes called the *bifurcation set* in the mathematical literature. The name is misleading, however, in that the set may include limit points; the name *critical set* would be more appropriate.

Remark 5.5. Showing bifurcation points on the λ versus u plane as in Figures 5.1 and 5.2 may be misleading, as it conceals the phenomenon of transition from one mode of deformation to another. A more realistic picture is provided in Figure 5.3, which shows the classical bifurcation behavior for a symmetrically loaded shallow arch; here u_1 and u_2 measure amplitude of symmetric and antisymmetric displacement shapes, respectively. At B_1 the arch, which had been previously deforming symmetrically, takes off along an antisymmetric deformation mode; at B_2 the latter disappears and the arch rejoin the symmetric path.

Remark 5.6. Physically the distinction between the two types of critical points is not so marked, inasmuch as imperfect structures display limit-point behavior. A bifurcation point may be viewed as the limit of a sequence of progressively sharper limit points realized as the structure strives towards mathematical perfection.

Remark 5.7. For a readable introduction to elastic structural stability along the lines of classical perturbation theory, the monographs by Thompson and Hunt [181,183] are still unsurpassed. The treatise by Knops and Wilkes [100] goes deeper into various mathematical questions, but is computationally useless. Brush and Almroth [37] give more information on computational methods. The survey and book by Bushnell [40] has more physics (*e.g.*, temperature, plasticity and creep effects) and a wider selection of practical problems. The connection between potential-based structural stability and modern catastrophe theory is presented in a highly readable manner by Poston and Steward [141] and Thompson [182].

§5.2. Limit Point Sensors

Scalar estimates of the overall stiffness of the structure as the control parameter varies are useful as limit points sensors. The following estimator is based on the Rayleigh quotient approximation to the fundamental eigenvalue of **K**:

$$k_{\mathbf{x}} = \frac{\mathbf{x}^T \mathbf{K}\mathbf{x}}{\mathbf{x}^T \mathbf{x}}, \tag{5.7}$$

where **x** is an arbitrary nonnull vector, and **K** is evaluated at an equilibrium position $\mathbf{u}(\lambda)$. An "equilibrium-path stiffness" estimator is obtained by taking **x** to be $\mathbf{v} = \mathbf{K}^{-1}\mathbf{q}$, in which case

$$k = k_{\mathbf{v}} = \frac{\mathbf{q}^T \mathbf{v}}{\mathbf{v}^T \mathbf{v}}. \tag{5.8}$$

This value of course depends on λ. It is convenient in practice to work with the dimensionless ratio

$$\kappa = k(\lambda)/k(0), \tag{5.9}$$

which takes the value 1 at the start of an analysis stage, and goes to zero as a limit point is approached. A stiffness estimator with this behavior (although computed in a different way) was introduced by Bergan and coworkers under the name *current stiffness parameter*. It should be noted, however, that no estimator of this type can reliably predict the occurrence of a bifurcation point. Sensors for such points are described later in the context of augmented equations.

§5.3. *Turning Points

Turning points are regular points at which the tangent is parallel to the λ axis so that $\mathbf{v} = \mathbf{0}$. The unit tangent takes the form

$$\mathbf{t}_u = \begin{bmatrix} \mathbf{0} \\ \pm 1 \end{bmatrix} \tag{5.10}$$

Although these points generally do not have physical meaning, they can cause special problems in path-following solution procedures because of "turnback" effects.

To detect the vicinity of a turning point one can check the two mathematical conditions: **v** becomes orthogonal to **q** and **u** tends to zero faster than **q**. For example:

$$|\cos(\mathbf{v}, \mathbf{q})| < \delta \qquad |\kappa| > \kappa_{min}, \tag{5.11}$$

where κ is the current stiffness parameter. Typical values may be $\delta = 0.01$, $\kappa_{min} = 100$.

