2

A Tour of Nonlinear Analysis

TABLE OF CONTENTS

§2.1. Introduction

This chapter reviews nonlinear structural problems by looking at the manifestation and physical sources of nonlinear behavior.

We begin by introducing *response* as a pictorial characterization of nonlinearity of a structural system. Response is a graphical representation of the fundamental concept of *equilibrium path*. This concept permeates the entire course because of both its intrinsic physical value and the fact that incremental solution methods (mentioned in Chapter 1) are based on it.

Finally, nonlinearities are classified according to their source in the mathematical model of continuum mechanics and correlated with the physical system. Examples of these nonlinearities in practical engineering applications are given.

§2.2. Equilibrium Path and Response Diagrams

The concept of *equilibrium path* plays a central role in explaining the mysteries of nonlinear structural analysis. This concept lends itself to graphical representation in the form of *response diagrams*. The most widely used form of these pictures is the *load-deflection* response diagram. Through this representation many key concepts can be illustrated and interpreted in physical, mathematical or computational terms.

§2.2.1. Load-deflection response

The *gross* or *overall* static behavior of many structures can be characterized by a *load-deflection* or *force-displacement* response. The response is usually drawn in two dimensions as a x-y plot as illustrated in Figure 2.1. In this figure a "representative" force quantity is plotted against a "representative" displacement quantity. If the response plot is nonlinear, the structure behavior is nonlinear.

Remark 2.1. We will see below that a response diagram generally depicts the relationship between inputs and outputs. Or, in more physical terms, between what is applied and what is measured. For structures the most common inputs are forces and the most common outputs are displacements or deflections[1]

Remark 2.2. The qualifier "representative" implies a choice among many possible candidates. For relatively simple structures the choice of load and deflection variables is often clear-cut from considerations such as the availability of experimental data. For more complex structures the choice may not be obvious, and many possibilities may exist. The load is not necessarily an applied force but may be an integrated quantity: for example the weight of traffic on a bridge, or the lift on an airplane wing.

Remark 2.3. This type of response should not be confused with what in structural dynamics is called the *response time history*. A response history involves time, which is the independent variable, plotted usually along the horizontal axis, with either inputs or outputs plotted vertically.

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[1] A deflection is the magnitude or amplitude of a displacement. Displacements are vector quantities whereas deflections are scalars.

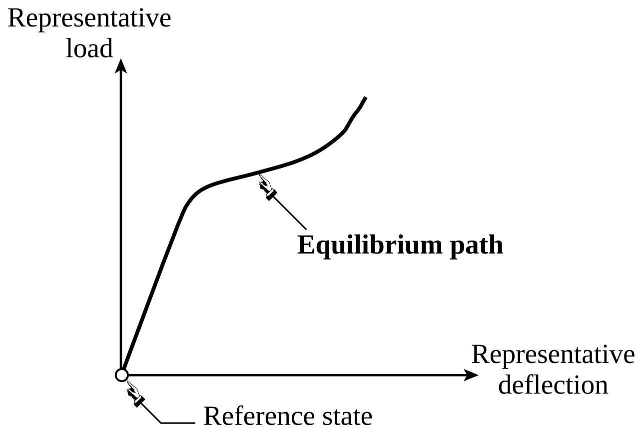

Figure 2.1. A load-deflection response diagram.

§2.2.2. Terminology

A *smooth* curve shown in a load-deflection diagram is called a *path*.[2] Each point in the path represents a possible *configuration* or *state* of the structure. If the path represents configurations of *static equilibrium* it is called an *equilibrium path*. Each point in an equilibrium path is called an *equilibrium point*. An equilibrium point is the graphical representation of an *equilibrium state* or *equilibrium configuration*.

The origin of the response plot (zero load, zero deflection) is called the *reference state* because it is the configuration from which loads and deflections are measured. However, the reference state may be in fact chosen rather arbitrarily, and this freedom is exploited in some nonlinear formulations and solution methods, as we shall see later.

For problems involving *perfect* structures[3] the reference state is unstressed and undeformed, and is also an equilibrium state. This means that an equilibrium path passes through the reference state, as in Figure 2.1.

The path that crosses the reference state is called the *fundamental equilibrium path* or *fundamental path* for short. (Many authors also call this a *primary path*.) The fundamental path extends from the reference state up to special states called *critical points* informally described in §2.3. Any path that is not a fundamental path but connects with it at a critical point is called a *secondary path*. See Figure 2.2.

§2.3. Special Equilibrium Points

Certain points of an equilibrium path have special significance in the applications and thus receive special names. Of interest to our subject are critical, turning and failure points.

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[2] The terms *branch* and *trajectory* are also used. "Branch" is commonly used in the treatment of bifurcation phenomena, whereas "trajectory" has temporal or historical connotation.

