

# 24

## Implementation of Iso-P Triangular Elements

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### §24.1. Introduction

This Chapter continues with the computer implementation of two-dimensional finite elements. It covers the programming of isoparametric *triangular* elements for the plane stress problem. Triangular elements bring two *sui generis* implementation quirks with respect to quadrilateral elements:

- (1) The numerical integration rules for triangles are not product of one-dimensional Gauss rules, as in the case of quadrilaterals. They are instead specialized to the triangle geometry.
- (2) The computation of  $x$ - $y$  partial derivatives and the element-of-area scaling by the Jacobian determinant must account for the fact that the triangular coordinates  $\zeta_1$ ,  $\zeta_2$  and  $\zeta_3$  do not form an independent set.

We deal with these issues in the next two sections.

### §24.2. Gauss Quadrature for Triangles

The numerical integration schemes for *quadrilaterals* introduced in §17.3 and implemented in §23.2 are built as “tensor products” of two one-dimensional Gauss formulas. On the other hand, Gauss rules for *triangles* are *not* derivable from one-dimensional rules, and must be constructed especially for the triangular geometry.

#### §24.2.1. Requirements for Gauss Rules

Gauss quadrature rules for triangles must possess *triangular symmetry* in the following sense:

If the sample point  $(\zeta_1, \zeta_2, \zeta_3)$  is present in a Gauss integration rule with weight  $w$ , then all other points obtainable by permuting the three triangular coordinates arbitrarily must appear in that rule, and have the same weight. (24.1)

This constraint guarantees that the result of the quadrature process will not depend on element node numbering.<sup>1</sup> If  $\zeta_1$ ,  $\zeta_2$ , and  $\zeta_3$  are different, (24.1) forces six equal-weight sample points to be present in the rule, because  $3! = 6$ . If two triangular coordinates are equal, the six points coalesce to three, and (24.1) forces three equal-weight sample points to be present. Finally, if the three coordinates are equal (which can only happen for the centroid  $\zeta_1 = \zeta_2 = \zeta_3 = 1/3$ ), the six points coalesce to one.<sup>2</sup>

Additional requirements for a Gauss rule to be numerically acceptable are:

All sample points must be inside the triangle (or on the triangle boundary) and all weights must be positive. (24.2)

This is called a *positivity* condition. It insures that the element internal energy evaluated by numerical quadrature is nonnegative definite.

<sup>1</sup> It would be disconcerting to users, to say the least, to have the FEM solution depend on how nodes are numbered.

<sup>2</sup> It follows that the number of sample points in triangle Gauss quadrature rules that satisfy (24.1) must be of the form  $6i + 3j + k$ , where  $i$  and  $j$  are nonnegative integers and  $k$  is 0 or 1. Consequently there are no rules with 2, 5 or 8 points.

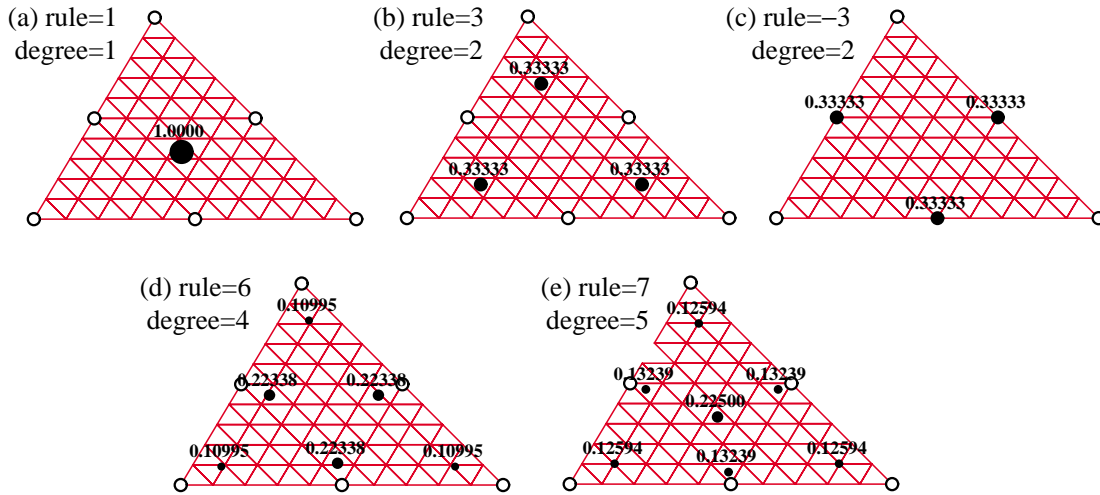


FIGURE 24.1. Location of sample points (dark circles) of five Gauss quadrature rules for straight sided (superparametric) 6-node triangles. Weight written to 5 places near each sample point; sample-point circle areas are proportional to weight.

A rule is said to be of degree  $n$  if it integrates exactly all polynomials in the triangular coordinates of order  $n$  or less when the Jacobian determinant is constant, and there is at least one polynomial of order  $n + 1$  that is not exactly integrated by the rule.

**Remark 24.1.** The positivity requirement (24.2) is automatically satisfied in quadrilaterals by using Gauss product rules, since the points are always inside while weights are positive. Consequently it was not necessary to call attention to it. On the other hand, for triangles there are Gauss rules with as few as 4 points that violate positivity.

### §24.2.2. Superparametric Triangles

We first consider superparametric straight-sided triangles geometry defined by the three corner nodes. Over such triangles the Jacobian determinant defined below is constant. The five simplest Gauss rules that satisfy the requirements (24.1) and (24.2) have 1, 3, 3, 6 and 7 points, respectively. The two rules with 3 points differ in the location of the sample points. The five rules are depicted in Figure 24.1 over 6-node straight-sided triangles; for such triangles to be superparametric the side nodes must be located at the midpoint of the sides.

*One point rule.* The simplest Gauss rule for a triangle has *one* sample point located at the centroid. For a straight sided triangle,

$$\frac{1}{A} \int_{\Omega^e} F(\zeta_1, \zeta_2, \zeta_3) d\Omega \approx F\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \tag{24.3}$$

where  $A$  is the triangle area

$$A = \int_{\Omega^e} d\Omega = \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \frac{1}{2} [(x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1)]. \tag{24.4}$$

This rule is illustrated in Figure 24.1(a). It has degree 1, meaning that it integrates exactly up to linear polynomials in  $\{\zeta_1, \zeta_2, \zeta_3\}$ . For example,  $F = 4 - \zeta_1 + 2\zeta_2 - \zeta_3$  is exactly integrated by (24.3).