§5.4. *Derivatives of Energy Functions

If the residual $\mathbf{r}(\mathbf{u},\lambda)$ is derivable from a total potential energy function $\Pi(\mathbf{u},\lambda)$ as in (3.2), then the stiffness matrix and incremental load vector appear naturally as components of the following matrix of second derivatives:

$$\begin{bmatrix} \dfrac{\partial^2\Pi}{\partial\mathbf{u}\partial\mathbf{u}} & \dfrac{\partial^2\Pi}{\partial\mathbf{u}\,\partial\lambda} \\ \left(\dfrac{\partial^2\Pi}{\partial\lambda\partial\mathbf{u}}\right)^T & \dfrac{\partial^2\Pi}{\partial\lambda\partial\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{K} & -\mathbf{q} \\ -\mathbf{q}^T & a \end{bmatrix} \tag{5.12}$$

where $a = \partial^2\Pi/\partial\lambda^2$ has not been introduced previously. Obviously the tangent stiffness matrix $\mathbf{K}$ (the Hessian of Π) is now symmetric. Note also that

$$\frac{\partial\mathbf{q}}{\partial\mathbf{u}} = \frac{\partial^3\Pi}{\partial\mathbf{u}\,\partial\lambda\partial\mathbf{u}} = \frac{\partial}{\partial\lambda}\frac{\partial^2\Pi}{\partial\mathbf{u}\partial\mathbf{u}} = \frac{\partial\mathbf{K}}{\partial\lambda} = \mathbf{K}_\lambda, \tag{5.13}$$

is a symmetric matrix.

The complementary energy function Π^* may be defined from the dual Legendre transformation (see *e.g.*, Chapter 2.5 of Sewell's book [163]) as

$$\Pi + \Pi^* = u_i\,\frac{\partial\Pi}{\partial u_i} = \mathbf{u}^T\mathbf{r} = \mathbf{r}^T\mathbf{u}. \tag{5.14}$$

This gives $\Pi^*(\mathbf{r},\lambda) = \mathbf{r}^T\mathbf{u} - \Pi$ with $\mathbf{u}$ eliminated from $\mathbf{r}(\mathbf{u},\lambda) = 0$, so now the residual forces are the active variables. Obviously

$$\mathbf{u} = \frac{\partial\Pi^*}{\partial\mathbf{r}}, \qquad \text{or} \quad u_i = \frac{\partial\Pi^*}{\partial r_i}. \tag{5.15}$$

The matrix of second derivatives of Π^* is

$$\begin{bmatrix} \dfrac{\partial^2\Pi^*}{\partial\mathbf{r}\partial\mathbf{r}} & \dfrac{\partial^2\Pi^*}{\partial\mathbf{r}\partial\lambda} \\ \left(\dfrac{\partial^2\Pi^*}{\partial\lambda\partial\mathbf{r}}\right)^T & \dfrac{\partial^2\Pi^*}{\partial\lambda\partial\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{v} \\ \mathbf{v}^T & b \end{bmatrix} \tag{5.16}$$

These are linked to the quantities that appear in (5.12) by the matrix relations

$$\mathbf{F} = \mathbf{K}^{-1}, \qquad \mathbf{v} = \mathbf{K}^{-1}\mathbf{q} = \mathbf{F}\mathbf{q}, \qquad b = \mathbf{q}^T\mathbf{K}^{-1}\mathbf{q} - a. \tag{5.17}$$

The converse relations are

$$\mathbf{K} = \mathbf{F}^{-1}, \qquad \mathbf{q} = \mathbf{K}\mathbf{v}, \qquad a = \mathbf{v}^T\mathbf{K}\mathbf{v} - b. \tag{5.18}$$

The tangent flexibility matrix $\mathbf{F} = \mathbf{K}^{-1}$ (the Hessian of K) is now symmetric. Note also that

$$\frac{\partial\mathbf{u}}{\partial\mathbf{r}} = \frac{\partial^3\Pi^*}{\partial\mathbf{r}\,\partial\lambda\partial\mathbf{r}} = \frac{\partial}{\partial\lambda}\frac{\partial^2\Pi^*}{\partial\mathbf{r}\partial\mathbf{r}} = \frac{\partial\mathbf{F}}{\partial\lambda} = \mathbf{F}_\lambda, \tag{5.19}$$

is a symmetric matrix.