[3] A concept to be explained later in connection with stability analysis. A perfect structure involves some form of idealization such as perfectly centered loads or perfect fabrication. An imperfect structure is one that deviates from that idealization in measurable ways.

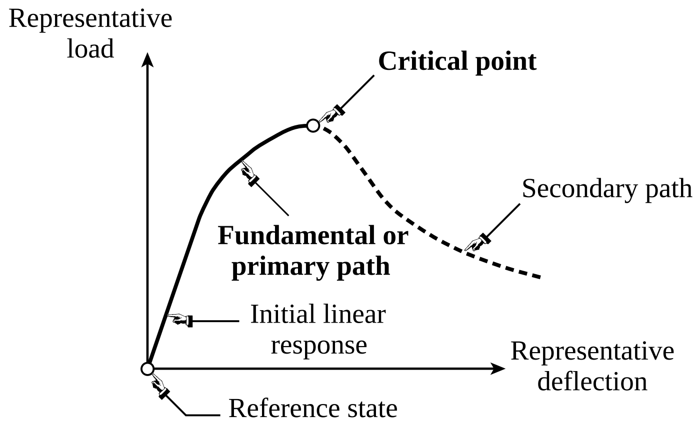

Figure 2.2. Fundamental (primary) and secondary equilibrium paths.

§2.3.1. Critical points

Critical points are characterized mathematically in later chapters. It is sufficient to note here that there are two types:

1. *Limit points*, at which the tangent to the equilibrium path is horizontal, i.e. parallel to the deflection axis, and

2. *Bifurcation points*, at which two or more equilibrium paths cross.

At critical points the relation between the given characteristic load and the associated deflection is not unique. Physically, the structure becomes *uncontrollable* or *marginally controllable* there. This property endows such points with engineering significance.

§2.3.2. Turning points

Points at which the tangent to the equilibrium path is vertical, *i.e.* parallel to the load axis, are called *turning points*. These are not critical points and have less physical significance, but are of interest for some structures. They have some computational significance, however, because they can affect the performance of certain solution methods.

§2.3.3. Failure points

Points at which a path suddenly stops or "breaks" because of physical failure are called *failure points*. The phenomenon of failure may be *local* or *global* in nature. In the first case (*e.g*, failure of a noncritical structure component) the structure may regain functional equilibrium after dynamically "jumping" to another equilibrium path. In the latter case the failure is catastrophic or destructive and the structure does not regain functional equilibrium.

In the present exposition, bifurcation, limit, turning and failure points are often identified by the letters `B`, `L`, `T` and `F`, respectively.

Equilibrium points that are not critical are called *regular*.

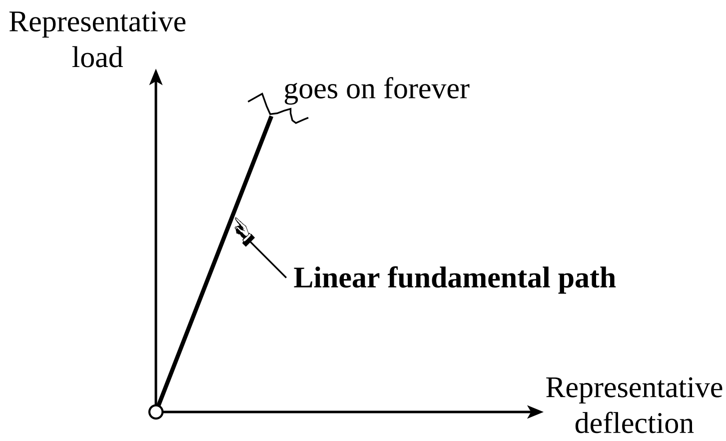

Figure 2.3. The response diagram for a purely linear structural model.

§2.4. Linear Response

A *linear* structure is a mathematical model characterized by a *linear* fundamental equilibrium path for *all* possible choices of load and deflection variables. This is shown schematically in Figure 2.3.

The consequences of such behavior are not difficult to foresee:

1. A linear structure can sustain any load whatsoever and undergo any displacement magnitude.
2. There are no critical, turning or failure points.
3. Response to different load systems can be obtained by superposition.
4. Removing all loads returns the structure to the reference position.

The requirements for such a model to be applicable are:

- Perfect linear elasticity for *any* deformation
- Infinitesimal deformations
- Infinite strength

These assumptions are not only physically unrealistic but mutually contradictory. For example, if the deformations are to remain infinitesimal for *any* load, the body must be rigid rather than elastic, which contradicts the first assumption. Thus, there are necessarily limits placed on the validity of the linear model.

Despite these obvious limitations, the linear model can be a good approximation of *portions* of the nonlinear response. In particular, the *fundamental path* response in the vicinity the reference state. See for instance Figure 2.2. Because for many structures this segment represents the operational or service range, the linear model is widely used in *design* calculations. The key advantage of this idealization is that the superposition-of-effects principle applies. Practical implications of the failure of the superposition principle are further discussed in Chapter 3.

