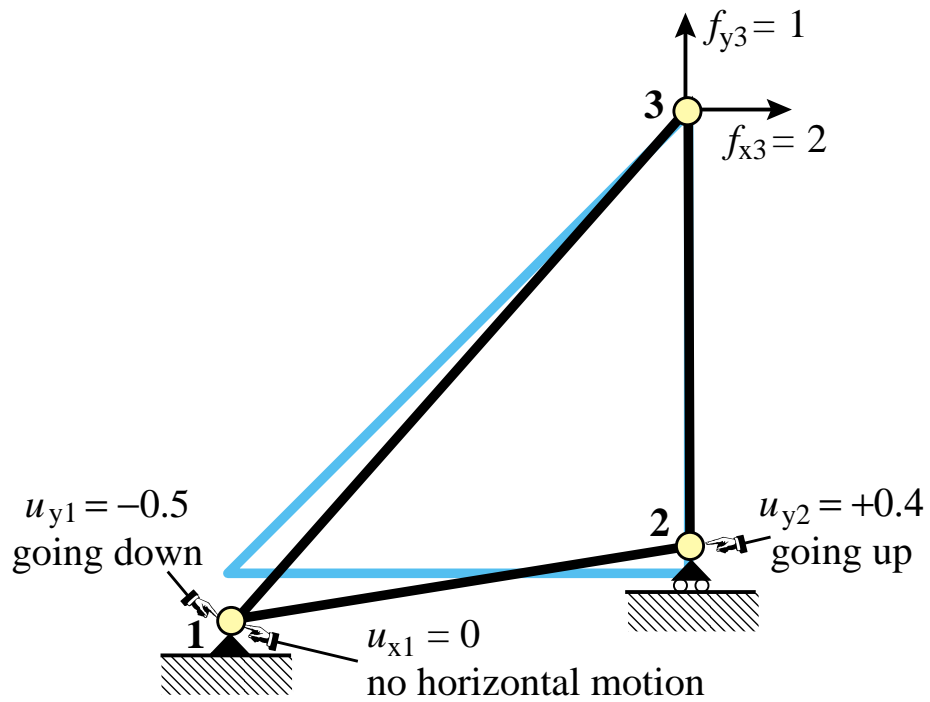


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Thermomechanical Effects

Prescribed Nonzero Displacements in Example Truss



Prescribed Nonzero Displacements

Recall the master stiffness equations

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix}$$

The displacement B.Cs are now

$$u_{x1} = 0, \quad u_{y1} = -0.5, \quad u_{y2} = 0.4$$

Prescribed NZ Displacements (cont'd)

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ u_{x2} \\ 0.4 \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ 0 \\ f_{y2} \\ 2 \\ 1 \end{bmatrix}$$

Remove rows 1,2,4 but (for now) keep columns

$$\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ u_{x2} \\ 0.4 \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Prescribed NZ Displacements (cont'd)

Transfer effect of known displacements to RHS, and delete columns:

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} -$$

$$\begin{bmatrix} (-10) \times 0 + 0 \times (-0.5) + 0 \times 0.4 \\ (-10) \times 0 + (-10) \times (-0.5) + 0 \times 0.4 \\ (-10) \times 0 + (-10) \times (-0.5) + (-5) \times 0.4 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}$$

Solving gives

$$\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0.2 \end{bmatrix}$$

Prescribed NZ Displacements (cont'd)

$$\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0.2 \end{bmatrix}$$

Complete the displacement vector with known values

$$\mathbf{u} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \\ 0.4 \\ -0.5 \\ 0.2 \end{bmatrix}$$

Prescribed NZ Displacements (cont'd)