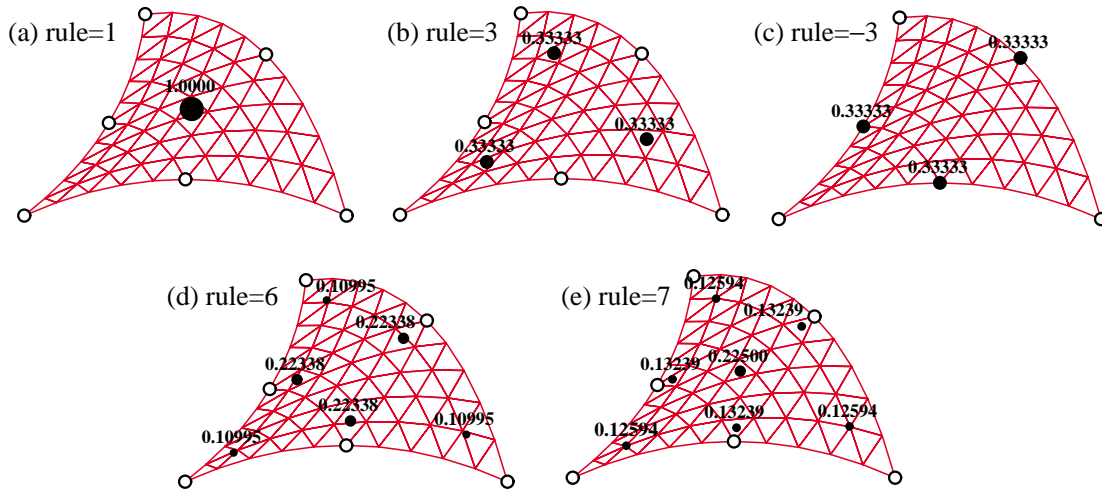


FIGURE 24.2. Location of sample points (dark circles) of five Gauss quadrature rules for curved sided 6-node triangles. Weight written to 5 places near each sample point; sample-point circle areas are proportional to weight.

*Three Point Rules.* The next two rules in order of simplicity contain *three* sample points:

$$\frac{1}{A} \int_{\Omega^e} F(\zeta_1, \zeta_2, \zeta_3) d\Omega \approx \frac{1}{3} F\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right) + \frac{1}{3} F\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right) + \frac{1}{3} F\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right). \quad (24.5)$$

$$\frac{1}{A} \int_{\Omega^e} F(\zeta_1, \zeta_2, \zeta_3) d\Omega \approx \frac{1}{3} F\left(\frac{1}{2}, \frac{1}{2}, 0\right) + \frac{1}{3} F\left(0, \frac{1}{2}, \frac{1}{2}\right) + \frac{1}{3} F\left(\frac{1}{2}, 0, \frac{1}{2}\right). \quad (24.6)$$

These are depicted in Figures 24.1(b,c). Both rules are of degree 2; that is, they integrate exactly up to quadratic polynomials in the triangular coordinates. For example, the function  $F = 6 + \zeta_1 + 3\zeta_3 + \zeta_2^2 - \zeta_3^2 + 3\zeta_1\zeta_3$  is integrated exactly by either rule. Formula (24.6) is called the *midpoint rule*.

*Six and Seven Point Rules.* There is a 4-point rule of degree 3, but it has a negative weight and so violates (24.2). There are no symmetric rules with 5 points. The next useful rules have six and seven points. There is a 6-point rule of degree 4 and a 7-point rule of degree 5, which integrate exactly up to quartic and quintic polynomials in  $\{\zeta_1, \zeta_2, \zeta_3\}$ , respectively. The 7-point rule includes the centroid as sample point. The abscissas and weights are expressible as rational combinations of square roots of integers and fractions. The expressions are listed in the *Mathematica* implementation discussed in §24.2.4. The sample point configurations are depicted in Figure 24.1(d,e).

### §24.2.3. Arbitrary Iso-P Triangles

If the triangle has variable metric, as in the curved sided 6-node triangle geometries shown in Figure 24.2, the foregoing formulas need adjustment because the element of area  $d\Omega$  becomes a function of position. Consider the more general case of an isoparametric element with  $n$  nodes and shape functions  $N_i$ . In §24.3 it is shown that the differential area element is given by

$$d\Omega = J d\zeta_1 d\zeta_2 d\zeta_3, \quad J = \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ \sum_{i=1}^n x_i \frac{\partial N_i}{\partial \zeta_1} & \sum_{i=1}^n x_i \frac{\partial N_i}{\partial \zeta_2} & \sum_{i=1}^n x_i \frac{\partial N_i}{\partial \zeta_3} \\ \sum_{i=1}^n y_i \frac{\partial N_i}{\partial \zeta_1} & \sum_{i=1}^n y_i \frac{\partial N_i}{\partial \zeta_2} & \sum_{i=1}^n y_i \frac{\partial N_i}{\partial \zeta_3} \end{bmatrix} \quad (24.7)$$

```

TrigGaussRuleInfo[{rule_,numer_},point_]:= Module[
  {zeta,p=rule,i=point,g1,g2,info={{Null,Null,Null},0} },
  If [p== 1, info={{1/3,1/3,1/3},1}];
  If [p== 3, info={{1,1,1}/6,1/3}; info[[1,i]]=2/3];
  If [p== -3, info={{1,1,1}/2,1/3}; info[[1,i]]=0 ];
  If [p== 6, g1=(8-Sqrt[10]+Sqrt[38-44*Sqrt[2/5]])/18;
    g2=(8-Sqrt[10]-Sqrt[38-44*Sqrt[2/5]])/18;
    If [i<4, info={{g1,g1,g1},(620+Sqrt[213125-
      53320*Sqrt[10]])/3720}; info[[1,i]]=1-2*g1];
    If [i>3, info={{g2,g2,g2},(620-Sqrt[213125-
      53320*Sqrt[10]])/3720}; info[[1,i-3]]=1-2*g2]];
  If [p== 7, g1=(6-Sqrt[15])/21; g2=(6+Sqrt[15])/21;
    If [i<4, info={{g1,g1,g1},(155-Sqrt[15])/1200};
      info[[1,i]]= 1-2*g1];
    If [i>3&& i<7, info={{g2,g2,g2},(155+Sqrt[15])/1200};
      info[[1,i-3]]=1-2*g2];
    If [i==7, info={{1/3,1/3,1/3},9/40} ];
  If [numer, Return[N[info]], Return[Simplify[info]]];
];

```

FIGURE 24.3. Module to get triangle Gauss quadrature rule information.

Here  $J$  is the Jacobian determinant, which plays the same role as  $J$  in the isoparametric quadrilaterals. If the metric is simply defined by the 3 corners, as in Figure 24.1, the geometry shape functions are  $N_1 = \zeta_1$ ,  $N_2 = \zeta_2$  and  $N_3 = \zeta_3$ . Then the foregoing determinant reduces to that of (24.4), and  $J = A$  everywhere. But for general (curved) geometries  $J = J(\zeta_1, \zeta_2, \zeta_3)$ , and the triangle area  $A$  cannot be factored out of the integration rules. For example the one point rule becomes

$$\int_{\Omega^e} F(\zeta_1, \zeta_2, \zeta_3) d\Omega \approx J\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) F\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \quad (24.8)$$

whereas the midpoint rule becomes

$$\int_{\Omega^e} F(\zeta_1, \zeta_2, \zeta_3) d\Omega \approx \frac{1}{3}J\left(\frac{1}{2}, \frac{1}{2}, 0\right) F\left(\frac{1}{2}, \frac{1}{2}, 0\right) + \frac{1}{3}J\left(0, \frac{1}{2}, \frac{1}{2}\right) F\left(0, \frac{1}{2}, \frac{1}{2}\right) + \frac{1}{3}J\left(\frac{1}{2}, 0, \frac{1}{2}\right) F\left(\frac{1}{2}, 0, \frac{1}{2}\right). \quad (24.9)$$

These can be expressed more compactly by saying that the Gauss integration rule is applied to  $JF$ .