Remark 5.8. The following matrix appears (as amplification matrix) in the study of the stability of incremental methods:

$$\mathbf{A} = \frac{\partial \mathbf{v}}{\partial \mathbf{u}} = \frac{\partial (\mathbf{F}\mathbf{q})}{\partial \mathbf{u}} = \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\mathbf{q} + \mathbf{F}\frac{\partial \mathbf{q}}{\partial \mathbf{u}} = \frac{\partial \mathbf{F}}{\partial \mathbf{r}}\mathbf{K}\mathbf{q} + \mathbf{F}\frac{\partial \mathbf{K}}{\partial \lambda}. \tag{5.20}$$

Although $\mathbf{A}$ is unsymmetric, under some general conditions it has real eigenvalues. To show that we express $\mathbf{A}$ as the product of two symmetric matrices:

$$\mathbf{A} = \frac{\partial \mathbf{v}}{\partial \mathbf{u}} = \frac{\partial \mathbf{v}}{\partial \mathbf{r}}\frac{\partial \mathbf{r}}{\partial \mathbf{u}} = \frac{\partial \mathbf{F}}{\partial \lambda}\mathbf{K} = \mathbf{F}_\lambda \mathbf{K}, \tag{5.21}$$

where the relation (5.19) has been used. If $\mathbf{F}_\lambda$ is nonsingular, the eigensystem $\mathbf{A}\mathbf{x}_i = \mu_i \mathbf{x}_i$ can be transformed to the generalized symmetric eigenproblem

$$\mathbf{K}\mathbf{x}_i = \mu_i \mathbf{F}_\lambda^{-1}\mathbf{x}_i. \tag{5.22}$$

If $\mathbf{K}$ is positive definite this system has nonzero real roots μ_i. If $\mathbf{F}_\lambda$ is singular but $\mathbf{K}$ positive definite, consideration of the alternative eigensystem

$$\mathbf{F}_\lambda \mathbf{y}_i = \mu_i \mathbf{K}^{-1}\mathbf{y}_i = \mu_i \mathbf{F}\mathbf{y}_i, \tag{5.23}$$

shows that such a singularity contributes only zero roots.

Remark 5.9. Another quantity that appears in the analysis of incremental methods is the vector

$$\mathbf{v}' = \frac{\partial \mathbf{v}}{\partial \lambda} = \frac{\partial \mathbf{v}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \lambda} = \mathbf{A}\mathbf{v} = \mathbf{F}_\lambda \mathbf{K}\mathbf{v} = \mathbf{F}_\lambda \mathbf{q}. \tag{5.24}$$

Remark 5.10. Two other Legendre transforms may be constructed: $X(\delta, \mathbf{u})$ and $Y(\delta, \mathbf{r})$, in which $\delta = \partial\Pi/\partial\lambda$ (a generalized displacement if λ is a load multiplier) is the active variable and either $\mathbf{u}$ or $\mathbf{r}$ take the role of passive variables. X and Y together with Π and K form a closed chain of Legendre transformations. The functions X and Y are, however, of limited interest in the present context.

§5.5. *Energy Increments

In this section we continue to assume that $\mathbf{r}$ is derivable from the potential $\Pi = U - W$. For questions such as positive path traversal it is interesting to obtain an expression of the energy increment on passing from an equilibrium position $(\mathbf{u}, \lambda)$ to a neighboring configuration $(\mathbf{u} + \Delta\mathbf{u}, \lambda + \Delta\lambda)$ on the equilibrium path:

$$\Delta\Pi = \Pi(\mathbf{u} + \Delta\mathbf{u}, \lambda + \Delta\lambda) - \Pi(\mathbf{u}, \lambda). \tag{5.25}$$

First we note that adding an arbitrary function of λ to Π

$$\Pi + F(\lambda), \tag{5.26}$$

does not change the equilibrium equations or rate forms. To second order in the increments we get

$$\Delta\Pi = \mathbf{r}^T \Delta\mathbf{u} + A\,\Delta\lambda + \tfrac{1}{2}\Delta\mathbf{u}^T \mathbf{K}\,\Delta\mathbf{u} - \mathbf{q}^T \Delta\mathbf{u}\,\Delta\lambda + \tfrac{1}{2}a(\Delta\lambda)^2, \tag{5.27}$$

with

$$A = \frac{\partial \Pi}{\partial \lambda}, \qquad a = \frac{\partial^2 \Pi}{\partial \lambda^2}, \tag{5.28}$$