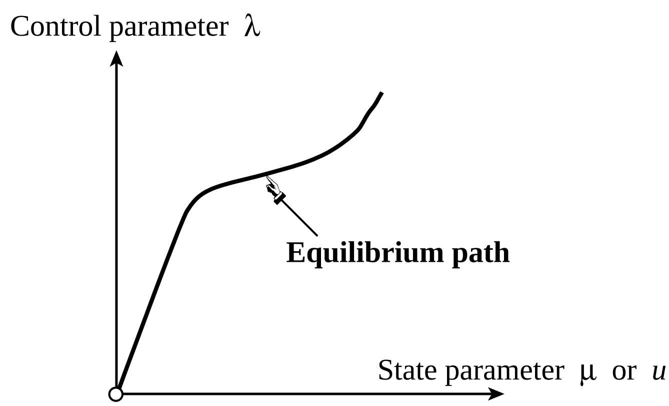

Figure 2.4. A control-state response diagram.

§2.5. Tangent Stiffness and Stability

The tangent to an equilibrium path may be informally viewed as the limit of the ratio

$$\frac{\text{force increment}}{\text{displacement increment}}$$

This is by definition a *stiffness* or, more precisely, the *tangent stiffness* associated with the representative force and displacement. The reciprocal ratio is called *flexibility* or *compliance*.

The sign of the tangent stiffness is closely associated with the question of *stability* of an equilibrium state. A negative stiffness is necessarily associated with *unstable* equilibrium. A positive stiffness is necessary but not sufficient for stability.[4]

If the load and deflection quantities are *conjugate* in the virtual work sense, the area under a load-deflection diagram may be interpreted as work performed by the system.

§2.6. Generalized Response

It is often useful to be able to generalize the load-displacement curve of Figure 2.1 in the following way. A *control-state* response involves two ingredients:

1. A *control parameter*, called λ, plotted along the vertical axis versus
2. A *state parameter*, called u or μ, plotted along the horizontal axis.[5]

We shall see in following Chapters that λ and u (or μ) characterize in some way the actions applied to the structure and the state of the structure, respectively.

A diagram such as that shown in Figure 2.4 is called a *control-state response*. Throughout this exposition the abbreviated term *response* is frequently used in this generalized sense. In practice the control parameter is often a *load amplitude* or *load factor*whereas the state parameter is a *displacement* amplitude. Thus the usual load-deflection response is a particular case of the control-state response.

[4] These sign criteria would be sufficient for a one-degree-of-freedom system.

[5] We shall use the symbol μ primarily for *dimensionless* state quantities.

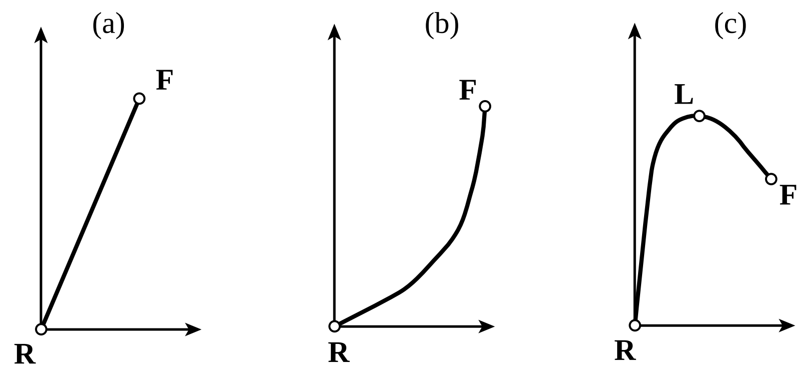

Figure 2.5. Basic flavors of nonlinear response: (a) Linear until brittle failure, (b) Stiffening or hardening, (c) Softening.

Remark 2.4. The interpretation of the tangent-to-the-path as stiffness discussed in §2.5 does not necessarily carry over to more general control-state diagrams. Similarly, the interpretations of the sign of the tangent and of the enclosed-area in terms of stability indicator and stored work, respectively, do not necessarily hold. This is because control and state are not necessarily conjugate in the virtual work sense.

§2.7. Response Flavors

The response diagrams in Figure 2.5 illustrate three "monotonic" types of response: linear, hardening, and softening. Symbols `F` and `L` identify failure and limit points, respectively.

A response such as in (a) is characteristic of pure crystals, glassy, and certain high strength composite materials.

A response such as in (b) is typical of cable, netted and pneumatic (inflatable) structures, which may be collectively called *tensile structures*. The stiffening effect comes from geometry "adaptation" to the applied loads. Some flat-plate assemblies also display this behavior initially.

A response such as in (c) is more common for structure materials than the previous two. A linear response is followed by a softening regime that may occur slowly or suddenly. More "softening flavors" are given in Figure 2.6.