#### §24.2.4. Implementation

The five rules pictured in Figures 24.1 and 24.2 are implemented in the module TrigGaussRuleInfo listed in Figure 24.3. The module is invoked as

$$\{\{zeta_1, zeta_2, zeta_3\}, w\} = \text{TrigGaussRuleInfo}[\{\text{rule}, \text{numer}\}, \text{point}] \quad (24.10)$$

The module has three arguments: rule, numer and i. The first two are grouped in a two-item list. Argument rule, which can be 1, 3, -3, 6 or 7, defines the integration formula as follows. Abs[rule] is the number of sample points. Of the two 3-point choices, if rule is -3 the midpoint rule is picked, else if +3 the 3-interior point rule is chosen. Logical flag numer is set to True or False to request floating-point or exact information, respectively

Argument point is the index of the sample point, which may range from 1 through Abs[rule].

The module returns the list  $\{\{\zeta_1, \zeta_2, \zeta_3\}, w\}$ , where  $\zeta_1, \zeta_2, \zeta_3$  are the triangular coordinates of the sample point, and  $w$  is the integration weight. For example, TrigGaussRuleInfo[{3, False}, 1] returns  $\{\{2/3, 1/6, 1/6\}, 1/3\}$ . If rule is not implemented, the module returns  $\{\{Null, Null, Null\}, 0\}$ .

### §24.3. Partial Derivative Computation

The calculation of Cartesian partial derivatives is illustrated in this section for the 6-node triangle shown in Figure 24.4. The results are applicable, however, to iso-P triangles with any number of nodes.

The element geometry is defined by the corner coordinates  $\{x_i, y_i\}$ , with  $i = 1, 2, \dots, 6$ . Corners are numbered 1,2,3 in counterclockwise sense. Side nodes are numbered 4,5,6 opposite to corners 3,1,2, respectively. Side nodes may be arbitrarily located within positive Jacobian constraints as discussed in §19.4.2. The triangular coordinates are as usual denoted by  $\zeta_1, \zeta_2$  and  $\zeta_3$ , which satisfy  $\zeta_1 + \zeta_2 + \zeta_3 = 1$ . The quadratic displacement field  $\{u_x(\zeta_1, \zeta_2, \zeta_3), u_y(\zeta_1, \zeta_2, \zeta_3)\}$  is defined by the 12 node displacements  $\{u_{xi}, u_{yi}\}$ ,  $i = 1, 2, \dots, 6$ , as per the iso-P quadratic interpolation formula (16.10–11). That formula is repeated here for convenience:

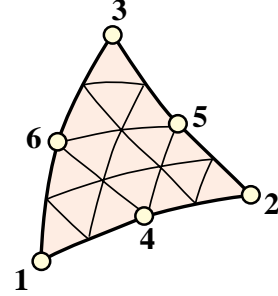


FIGURE 24.4. The 6-node iso-P triangle.

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ N_5^e \\ N_6^e \end{bmatrix} \quad (24.11)$$

in which the shape functions and their natural derivatives are

$$\mathbf{N}^T = \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ N_5^e \\ N_6^e \end{bmatrix} = \begin{bmatrix} \zeta_1(2\zeta_1-1) \\ \zeta_2(2\zeta_2-1) \\ \zeta_3(2\zeta_3-1) \\ 4\zeta_1\zeta_2 \\ 4\zeta_2\zeta_3 \\ 4\zeta_3\zeta_1 \end{bmatrix}, \quad \frac{\partial \mathbf{N}^T}{\partial \zeta_1} = \begin{bmatrix} 4\zeta_1-1 \\ 0 \\ 0 \\ 4\zeta_2 \\ 0 \\ 4\zeta_3 \end{bmatrix}, \quad \frac{\partial \mathbf{N}^T}{\partial \zeta_2} = \begin{bmatrix} 0 \\ 4\zeta_2-1 \\ 0 \\ 4\zeta_1 \\ 4\zeta_3 \\ 0 \end{bmatrix}, \quad \frac{\partial \mathbf{N}^T}{\partial \zeta_3} = \begin{bmatrix} 0 \\ 0 \\ 4\zeta_3-1 \\ 0 \\ 4\zeta_2 \\ 4\zeta_1 \end{bmatrix}. \quad (24.12)$$

#### §24.3.1. Triangle Coordinate Partial

In this and following sections the superscript  $e$  of shape functions will be omitted for brevity. The bulk of the shape function logic is concerned with the computation of the partial derivatives of the shape functions (24.12) with respect to  $x$  and  $y$  at any point in the element. For this purpose consider a *generic* scalar function  $w(\zeta_1, \zeta_2, \zeta_3)$  that is quadratically interpolated over the triangle by

$$w = w_1 N_1 + w_2 N_2 + w_3 N_3 + w_4 N_4 + w_5 N_5 + w_6 N_6 = [w_1 \ w_2 \ w_3 \ w_4 \ w_5 \ w_6] \mathbf{N}^T. \quad (24.13)$$

Symbol  $w$  may stand for 1,  $x$ ,  $y$ ,  $u_x$  or  $u_y$ , which are interpolated in the iso-P representation (24.11), or other element-varying quantities such as thickness, temperature, etc. Taking partials of (24.13) with respect to  $x$  and  $y$  and applying the chain rule twice yields

$$\begin{aligned} \frac{\partial w}{\partial x} &= \sum w_i \frac{\partial N_i}{\partial x} = \sum w_i \left( \frac{\partial N_i}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial x} + \frac{\partial N_i}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial x} + \frac{\partial N_i}{\partial \zeta_3} \frac{\partial \zeta_3}{\partial x} \right), \\ \frac{\partial w}{\partial y} &= \sum w_i \frac{\partial N_i}{\partial y} = \sum w_i \left( \frac{\partial N_i}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial y} + \frac{\partial N_i}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial y} + \frac{\partial N_i}{\partial \zeta_3} \frac{\partial \zeta_3}{\partial y} \right), \end{aligned} \quad (24.14)$$

where all sums are understood to run over  $i = 1, \dots, 6$ . In matrix form:

$$\begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_3}{\partial x} \\ \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_3}{\partial y} \end{bmatrix} \begin{bmatrix} \sum w_i \frac{\partial N_i}{\partial \zeta_1} \\ \sum w_i \frac{\partial N_i}{\partial \zeta_2} \\ \sum w_i \frac{\partial N_i}{\partial \zeta_3} \end{bmatrix}. \quad (24.15)$$

Transposing both sides of (24.15) while exchanging sides yields

$$\begin{bmatrix} \sum w_i \frac{\partial N_i}{\partial \zeta_1} & \sum w_i \frac{\partial N_i}{\partial \zeta_2} & \sum w_i \frac{\partial N_i}{\partial \zeta_3} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}. \quad (24.16)$$

Now make  $w \equiv 1, x, y$  and stack the results row-wise:

$$\begin{bmatrix} \sum \frac{\partial N_i}{\partial \zeta_1} & \sum \frac{\partial N_i}{\partial \zeta_2} & \sum \frac{\partial N_i}{\partial \zeta_3} \\ \sum x_i \frac{\partial N_i}{\partial \zeta_1} & \sum x_i \frac{\partial N_i}{\partial \zeta_2} & \sum x_i \frac{\partial N_i}{\partial \zeta_3} \\ \sum y_i \frac{\partial N_i}{\partial \zeta_1} & \sum y_i \frac{\partial N_i}{\partial \zeta_2} & \sum y_i \frac{\partial N_i}{\partial \zeta_3} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial 1}{\partial x} & \frac{\partial 1}{\partial y} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{bmatrix}. \quad (24.17)$$