Recovery of reaction forces and internal member forces proceeds as before

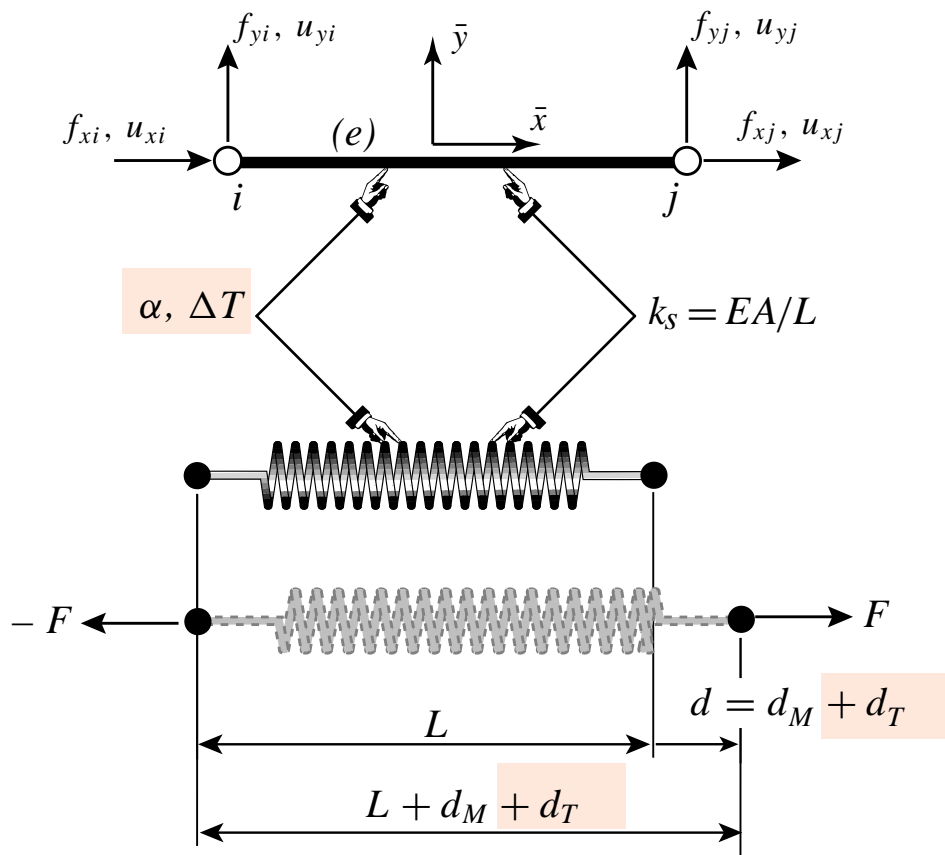
In summary, the only changes to the DSM is in the application of displacement boundary conditions before solve

Initial Force Effects (also called Initial Strain & Initial Stress Effects by FEM authors)

- **Thermomechanical effects**
- **Moisture effects**
- **Prestress effects**
- **Lack of fit**
- **Residual stresses**
- **Internal actuators ("smart" structures)**

For specificity we will study in detail only the first one: *thermomechanical effects*

Thermomechanical Effects on Bar Element



Thermomechanical Effects on Bar Element (Cont'd)

Axial strain is sum of mechanical and thermal:

$$e_T = d_T / L = \alpha \Delta T$$

$$e = e_M + e_T = \frac{\sigma}{E} + \alpha \Delta T$$

Thermomechanical Effects on Bar Element (cont'd)

$$e = d/L \quad d = \bar{u}_{xj} - \bar{u}_{xi}$$

$$\frac{\bar{u}_{xj} - \bar{u}_{xi}}{L} = \frac{\sigma}{E} + \alpha \Delta T$$

$$\frac{EA}{L} (\bar{u}_{xj} - \bar{u}_{xi}) = A\sigma + EA\alpha \Delta T = p_M + p_T = F$$

$$F = \frac{EA}{L} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{yj} \end{bmatrix}$$