The diagrams of Figure 2.6 illustrate a "combination of basic flavors" that can complicate the response as well as the task of the analyst. Here `B` and `T` denote bifurcation and turning points, respectively.

The snap-through response (d) combines softening with hardening following the second limit point. The response branch between the two limit points has a negative stiffness and is therefore unstable. (If the structure is subject to a prescribed constant load, the structure "takes off" dynamically when the first limit point is reached.) A response of this type is typical of slightly curved structures such as shallow arches.

The snap-back response (e) is an exaggerated snap-through, in which the response curve "turns back" in itself with the consequent appearance of turning points. The equilibrium between the two turning points may be stable and consequently physically realizable. This type of response is exhibited by trussed-dome, folded and thin-shell structures in which "moving arch" effects occur

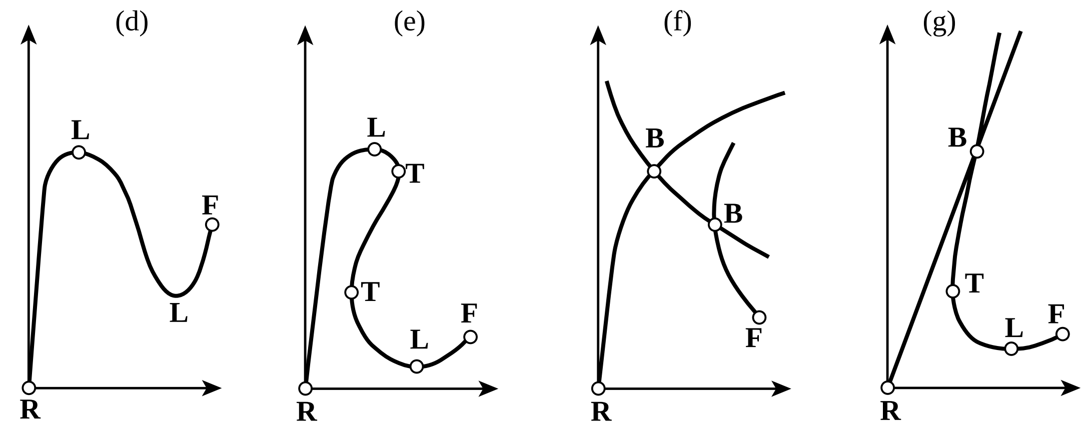

Figure 2.6. More complex response patterns: (d) snap-through, (e) snap-back, (f) bifurcation, (g) bifurcation combined with limit points and snap-back.

following the first limit point; for example cylindrical shells with free edges and supported by end diaphragms.

In all previous diagrams the response was a unique curve. The presence of bifurcation (popularly known as “buckling” by structural engineers) points as in (f) and (g) introduces more features. At such points more than one response path is possible. The structure takes the path that is dynamically preferred (in the sense of having a lower energy) over the others. Bifurcation points may occur in any sufficiently thin structure that experiences compressive stresses.

Bifurcation, limit and turning points may occur in many combinations as illustrated in (g). A striking example of such a complicated response is provided by thin cylindrical shells under axial compression.

§2.8. Engineering Applications

Nonlinear Structural Analysis is the prediction of the response of nonlinear structures by model-based simulation. Simulation involves a combination of mathematical modeling, discretization methods and numerical techniques. As noted in Chapter 1, finite element methods dominate the discretization step.

Table 2.1 summarizes the most important applications of nonlinear structural analysis.

§2.9. Sources of Nonlinearities

A response diagram characterizes only the *gross* behavior of a structure, as it might be observed simply by conducting an experiment on a mechanical testing machine. Further insight into the source of nonlinearity is required to capture such physical behavior with mathematical and computational models for computer simulation.

For structural analysis there are four sources of nonlinear behavior. The corresponding nonlinear effects are identified by the terms *material*, *geometric*, *force B.C.* and *displacement B.C.*, in which B.C. means “boundary conditions.” In this course we shall be primarily concerned with the last

Table 2.1 Engineering Applications of Nonlinear Structural Analysis

Application	*Explanation*
Strength analysis	How much load can the structure support before global failure occurs?
Deflection analysis	When deflection control is of primary importance
Stability analysis	Finding critical points (limit points or bifurcation points) closest to operational range
Service configuration analysis	Finding the "operational" equilibrium form of certain slender structures when the fabrication and service configurations are quite different (e.g. cables, inflatable structures, helicoids)
Reserve strength analysis	Finding the load carrying capacity beyond critical points to assess safety under abnormal conditions.
Progressive failure analysis	A variant of stability and strength analysis in which progressive deterioration (e.g. cracking) is considered.
Envelope analysis	A combination of previous analyses in which multiple parameters are varied and the strength information thus obtained is condensed into failure envelopes.

three types of nonlinearity, with emphasis on the geometric one.[6]

The four sources are discussed in more detail in following sections. To remember where the nonlinear terms appear in the governing equations, it is useful to recall the fields that continuum mechanics deals with, and the relationships among these fields. For *linear* solid continuum mechanics information is presented in Figures 2.7 and 2.8.[7]

In linear solid mechanics or linear structural mechanics the connecting relationships shown in Figure 2.8 are *linear*, and so are the governing equations obtained by eliminating all fields but one.