But  $\partial x/\partial x = \partial y/\partial y = 1$  and  $\partial 1/\partial x = \partial 1/\partial y = \partial x/\partial y = \partial y/\partial x = 0$  because  $x$  and  $y$  are independent coordinates. It is shown in Remark 24.2 below that, if  $\sum N_i = 1$ , the entries of the first row of the coefficient matrix are equal to a constant  $C$ . These entries can be scaled to unity because the first row of the right-hand side is null. Consequently we arrive at a system of linear equations of order 3 with two right-hand sides:

$$\begin{bmatrix} 1 & 1 & 1 \\ \sum x_i \frac{\partial N_i}{\partial \zeta_1} & \sum x_i \frac{\partial N_i}{\partial \zeta_2} & \sum x_i \frac{\partial N_i}{\partial \zeta_3} \\ \sum y_i \frac{\partial N_i}{\partial \zeta_1} & \sum y_i \frac{\partial N_i}{\partial \zeta_2} & \sum y_i \frac{\partial N_i}{\partial \zeta_3} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (24.18)$$

### §24.3.2. Solving the Jacobian System

By analogy with quadrilateral elements, the coefficient matrix of (24.18) will be called the *Jacobian matrix* and denoted by  $\mathbf{J}$ . Its determinant scaled by one half is equal to the Jacobian  $J = \frac{1}{2} \det \mathbf{J}$  used in the expression of the area element introduced in §24.2.3. For compactness (24.18) is rewritten

$$\mathbf{J}\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ J_{x1} & J_{x2} & J_{x3} \\ J_{y1} & J_{y2} & J_{y3} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (24.19)$$

If  $J \neq 0$ , solving this system gives

$$\begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} \end{bmatrix} = \frac{1}{2J} \begin{bmatrix} J_{y23} & J_{x32} \\ J_{y31} & J_{x13} \\ J_{y12} & J_{x21} \end{bmatrix} = \mathbf{P}, \quad (24.20)$$

in which  $J_{xji} = J_{xj} - J_{xi}$ ,  $J_{yji} = J_{yj} - J_{yi}$  and  $J = \frac{1}{2} \det \mathbf{J} = \frac{1}{2}(J_{x21}J_{y31} - J_{y12}J_{x13})$ . Substituting into (24.14) we arrive at

$$\begin{aligned}\frac{\partial w}{\partial x} &= \sum w_i \frac{\partial N_i}{\partial x} = \sum \frac{w_i}{2J} \left( \frac{\partial N_i}{\partial \zeta_1} J_{y23} + \frac{\partial N_i}{\partial \zeta_2} J_{y31} + \frac{\partial N_i}{\partial \zeta_3} J_{y12} \right), \\ \frac{\partial w}{\partial y} &= \sum w_i \frac{\partial N_i}{\partial y} = \sum \frac{w_i}{2J} \left( \frac{\partial N_i}{\partial \zeta_1} J_{x32} + \frac{\partial N_i}{\partial \zeta_2} J_{x13} + \frac{\partial N_i}{\partial \zeta_3} J_{x21} \right).\end{aligned}\quad (24.21)$$

In particular, the shape function derivatives are

$$\begin{aligned}\frac{\partial N_i}{\partial x} &= \frac{1}{2J} \left( \frac{\partial N_i}{\partial \zeta_1} J_{y23} + \frac{\partial N_i}{\partial \zeta_2} J_{y31} + \frac{\partial N_i}{\partial \zeta_3} J_{y12} \right), \\ \frac{\partial N_i}{\partial y} &= \frac{1}{2J} \left( \frac{\partial N_i}{\partial \zeta_1} J_{x32} + \frac{\partial N_i}{\partial \zeta_2} J_{x13} + \frac{\partial N_i}{\partial \zeta_3} J_{x21} \right).\end{aligned}\quad (24.22)$$

in which the natural derivatives  $\partial N_i / \partial \zeta_j$  can be read off (24.12). Using the  $3 \times 2$   $\mathbf{P}$  matrix defined in (24.20) yields finally the compact form

$$\left[ \frac{\partial N_i}{\partial x} \quad \frac{\partial N_i}{\partial y} \right] = \left[ \frac{\partial N_i}{\partial \zeta_1} \quad \frac{\partial N_i}{\partial \zeta_2} \quad \frac{\partial N_i}{\partial \zeta_3} \right] \mathbf{P}. \quad (24.23)$$

**Remark 24.2.** Here is the proof that each first row entry of the  $3 \times 3$  matrix in (24.17) is a numerical constant, say  $C$ . Suppose the shape functions are polynomials of order  $n$  in the triangular coordinates, and let  $Z = \zeta_1 + \zeta_2 + \zeta_3$ . The completeness identity is

$$S = \sum N_i = 1 = c_1 Z + c_2 Z^2 + \dots + c_n Z^n, \quad c_1 + c_2 + \dots + c_n = 1. \quad (24.24)$$

where the  $c_i$  are element dependent scalar coefficients. Differentiating  $S$  with respect to the  $\zeta_i$ 's and setting  $Z = 1$  yields

$$\begin{aligned}C = \sum \frac{\partial N_i}{\partial \zeta_1} &= \sum \frac{\partial N_i}{\partial \zeta_2} = \sum \frac{\partial N_i}{\partial \zeta_3} = c_1 + 2c_2 Z + c_3 Z^3 + \dots + (n-1)Z^{n-1} = c_1 + 2c_2 + 3c_3 + \dots + c_{n-1} \\ &= 1 + c_2 + 2c_3 + \dots + (n-2)c_n,\end{aligned}\quad (24.25)$$

which proves the assertion. For the 3-node linear triangle,  $S = Z$  and  $C = 1$ . For the 6-node quadratic triangle,  $S = 2Z^2 - Z$  and  $C = 3$ . For the 10-node cubic triangle,  $S = 9Z^3/2 - 9Z^2/2 + Z$  and  $C = 11/2$ . Because the first equation in (24.18) is homogeneous, the  $C$ 's can be scaled to unity.