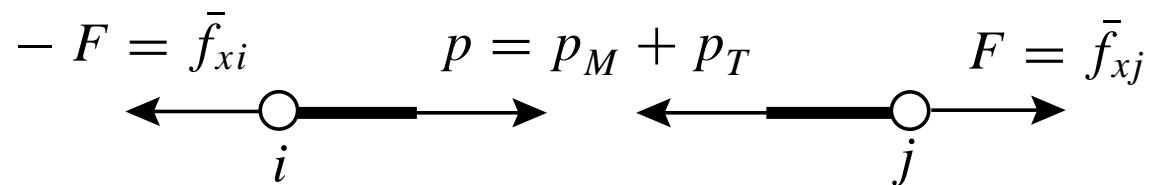
Incorporating Thermomechanical Effects into the Element Stiffness Equations

$$F = \bar{f}_{xj} = -\bar{f}_{xi}, \quad \bar{f}_{yi} = \bar{f}_{yj} = 0$$

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} \bar{f}_{Mxi} \\ \bar{f}_{Myi} \\ \bar{f}_{Mxj} \\ \bar{f}_{Myj} \end{bmatrix} + \begin{bmatrix} \bar{f}_{Txi} \\ \bar{f}_{Tyi} \\ \bar{f}_{Txj} \\ \bar{f}_{Tyj} \end{bmatrix} = \begin{bmatrix} \bar{f}_{Mxi} \\ \bar{f}_{Myi} \\ \bar{f}_{Mxj} \\ \bar{f}_{Myj} \end{bmatrix} + EA \alpha \Delta T \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

Physical Interpretation of Thermal Force Vector



Think of a fully restrained bar when **heated**

Matrix Form of Element Stiffness Equations in Local System

$$\bar{\mathbf{f}} = \bar{\mathbf{f}}_M + \bar{\mathbf{f}}_T = \bar{\mathbf{K}} \bar{\mathbf{u}}$$

or

$$\bar{\mathbf{K}} \bar{\mathbf{u}} = \bar{\mathbf{f}}_M + \bar{\mathbf{f}}_T$$

Here $\bar{\mathbf{f}}_T$ is the element thermal force vector

$$\bar{\mathbf{f}}_T = EA\alpha\Delta T \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Assembly Rules with Thermomechanical Effects

- 1. *Compatibility*: The joint displacements of all members meeting at a joint must be the same**
- 2. *Equilibrium*: The sum of *effective forces* exerted by all members that meet at a joint must balance the external force applied to that joint.**

No change in application of 1. To account for 2, the thermal forces are globalized and added to the mechanical forces during the merge process

Master Stiffness Equations with Thermomechanical Effects

$$\mathbf{Ku} = \mathbf{f}_M + \mathbf{f}_T = \mathbf{f}$$

 **effective force vector**

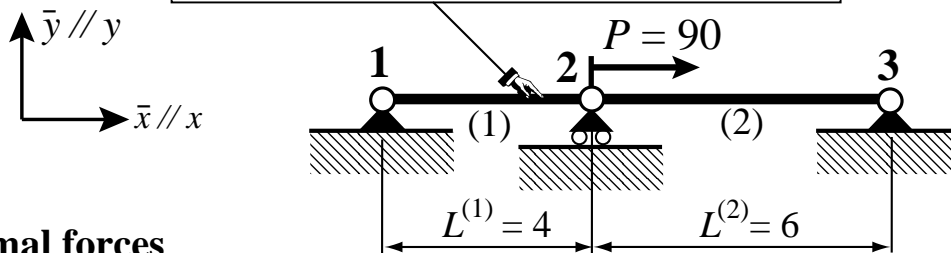
Solve for node displacements \mathbf{u} . If you want to retrieve the mechanical node forces including reactions subtract the thermal force:

$$\mathbf{f}_M = \mathbf{Ku} - \mathbf{f}_T$$

For computation of internal forces and stresses see worked out Examples

Worked Out Example #1 in Notes

$E = 1000, A = 12, \alpha = 0.0005$ for both members; $\Delta T^{(1)} = 25^\circ, \Delta T^{(2)} = -10^\circ$