Any of these relations, however, may be nonlinear. Tracing this fact back to physics gives rise to the types of nonlinearities depicted in Figure 2.9. Relations between body force and stress (the equilibrium equations) and between strains and displacements (the kinematic equations) are closely

[6] the exclusion of constitutive or material nonlinearities does not imply that there are less important than the others. Quite the contrary. But the topic is covered in separate courses offered in Civil Engineering departments.

[7] These are the Tonti diagrams introduced in the IFEM course.

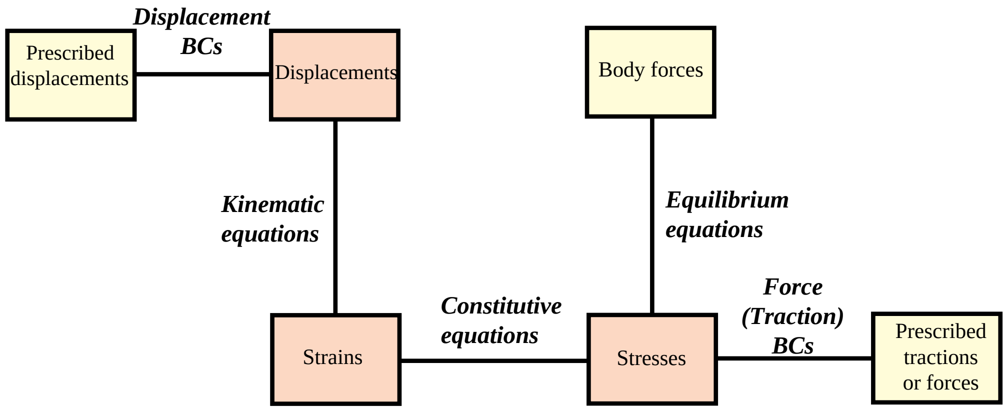

Figure 2.7. Fields in solid continuum mechanics and connecting relationships.

linked in a "duality" sense, and so the term *geometric nonlinearities* applies collectively to both sets of relations. The *force BC* nonlinearities couple displacements and applied forces (surface tractions and/or body forces) and thus bring the additional links drawn in Figure 2.8.

In the following sections these sources of nonlinearities are correlated to the physics in more detail.

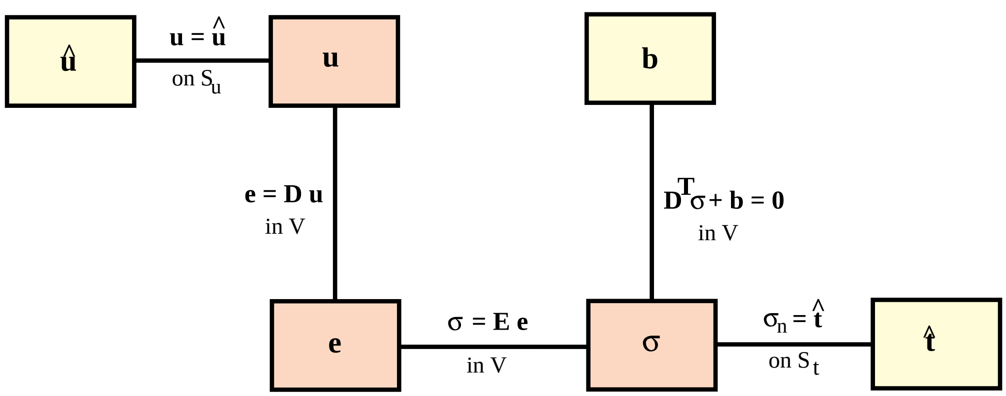

Figure 2.8. Same as Figure 2.7, with symbols and equations written down for the *linear* case.

§2.10. Geometric Nonlinearity

Physical source

Change in geometry as the structure deforms is taken into account in setting up the strain-displacement and equilibrium equations.