## §24.4. The Quadratic Triangle

### §24.4.1. Shape Function Module

We specialize now the results of §24.3 to obtain the stiffness matrix of the 6-node quadratic triangle in plane stress. Taking the dot products of the natural-coordinate partials (24.12) with the node coordinates we obtain the entries of the Jacobian matrix of (24.19):

$$\begin{aligned}J_{x1} &= x_1(4\zeta_1 - 1) + 4(x_4\zeta_2 + x_6\zeta_3), & J_{x2} &= x_2(4\zeta_2 - 1) + 4(x_5\zeta_3 + x_4\zeta_1), & J_{x3} &= x_3(4\zeta_3 - 1) + 4(x_6\zeta_1 + x_5\zeta_2), \\ J_{y1} &= y_1(4\zeta_1 - 1) + 4(y_4\zeta_2 + y_6\zeta_3), & J_{y2} &= y_2(4\zeta_2 - 1) + 4(y_5\zeta_3 + y_4\zeta_1), & J_{y3} &= y_3(4\zeta_3 - 1) + 4(y_6\zeta_1 + y_5\zeta_2).\end{aligned}\quad (24.26)$$

```

Trig6IsoPShapeFunDer[ncoor_,tcoor_] := Module[
  {ζ1,ζ2,ζ3,x1,x2,x3,x4,x5,x6,y1,y2,y3,y4,y5,y6,
  dx4,dx5,dx6,dy4,dy5,dy6,Jx21,Jx32,Jx13,Jy12,Jy23,Jy31,
  Nf,dNx,dNy,Jdet}, {ζ1,ζ2,ζ3}=tcoor;
  {{x1,y1},{x2,y2},{x3,y3},{x4,y4},{x5,y5},{x6,y6}}=ncoor;
  dx4=x4-(x1+x2)/2; dx5=x5-(x2+x3)/2; dx6=x6-(x3+x1)/2;
  dy4=y4-(y1+y2)/2; dy5=y5-(y2+y3)/2; dy6=y6-(y3+y1)/2;
  Nf={ζ1*(2*ζ1-1),ζ2*(2*ζ2-1),ζ3*(2*ζ3-1),4*ζ1*ζ2,4*ζ2*ζ3,4*ζ3*ζ1};
  Jx21= x2-x1+4*(dx4*(ζ1-ζ2)+(dx5-dx6)*ζ3);
  Jx32= x3-x2+4*(dx5*(ζ2-ζ3)+(dx6-dx4)*ζ1);
  Jx13= x1-x3+4*(dx6*(ζ3-ζ1)+(dx4-dx5)*ζ2);
  Jy12= y1-y2+4*(dy4*(ζ2-ζ1)+(dy6-dy5)*ζ3);
  Jy23= y2-y3+4*(dy5*(ζ3-ζ2)+(dy4-dy6)*ζ1);
  Jy31= y3-y1+4*(dy6*(ζ1-ζ3)+(dy5-dy4)*ζ2);
  Jdet = Jx21*Jy31-Jy12*Jx13;
  dNx= {(4*ζ1-1)*Jy23,(4*ζ2-1)*Jy31,(4*ζ3-1)*Jy12,4*(ζ2*Jy23+ζ1*Jy31),
  4*(ζ3*Jy31+ζ2*Jy12),4*(ζ1*Jy12+ζ3*Jy23)}/Jdet;
  dNy= {(4*ζ1-1)*Jx32,(4*ζ2-1)*Jx13,(4*ζ3-1)*Jx21,4*(ζ2*Jx32+ζ1*Jx13),
  4*(ζ3*Jx13+ζ2*Jx21),4*(ζ1*Jx21+ζ3*Jx32)}/Jdet;
  Return[Simplify[{Nf,dNx,dNy,Jdet}]]
];

```

FIGURE 24.5. Shape function module for 6-node quadratic triangle.

From these  $\mathbf{J}$  can be constructed, and shape function partials  $\partial N_i/\partial x$  and  $\partial N_i/\partial y$  explicitly obtained from (24.22). Somewhat simpler expressions, however, result by using the following “hierarchical” side node coordinates:

$$\begin{aligned} \Delta x_4 &= x_4 - \frac{1}{2}(x_1 + x_2), & \Delta x_5 &= x_5 - \frac{1}{2}(x_2 + x_3), & \Delta x_6 &= x_6 - \frac{1}{2}(x_3 + x_1), \\ \Delta y_4 &= y_4 - \frac{1}{2}(y_1 + y_2), & \Delta y_5 &= y_5 - \frac{1}{2}(y_2 + y_3), & \Delta y_6 &= y_6 - \frac{1}{2}(y_3 + y_1). \end{aligned} \quad (24.27)$$

Geometrically these represent the *deviations from midpoint positions*; thus for a superparametric element  $\Delta x_4 = \Delta x_5 = \Delta x_6 = \Delta y_4 = \Delta y_5 = \Delta y_6 = 0$ . The Jacobian coefficients become

$$\begin{aligned} J_{x21} &= x_{21} + 4(\Delta x_4(\zeta_1 - \zeta_2) + (\Delta x_5 - \Delta x_6)\zeta_3), & J_{x32} &= x_{32} + 4(\Delta x_5(\zeta_2 - \zeta_3) + (\Delta x_6 - \Delta x_4)\zeta_1), \\ J_{x13} &= x_{13} + 4(\Delta x_6(\zeta_3 - \zeta_1) + (\Delta x_4 - \Delta x_5)\zeta_2), & J_{y12} &= y_{12} + 4(\Delta y_4(\zeta_2 - \zeta_1) + (\Delta y_6 - \Delta y_5)\zeta_3), \\ J_{y23} &= y_{23} + 4(\Delta y_5(\zeta_3 - \zeta_2) + (\Delta y_4 - \Delta y_6)\zeta_1), & J_{y31} &= y_{31} + 4(\Delta y_6(\zeta_1 - \zeta_3) + (\Delta y_5 - \Delta y_4)\zeta_2). \end{aligned} \quad (24.28)$$

(Note that if all midpoint deviations vanish,  $J_{xji} = x_{ji}$  and  $J_{yji} = y_{ji}$ .) From this one gets  $J = \frac{1}{2} \det \mathbf{J} = \frac{1}{2}(J_{x21}J_{y31} - J_{y12}J_{x13})$  and

$$\mathbf{P} = \frac{1}{2J} \begin{bmatrix} y_{23} + 4(\Delta y_5(\zeta_3 - \zeta_2) + (\Delta y_4 - \Delta y_6)\zeta_1) & x_{32} + 4(\Delta x_5(\zeta_2 - \zeta_3) + (\Delta x_6 - \Delta x_4)\zeta_1) \\ y_{31} + 4(\Delta y_6(\zeta_1 - \zeta_3) + (\Delta y_5 - \Delta y_4)\zeta_2) & x_{13} + 4(\Delta x_6(\zeta_3 - \zeta_1) + (\Delta x_4 - \Delta x_5)\zeta_2) \\ y_{12} + 4(\Delta y_4(\zeta_2 - \zeta_1) + (\Delta y_6 - \Delta y_5)\zeta_3) & x_{21} + 4(\Delta x_4(\zeta_1 - \zeta_2) + (\Delta x_5 - \Delta x_6)\zeta_3) \end{bmatrix}. \quad (24.29)$$