Thermal forces

$$\bar{\mathbf{f}}_T^{(1)} = \begin{bmatrix} \bar{f}_{T1}^{(1)} \\ \bar{f}_{T2}^{(1)} \end{bmatrix} = E^{(1)} A^{(1)} \alpha^{(1)} \Delta T^{(1)} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -150 \\ 150 \end{bmatrix}$$

$$\bar{\mathbf{f}}_T^{(2)} = \begin{bmatrix} \bar{f}_{T2}^{(2)} \\ \bar{f}_{T3}^{(2)} \end{bmatrix} = E^{(2)} A^{(2)} \alpha^{(2)} \Delta T^{(2)} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 60 \\ -60 \end{bmatrix}$$

Element stiffness equations

$$3000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_1^{(1)} \\ \bar{u}_2^{(1)} \end{bmatrix} = \begin{bmatrix} \bar{f}_{M1}^{(1)} \\ \bar{f}_{M2}^{(1)} \end{bmatrix} + \begin{bmatrix} -150 \\ 150 \end{bmatrix}$$

$$2000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_2^{(2)} \\ \bar{u}_3^{(2)} \end{bmatrix} = \begin{bmatrix} \bar{f}_{M2}^{(2)} \\ \bar{f}_{M3}^{(2)} \end{bmatrix} + \begin{bmatrix} 60 \\ -60 \end{bmatrix}$$

Worked Out Example #1 in Notes (cont'd)

Assembled master stiffness equations

$$1000 \begin{bmatrix} 3 & -3 & 0 \\ -3 & 5 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_{M1} \\ f_{M2} \\ f_{M3} \end{bmatrix} + \begin{bmatrix} -150 \\ 150 + 60 \\ -60 \end{bmatrix} = \begin{bmatrix} f_{M1} \\ f_{M2} \\ f_{M3} \end{bmatrix} + \begin{bmatrix} -150 \\ 210 \\ -60 \end{bmatrix}$$

Reduced stiffness equations upon applying BCs:

$$5000 u_2 = f_{M2} + 210 = 90 + 210 = 300$$

Solution for unknown displacement

$$u_2 = 300/5000 = +0.06$$

Recovery of internal member forces

$$p^{(1)} = \frac{E^{(1)}A^{(1)}}{L^{(1)}}(u_2 - u_1) - E^{(1)}A^{(1)}\alpha^{(1)}\Delta T^{(1)} = 3000 \times 0.06 - 12000 \times 0.0005 \times 25 = 60$$

$$p^{(2)} = \frac{E^{(2)}A^{(2)}}{L^{(2)}}(u_3 - u_2) - E^{(2)}A^{(2)}\alpha^{(2)}\Delta T^{(2)} = 2000 \times (-0.06) - 12000 \times 0.0005 \times (-10) = -72$$

Member stresses

$$\sigma^{(1)} = 60/12 = 5 \quad (\text{T})$$

$$\sigma^{(2)} = -72/12 = -6 \quad (\text{C})$$

Generalization: Initial Force Effects in the DSM

$$\mathbf{K}\mathbf{u} = \mathbf{f}_M + \mathbf{f}_I = \mathbf{f}$$

initial force vector

effective force vector

Where does \mathbf{f}_I come from? Thermal effects, moisture, prestress, lack of fit, residual stresses, some nonlinearities.

Common property: if displacements \mathbf{u} vanish

$$\mathbf{f}_M + \mathbf{f}_I = \mathbf{0}$$

$$\mathbf{f}_M = -\mathbf{f}_I$$

mechanical reactions

There are (self-equilibrating) mechanical reactions in the absence of displacements

Summary: Treating Initial Force Effects in the DSM

Disconnection

Localization

Member (Element) Relations

➡ Include \bar{f}_I^e

Globalization

Merge

Application of BCs

Solution

Recovery of Derived Quantities

➡ Transform to f_I^e

➡ Assemble into f_I

➡ Subtract \bar{f}_I^e