Applications

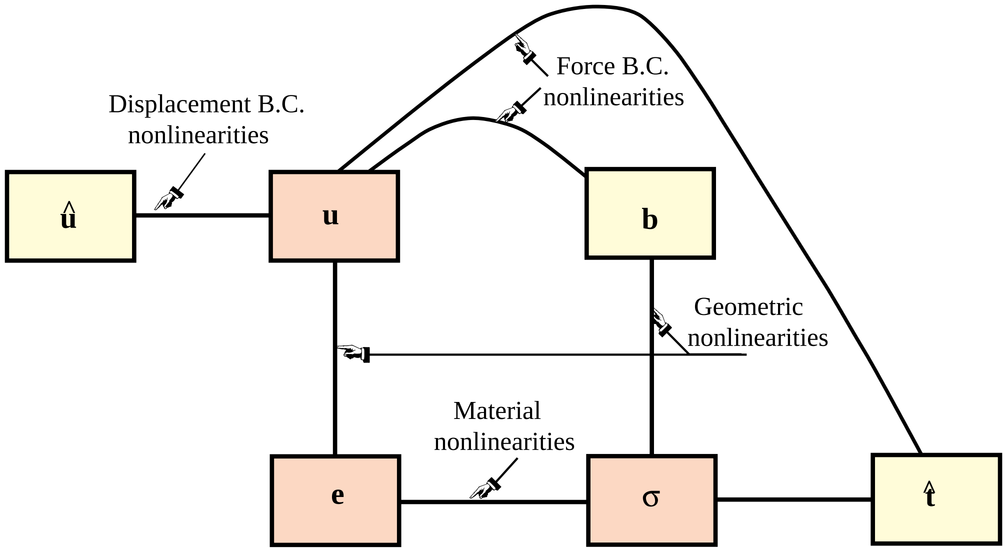

Figure 2.9. Graphical depiction of sources of nonlinearities in solid continuum mechanics.

Slender structures in aerospace, civil and mechanical engineering applications. Tensile structures such as cables and inflatable membranes. Metal and plastic forming. Stability analysis of all types.

Mathematical source

Strain-displacement equations:

$$\mathbf{e} = \mathbf{D}\mathbf{u} \tag{2.1}$$

The operator $\mathbf{D}$ is nonlinear when finite strains (as opposed to infinitesimal strains) are expressed in terms of displacements. Internal equilibrium equations:

$$\mathbf{b} = -\mathbf{D}^{*}\boldsymbol{\sigma} \tag{2.2}$$

In the classical linear theory of elasticity, $\mathbf{D}^{*} = \mathbf{D}^{T}$ is the formal adjoint of $\mathbf{D}$, but that is not necessarily true if geometric nonlinearities are considered.

Remark 2.5. The term *geometric nonlinerities* models a myriad of physical problems:

Large strain. The strains themselves may be large, say over 5%. Examples: rubber structures (tires, membranes), metal forming. These are frequently associated with material nonlinearities.

Small strains but finite displacements and/or rotations. Slender structures undergoing finite displacements and rotations although the deformational strains may be treated as infinitesimal. Example: cables, springs, arches, bars, thin plates.

Linearized prebucking. When both strains and displacements may be treated as infinitesimal before loss of stability by buckling. These may be viewed as initially stressed members. Example: many civil engineering structures such as buildings and stiff (non-suspended) bridges.

§2.11. Material Nonlinearity

Physical source

Material behavior depends on current deformation state and possibly past history of the deformation. Other constitutive variables (prestress, temperature, time, moisture, electromagnetic fields, etc.) may be involved.

Applications

Structures undergoing nonlinear elasticity, plasticity, viscoelasticity, creep, or inelastic rate effects.

Mathematical source

The constitutive equations that relate stresses and strains. For a linear elastic material

$$\boldsymbol{\sigma} = \mathbf{E}\mathbf{e} \tag{2.3}$$

where the matrix $\mathbf{E}$ contains elastic moduli. If the material does not fit the elastic model, generalizations of this equation are necessary, and a whole branch of continuum mechanics is devoted to the formulation, study and validation of constitutive equations.

Remark 2.6. The engineering significance of material nonlinearities varies greatly across disciplines. They seem to occur most often in civil engineering, that deals with inherently nonlinear materials such as concrete, soils and low-strength steel. In mechanical engineering creep and plasticity are most important, frequently occurring in combination with strain-rate and thermal effects. In aerospace engineering material nonlinearities are less important and tend to be local in nature (for example, cracking and "localization" failures of composite materials).

Remark 2.7. Material nonlinearities may give rise to very complex phenomena such as *path dependence*, *hysteresis*, *localization*, *shakedown*, *fatigue*, *progressive failure*. The detailed numerical simulation of these phenomena in three dimensions is still beyond the capabilities of the most powerful computers.

§2.12. Force BC Nonlinearity

Physical Source

Applied forces depend on deformation.

Applications

The most important engineering application concerns *pressure loads* of fluids. These include hydrostatic loads on submerged or container structures; aerodynamic and hydrodynamic loads caused by the motion of aeriform and hydroform fluids (wind loads, wave loads, drag forces). Of more mathematical interest are gyroscopic and non-conservative follower forces, but these are of interest only in a limited class of problems, particularly in aerospace engineering.