and the Cartesian derivatives of the shape functions are

$$\frac{\partial \mathbf{N}^T}{\partial x} = \frac{1}{2J} \begin{bmatrix} (4\zeta_1 - 1)J_{y23} \\ (4\zeta_2 - 1)J_{y31} \\ (4\zeta_3 - 1)J_{y12} \\ 4(\zeta_2 J_{y23} + \zeta_1 J_{y31}) \\ 4(\zeta_3 J_{y31} + \zeta_2 J_{y12}) \\ 4(\zeta_1 J_{y12} + \zeta_3 J_{y23}) \end{bmatrix}, \quad \frac{\partial \mathbf{N}^T}{\partial y} = \frac{1}{2J} \begin{bmatrix} (4\zeta_1 - 1)J_{x32} \\ (4\zeta_2 - 1)J_{x13} \\ (4\zeta_3 - 1)J_{x21} \\ 4(\zeta_2 J_{x32} + \zeta_1 J_{x13}) \\ 4(\zeta_3 J_{x13} + \zeta_2 J_{x21}) \\ 4(\zeta_1 J_{x21} + \zeta_3 J_{x32}) \end{bmatrix}. \quad (24.30)$$

```

Trig6IsoPMembraneStiffness[ncoor_,Emat_,th_,options_]:=
Module[{i,k,p=3,numer=False,h=th,tcoor,w,c,
Nf,dNx,dNy,Jdet,Be,Ke=Table[0,{12},{12}]},
If [Length[options]>=1, numer=options[[1]]];
If [Length[options]>=2, p= options[[2]]];
If [!MemberQ[{1,-3,3,6,7},p], Print["Illegal p"]; Return[Null]];
For [k=1, k<=Abs[p], k++,
{tcoor,w}= TrigGaussRuleInfo[{p,numer},k];
{Nf,dNx,dNy,Jdet}= Trig6IsoPShapeFunDer[ncoor,tcoor];
If [numer, {Nf,dNx,dNy,Jdet}=N[{Nf,dNx,dNy,Jdet}]];
If [Length[th]==6, h=th.Nf]; c=w*Jdet*h/2;
Be= {Flatten[Table[{dNx[[i]],0},{i,6}]],
Flatten[Table[{0,dNy[[i]]},{i,6}]],
Flatten[Table[{dNy[[i]],dNx[[i]]},{i,6}]]};
Ke+=c*Transpose[Be].(Emat.Be);
]; If[!numer,Ke=Simplify[Ke]]; Return[Ke]
];

```

FIGURE 24.6. Stiffness matrix module for 6-node plane stress triangle.

A *Mathematica* shape function module Trig6IsoPShapeFunDer is shown in Figure 24.5. It receives two arguments: ncoor and tcoor. The first one is the list of  $\{x_i, y_i\}$  coordinates of the six nodes. The second is the list of three triangular coordinates  $\{\zeta_1, \zeta_2, \zeta_3\}$  of the location at which the shape functions and their Cartesian derivatives are to be computed.

The module returns  $\{Nf, dNx, dNy, Jdet\}$  as module value. Here Nf collects the shape function values, dNx and dNy the  $x$  and  $y$  shape function derivatives, respectively, and Jdet is the determinant of matrix  $\mathbf{J}$ , equal to  $2J$  in the notation used here.

#### §24.4.2. Stiffness Module

The numerically integrated stiffness matrix is

$$\mathbf{K}^e = \int_{\Omega^e} h \mathbf{B}^T \mathbf{E} \mathbf{B} d\Omega \approx \sum_{i=1}^p w_i \mathbf{F}(\zeta_{1i}, \zeta_{2i}, \zeta_{3i}), \quad \text{where } \mathbf{F}(\zeta_1, \zeta_2, \zeta_3) = h \mathbf{B}^T \mathbf{E} \mathbf{B} J. \quad (24.31)$$

Here  $p$  denotes the number of sample points of the Gauss rule being used,  $w_i$  is the integration weight for the  $i^{\text{th}}$  sample point,  $\zeta_{1i}, \zeta_{2i}, \zeta_{3i}$  are the sample point triangular coordinates and  $J = \frac{1}{2} \det \mathbf{J}$ . This data is provided by TrigGaussRuleInfo. For the 6-node triangle (24.31) is implemented in module Trig6IsoPMembraneStiffness, listed in Figure 24.6. The module is invoked as

$$\text{Ke} = \text{Trig6IsoPMembraneStiffness}[\text{ncoor}, \text{Emat}, \text{th}, \text{options}] \quad (24.32)$$

The arguments are:

- ncoor      The list of node coordinates arranged as  $\{\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_6, y_6\}\}$ .  
Emat        The plane stress elasticity matrix

$$\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \quad (24.33)$$

arranged as  $\{\{E_{11}, E_{12}, E_{33}\}, \{E_{12}, E_{22}, E_{23}\}, \{E_{13}, E_{23}, E_{33}\}\}$ .

- th          Plate thickness specified as either a scalar h or a six-entry list:  $\{h_1, h_2, h_3, h_4, h_5, h_6\}$ .

The one-entry form specifies uniform thickness  $h$ . The six-entry form is used to specify an element of variable thickness, in which case the entries are the six node thicknesses and  $h$  is interpolated quadratically.

- options Processing options list. May contain two items: { numer , rule } or one: { numer }.
- numer is a logical flag. If True, the computations are forced to proceed in floating-point arithmetic. For symbolic or exact arithmetic work set numer to False.
- rule specifies the triangle Gauss rule as described in §24.2.4; rule may be 1, 3, -3, 6 or 7. For the 6-node element the three point rules are sufficient to get the correct rank. If omitted rule = 3 is assumed.

The module returns  $\mathbf{K}^e$  as an  $12 \times 12$  symmetric matrix pertaining to the following arrangement of nodal displacements:

$$\mathbf{u}^e = [u_{x1} \ u_{y1} \ u_{x2} \ u_{y2} \ u_{x3} \ u_{y3} \ u_{x4} \ u_{y4} \ u_{x5} \ u_{y5} \ u_{x6} \ u_{y6}]^T. \quad (24.34)$$

### §24.4.3. Test on Straight-Sided Triangle

The stiffness module of Figure 24.6 is tested on two triangle geometries, one with straight sides and constant metric and one with curved sides and highly variable metric. The straight sided triangle geometry, shown in Figure 24.7, has the corner nodes placed at  $(0, 0)$ ,  $(6, 2)$  and  $(4, 4)$  with side nodes 4,5,6 at the midpoints of the sides. The element has unit plate thickness. The material is isotropic with  $E = 288$  and  $\nu = 1/3$ .

The element is tested with the script shown in Figure 24.8. The stiffness matrix  $\mathbf{K}^e$  is computed by the three-interior-point Gauss rule, specified as  $p=3$ .

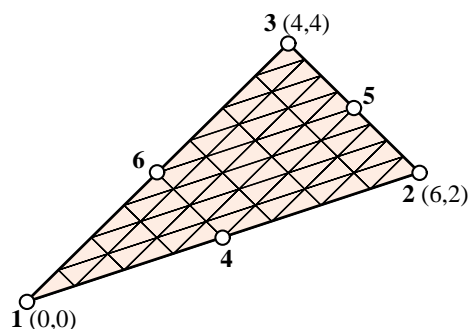


FIGURE 24.7. Straight sided 6-node triangle test element.