evaluated at $(\mathbf{u}, \lambda)$. But we can always adjust $F(\lambda)$ in (5.25) so that $A = a = 0$. Furthermore at an equilibrium position $\mathbf{r} = \mathbf{0}$, and along the equilibrium path $\Delta\mathbf{u} = \mathbf{K}^{-1}\mathbf{q}\,\Delta\lambda = \mathbf{u}^T \Delta\lambda$. Substituting we find for the energy increment

$$\Delta\Pi = \Delta U - \Delta W = \tfrac{1}{2}\mathbf{q}^T \mathbf{u}(\Delta\lambda)^2 - \mathbf{q}^T \mathbf{u}(\Delta\lambda)^2 = -\tfrac{1}{2}\mathbf{q}^T \mathbf{u}(\Delta\lambda)^2. \tag{5.29}$$

This formula displays the important function of the product $\mathbf{q}^T \mathbf{u}$ in the energy increment. By extension we may call

$$\Delta W = \mathbf{q}^T \mathbf{u}(\Delta\lambda)^2 \tag{5.30}$$

the *external work increment* even if $\mathbf{r}$ does not derive from a potential.

To fix the ideas assume that $\mathbf{r}$ derives from a *quadratic* potential

$$\Pi = \tfrac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{q}^T\mathbf{u}\lambda + C\lambda + D, \tag{5.31}$$

where C and D are arbitrary constants. Then the increment $\Delta\Pi$ from an equilibrium position $(\mathbf{u}, \lambda)$ that satisfies the linear relation $\mathbf{K}\mathbf{u} = \mathbf{q}^T\lambda$, to an arbitrary configuration $(\mathbf{u} + \Delta\mathbf{u}, \lambda + \Delta\lambda)$ is

$$\Delta\Pi = \Delta\mathbf{u}^T(\mathbf{K}\mathbf{u} - \mathbf{q}^T\lambda) + \Delta\lambda(\mathbf{q}^T\mathbf{u} - C) = -\Delta\lambda(\mathbf{q}^T\mathbf{u} - C) = -(\mathbf{q}^T\mathbf{v}\lambda - C)\Delta\lambda. \tag{5.32}$$

Since C is arbitrary, chose it so that $\partial\Pi/\partial\lambda = -\mathbf{q}^T\mathbf{u} + C = 0$. Then

$$\Delta\Pi = -\mathbf{q}^T\mathbf{v}\Delta(\tfrac{1}{2}\lambda^2). \tag{5.33}$$

Homework Exercise for Chapter 5

Critical Points and Related Properties

EXERCISE 5.1 Given the one-parameter, two-degree-of-freedom residual-force system

$$\mathbf{r}(u_1, u_2, \lambda) = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 6u_1 - 2u_2 - u_1^2 - 12\lambda \\ -2u_1 + 4u_2 - u_2^2 + 2\lambda \end{bmatrix} \tag{E5.1}$$

Consider the point $P(u_1, u_2, \lambda)$ located at

$$u_1 = 2, \quad u_2 = 1, \quad \lambda = \tfrac{1}{2}, \tag{E5.2}$$

(a) Show that P is on an equilibrium path,

(b) Show that P is a critical point,

(c) Determine whether it is a limit or a bifurcation point. [Compute the null eigenvector $\mathbf{z}$ of $\mathbf{K}$ at that point].

(d) Verify whether the limit point sensor κ is zero at P.

EXERCISE 5.2 Show that all critical points of (E5.1) satisfy either of the equations

$$63 - u_1 - 36u_2 = 0, \quad 5 - 2u_1 - 3u_2 + u_1 u_2 = 0 \tag{E5.3}$$

called *critical point surfaces*, and that the only intersection of these surfaces and the equilibrium path is at (E5.2).

EXERCISE 5.3 Show that the critical point surface defined by $\det(\mathbf{K}) = 0$ is independent of λ if the residual force system is separable.

EXERCISE 5.4 Show that $\mathbf{q}^T\mathbf{z}$ is independent of λ if the residual force system is separable and the load is proportional.

EXERCISE 5.5 (Advanced, requires knowledge of matrix eigensystem theory). If $\mathbf{K}$ is not symmetric, the critical point classification argument based on $\mathbf{q}^T\mathbf{z}$ fails. Explain why.