Mathematical source

The applied forces (prescribed surface tractions $\widehat{\mathbf{t}}$ and/or body forces $\mathbf{b}$) depend on the displacements:

$$\widehat{\mathbf{t}} = \widehat{\mathbf{t}}(\mathbf{u}), \qquad \mathbf{b} = \mathbf{b}(\mathbf{u}), \tag{2.4}$$

the former being more important in practice.

§2.13. Displacement BC Nonlinearity

Physical source

Displacement boundary conditions depend on the deformation of the structure.

Applications

The most important application is the *contact* problem,[8] in which no-interpenetration conditions are enforced on flexible bodies while the extent of the contact area is unknown. Non-structural applications of this problem pertain to the more general class of *free boundary problems*, for example: ice melting, phase changes, flow in porous media. The determination of the essential boundary conditions is a key part of the solution process.

Mathematical source

For the contact problem: prescribed displacements $\widehat{\mathbf{d}}$ depend on internal displacements $\mathbf{u}$:

$$\widehat{\mathbf{d}} = \widehat{\mathbf{d}}(\mathbf{u}) \tag{2.5}$$

More complicated dependencies can occur in the free-boundary problems mentioned above.

[8] Contact-impact in dynamics.

Homework Exercises for Chapter 2

A Tour of Nonlinear Analysis

EXERCISE 2.1 [D:10 Explain the difference, if any, between a load-deflection response and a control-state response.

EXERCISE 2.2 [D:20] Can the following occur simultaneously: (a) a limit and a bifurcation point, (b) a bifurcation and a turning point, (c) a limit and a turning point, (d) two bifurcation points coalescing into one. If you answer "yes" to an item, sketch a response diagram to justify that reply.

EXERCISE 2.3 [D:25] In §2.10–13, nonlinearities are classified according to physical source into geometric, material, force boundary conditions, and displacement boundary conditions. For each of the following mechanical systems indicate the source(s) of nonlinearity that you think are significant; note that there may be more than one. (If you are not familiar with the underlying concepts, read those sections.)

(a) a long, slender elastic pipe bent under end couples while the pipe material stays elastic. See Figure E2.1.

(b) an inflating balloon. See Figure E2.2.

(c) a cable deflecting under action of wind forces while its material stays elastic. See Figure E2.3.

(d) a forming process in which hot metal is extruded through a rigid die. See Figure E2.4.

(e) a metal anchor is drilled into the soil to serve as a cable support; the hole is then filled with concrete. See Figures E2.5 and E2.6. The question refers to the soil-drilling process, ignoring dynamics.

(f) a hefty bird — say a condor — sucked into an aircraft jet engine. Ignore dynamics; engine is the structure, bird the load.

EXERCISE 2.4 [D:15] Can you think of a mechanical component that has the load-deflection response diagram pictured in Figure E2.7? (Explain why). *Hint:* Think of a helicoidal spring.

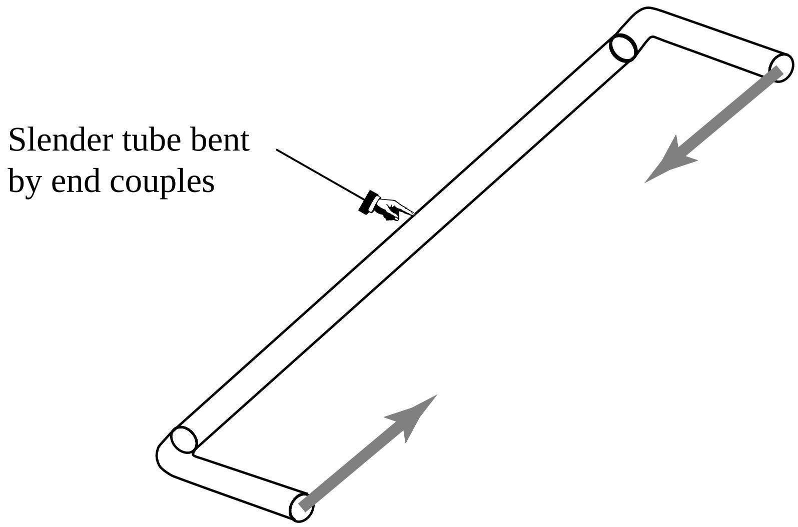

Figure E2.1. Slender elastic pipe bent under end couples for Exercise 2.3(a).

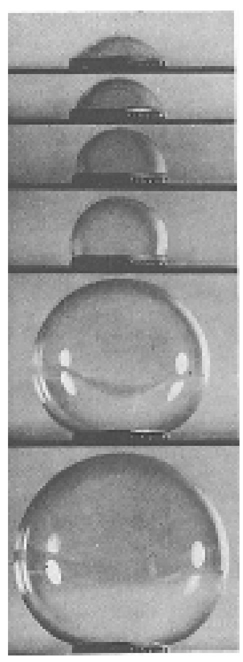

Figure E2.2. Inflating balloon for Exercise 3(b).