The computed  $\mathbf{K}^e$  for integration rules with 3 or more points are identical since those rules are exact for this element if the metric is constant, which is the case here. That stiffness is

$$\begin{bmatrix} 54 & 27 & 18 & 0 & 0 & 9 & -72 & 0 & 0 & 0 & 0 & -36 \\ 27 & 54 & 0 & -18 & 9 & 36 & 0 & 72 & 0 & 0 & -36 & -144 \\ 18 & 0 & 216 & -108 & 54 & -36 & -72 & 0 & -216 & 144 & 0 & 0 \\ 0 & -18 & -108 & 216 & -36 & 90 & 0 & 72 & 144 & -360 & 0 & 0 \\ 0 & 9 & 54 & -36 & 162 & -81 & 0 & 0 & -216 & 144 & 0 & -36 \\ 9 & 36 & -36 & 90 & -81 & 378 & 0 & 0 & 144 & -360 & -36 & -144 \\ -72 & 0 & -72 & 0 & 0 & 0 & 576 & -216 & 0 & -72 & -432 & 288 \\ 0 & 72 & 0 & 72 & 0 & 0 & -216 & 864 & -72 & -288 & 288 & -720 \\ 0 & 0 & -216 & 144 & -216 & 144 & 0 & -72 & 576 & -216 & -144 & 0 \\ 0 & 0 & 144 & -360 & 144 & -360 & -72 & -288 & -216 & 864 & 0 & 144 \\ 0 & -36 & 0 & 0 & 0 & -36 & -432 & 288 & -144 & 0 & 576 & -216 \\ -36 & -144 & 0 & 0 & -36 & -144 & 288 & -720 & 0 & 144 & -216 & 864 \end{bmatrix} \quad (24.35)$$

The eigenvalues are:

$$[1971.66 \ 1416.75 \ 694.82 \ 545.72 \ 367.7 \ 175.23 \ 157.68 \ 57.54 \ 12.899 \ 0 \ 0 \ 0] \quad (24.36)$$

The 3 zero eigenvalues pertain to the three independent rigid-body modes. The 9 other ones are positive. Consequently the computed  $\mathbf{K}^e$  has the correct rank of 9.

```

ClearAll[Em,v,a,b,e,h]; h=1; Em=288; v=1/3;
ncoor={{0,0},{6,2},{4,4},{3,1},{5,3},{2,2}};
Emat=Em/(1-v^2)*{{1,v,0},{v,1,0},{0,0,(1-v)/2}};
Print["Emat=",Emat//MatrixForm]
Ke=Trig6IsoPMembraneStiffness[ncoor,Emat,h,{False,3}];
Ke=Simplify[Ke]; Print[Chop[Ke]//MatrixForm];
Print["eigs of Ke=",Chop[Eigenvalues[N[Ke]]]];

```

FIGURE 24.8. Script to form  $\mathbf{K}^e$  of test triangle of Figure 24.7.

#### §24.4.4. Test on Highly Distorted Triangle

A highly distorted test geometry places the 3 corners at the vertices of an equilateral triangle:  $\{-1/2, 0\}$ ,  $\{1/2, 0\}$ ,  $\{0, \sqrt{3}/2\}$  whereas the 3 side nodes are located at  $\{0, -1/(2\sqrt{3})\}$ ,  $\{1/2, 1/\sqrt{3}\}$  and  $\{-1/2, 1/\sqrt{3}\}$ . The result is that the six nodes lie on a circle of radius  $1/\sqrt{3}$ , as depicted in Figure 24.9. The element has unit thickness. The material is isotropic with  $E = 504$  and  $\nu = 0$ . The stiffness  $\mathbf{K}^e$  is evaluated for four rank-sufficient rules: 3, -3, 6, 7, using the script of Figure 24.10.

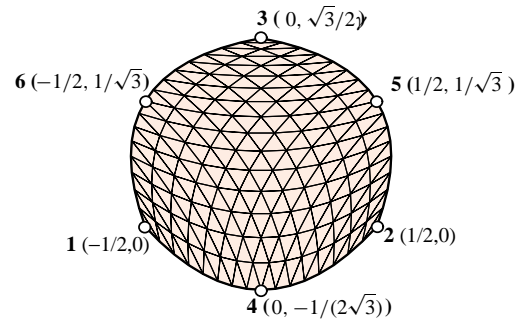


FIGURE 24.9. Test curved sided quadratic triangle, with the 6 nodes lying on a circle.

```

ClearAll[Em,v,h]; h=1; Em=7*72; v=0; h=1;
{x1,y1}={-1,0}/2; {x2,y2}={1,0}/2; {x3,y3}={0,Sqrt[3]}/2;
{x4,y4}={0,-1/Sqrt[3]}/2; {x5,y5}={1/2,1/Sqrt[3]};
{x6,y6}={-1/2,1/Sqrt[3]};
ncoor= {{x1,y1},{x2,y2},{x3,y3},{x4,y4},{x5,y5},{x6,y6}};
Emat=Em/(1-v^2)*{{1,v,0},{v,1,0},{0,0,(1-v)/2}};
For [i=2,i<=5,i++, p={1,-3,3,6,7}][[i]];
Ke=Trig6IsoPMembraneStiffness[ncoor,Emat,h,{True,p}];
Ke=Chop[Simplify[Ke]];
Print["Ke=",SetPrecision[Ke,4]//MatrixForm];
Print["Eigenvalues of Ke=",Chop[Eigenvalues[N[Ke]],.0000001]];
];

```

FIGURE 24.10. Script to form  $\mathbf{K}^e$  for the triangle of Figure 24.9, using four integration rules.

For rule=3 the result (printed to 4 places because of the SetPrecision statement in script) is

$$\begin{bmatrix}
 344.7 & 75.00 & -91.80 & 21.00 & -86.60 & -24.00 & -124.7 & -72.00 & -20.78 & -36.00 & -20.78 & 36.00 \\
 75.00 & 258.1 & -21.00 & -84.87 & 18.00 & -90.07 & 96.00 & 0 & -36.00 & 20.78 & -132.0 & -103.9 \\
 -91.80 & -21.00 & 344.7 & -75.00 & -86.60 & 24.00 & -124.7 & 72.00 & -20.78 & -36.00 & -20.78 & 36.00 \\
 21.00 & -84.87 & -75.00 & 258.1 & -18.00 & -90.07 & -96.00 & 0 & 132.0 & -103.9 & 36.00 & 20.78 \\
 -86.60 & 18.00 & -86.60 & -18.00 & 214.8 & 0 & 41.57 & 0 & -41.57 & 144.0 & -41.57 & -144.0 \\
 -24.00 & -90.07 & 24.00 & -90.07 & 0 & 388.0 & 0 & -41.57 & -24.00 & -83.14 & 24.00 & -83.14 \\
 -124.7 & 96.00 & -124.7 & -96.00 & 41.57 & 0 & 374.1 & 0 & -83.14 & -72.00 & -83.14 & 72.00 \\
 -72.00 & 0 & 72.00 & 0 & 0 & -41.57 & 0 & 374.1 & -72.00 & -166.3 & 72.00 & -166.3 \\
 -20.78 & -36.00 & -20.78 & 132.0 & -41.57 & -24.00 & -83.14 & -72.00 & 374.1 & 0 & -207.8 & 0 \\
 -36.00 & 20.78 & -36.00 & -103.9 & 144.0 & -83.14 & -72.00 & -166.3 & 0 & 374.1 & 0 & -41.57 \\
 -20.78 & -132.0 & -20.78 & 36.00 & -41.57 & 24.00 & -83.14 & 72.00 & -207.8 & 0 & 374.1 & 0 \\
 36.00 & -103.9 & 36.00 & 20.78 & -144.0 & -83.14 & 72.00 & -166.3 & 0 & -41.57 & 0 & 374.1
 \end{bmatrix}$$

(24.37)