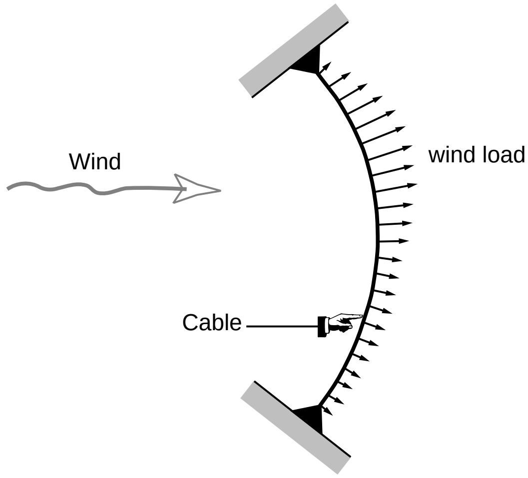

Figure E2.3. Cable deflecting under wind forces for Exercise 2.3(c).

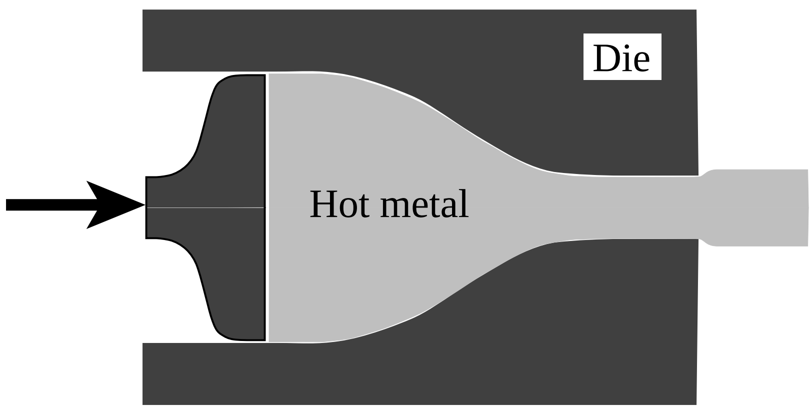

Figure E2.4. Hot metal extruded trough a rigid die for Exercise 2.3(d).

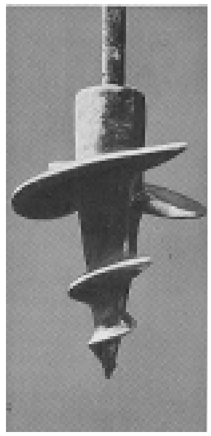

Figure E2.5. Drill element of a cable anchor, for Exercise 2.3(e).

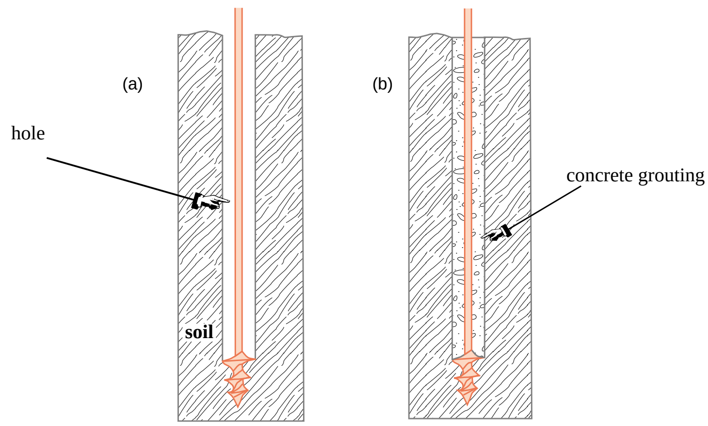

Figure E2.6. Configuration of cable anchor after drilling in the soil, for Exercise 2.3(e).

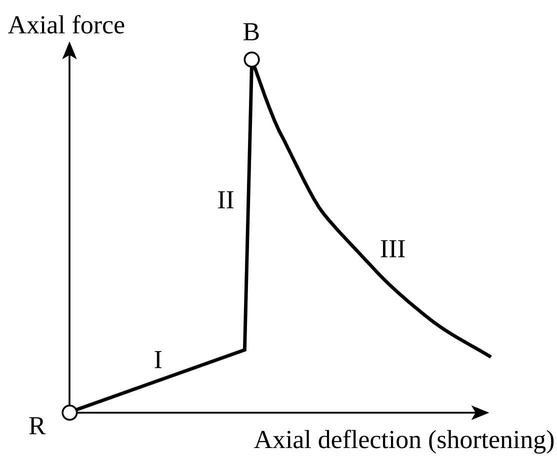

Figure E2.7. A “mystery” response diagram for Exercise 2.4.