For rule=-3:

$$\begin{bmatrix} 566.4 & 139.0 & 129.9 & 21.00 & 79.67 & 8.000 & -364.9 & -104.0 & -205.5 & -36.00 & -205.5 & -28.00 \\ 139.0 & 405.9 & -21.00 & 62.93 & 50.00 & 113.2 & 64.00 & -129.3 & -36.00 & -164.0 & -196.0 & -288.7 \\ 129.9 & -21.00 & 566.4 & -139.0 & 79.67 & -8.000 & -364.9 & 104.0 & -205.5 & 28.00 & -205.5 & 36.00 \\ 21.00 & 62.93 & -139.0 & 405.9 & -50.00 & 113.2 & -64.00 & -129.3 & 196.0 & -288.7 & 36.00 & -164.0 \\ 79.67 & 50.00 & 79.67 & -50.00 & 325.6 & 0 & -143.2 & 0 & -170.9 & 176.0 & -170.9 & -176.0 \\ 8.000 & 113.2 & -8.000 & 113.2 & 0 & 646.6 & 0 & -226.3 & 8.000 & -323.3 & -8.000 & -323.3 \\ -364.9 & 64.00 & -364.9 & -64.00 & -143.2 & 0 & 632.8 & 0 & 120.1 & -104.0 & 120.1 & 104.0 \\ -104.0 & -129.3 & 104.0 & -129.3 & 0 & -226.3 & 0 & 485.0 & -104.0 & 0 & 104.0 & 0 \\ -205.5 & -36.00 & -205.5 & 196.0 & -170.9 & 8.000 & 120.1 & -104.0 & 521.9 & -64.00 & -60.04 & 0 \\ -36.00 & -164.0 & 28.00 & -288.7 & 176.0 & -323.3 & -104.0 & 0 & -64.00 & 595.8 & 0 & 180.1 \\ -205.5 & -196.0 & -205.5 & 36.00 & -170.9 & -8.000 & 120.1 & 104.0 & -60.04 & 0 & 521.9 & 64.00 \\ -28.00 & -288.7 & 36.00 & -164.0 & -176.0 & -323.3 & 104.0 & 0 & 0 & 180.1 & 64.00 & 595.8 \end{bmatrix} \quad (24.38)$$

The stiffness for rule=6 is omitted to save space. For rule=7:

$$\begin{bmatrix} 661.9 & 158.5 & 141.7 & 21.00 & 92.53 & 7.407 & -432.1 & -117.2 & -190.2 & -29.10 & -273.8 & -40.61 \\ 158.5 & 478.8 & -21.00 & 76.13 & 49.41 & 125.3 & 50.79 & -182.7 & -29.10 & -156.6 & -208.6 & -341.0 \\ 141.7 & -21.00 & 661.9 & -158.5 & 92.53 & -7.407 & -432.1 & 117.2 & -273.8 & 40.61 & -190.2 & 29.10 \\ 21.00 & 76.13 & -158.5 & 478.8 & -49.41 & 125.3 & -50.79 & -182.7 & 208.6 & -341.0 & 29.10 & -156.6 \\ 92.53 & 49.41 & 92.53 & -49.41 & 387.3 & 0 & -139.8 & 0 & -216.3 & 175.4 & -216.3 & -175.4 \\ 7.407 & 125.3 & -7.407 & 125.3 & 0 & 753.4 & 0 & -207.0 & 7.407 & -398.5 & -7.407 & -398.5 \\ -432.1 & 50.79 & -432.1 & -50.79 & -139.8 & 0 & 723.6 & 0 & 140.2 & -117.2 & 140.2 & 117.2 \\ -117.2 & -182.7 & 117.2 & -182.7 & 0 & -207.0 & 0 & 562.6 & -117.2 & 4.884 & 117.2 & 4.884 \\ -190.2 & -29.10 & -273.8 & 208.6 & -216.3 & 7.407 & 140.2 & -117.2 & 602.8 & -69.71 & -62.78 & 0 \\ -29.10 & -156.6 & 40.61 & -341.0 & 175.4 & -398.5 & -117.2 & 4.884 & -69.71 & 683.3 & 0 & 207.9 \\ -273.8 & -208.6 & -190.2 & 29.10 & -216.3 & -7.407 & 140.2 & 117.2 & -62.78 & 0 & 602.8 & 69.71 \\ -40.61 & -341.0 & 29.10 & -156.6 & -175.4 & -398.5 & 117.2 & 4.884 & 0 & 207.9 & 69.71 & 683.3 \end{bmatrix} \quad (24.39)$$

The eigenvalues of these matrices are:

Rule	Eigenvalues of $\mathbf{K}^e$											
3	702.83	665.11	553.472	553.472	481.89	429.721	429.721	118.391	118.391	0	0	0
-3	1489.80	1489.80	702.833	665.108	523.866	523.866	481.890	196.429	196.429	0	0	0
6	1775.53	1775.53	896.833	768.948	533.970	533.970	495.570	321.181	321.181	0	0	0
7	1727.11	1727.11	880.958	760.719	532.750	532.750	494.987	312.123	312.123	0	0	0

(24.40)

Since the metric of this element is highly distorted near its boundary, the stiffness matrix entries and eigenvalues change significantly as the integration formulas are advanced from 3 to 6 and 7 points. However as can be seen the matrix remains rank-sufficient.

### §24.5. \*The Cubic Triangle

The 10-node cubic triangle, depicted in Figure 24.11, is rarely used as such in applications because of the difficulty of combining it with other elements. Nevertheless the derivation of the element modules is instructive as this triangle has other and more productive uses as a generator of more practical elements with “drilling” rotational degrees of freedom at corners for modeling shells. Such transformations are studied in advanced FEM courses.

The geometry of the triangle is defined by the coordinates of the ten nodes. A notational warning: the interior node is labeled as 0 instead of 10 (cf. Figure 24.8) to avoid confusion with notation such as  $x_{12} = x_1 - x_2$  for coordinate differences.

## §24.5.1. \*Shape Function Module

The shape functions obtained in Exercise 18.1 are  $N_1 = \frac{1}{2}\zeta_1(3\zeta_1 - 1)(3\zeta_1 - 2)$ ,  $N_2 = \frac{1}{2}\zeta_2(3\zeta_2 - 1)(3\zeta_2 - 2)$ ,  $N_3 = \frac{1}{2}\zeta_3(3\zeta_3 - 1)(3\zeta_3 - 2)$ ,  $N_4 = \frac{9}{2}\zeta_1\zeta_2(3\zeta_1 - 1)$ ,  $N_5 = \frac{9}{2}\zeta_1\zeta_2(3\zeta_2 - 1)$ ,  $N_6 = \frac{9}{2}\zeta_2\zeta_3(3\zeta_2 - 1)$ ,  $N_7 = \frac{9}{2}\zeta_2\zeta_3(3\zeta_3 - 1)$ ,  $N_8 = \frac{9}{2}\zeta_3\zeta_1(3\zeta_3 - 1)$ ,  $N_9 = \frac{9}{2}\zeta_3\zeta_1(3\zeta_1 - 1)$  and  $N_0 = 27\zeta_1\zeta_2\zeta_3$ . Their natural derivatives are

$$\frac{\partial \mathbf{N}^T}{\partial \zeta_1} = \frac{9}{2} \begin{bmatrix} \frac{2}{9} - 2\zeta_1 + 3\zeta_1^2 \\ 0 \\ 0 \\ \zeta_2(6\zeta_1 - 1) \\ \zeta_2(3\zeta_2 - 1) \\ 0 \\ 0 \\ \zeta_3(3\zeta_3 - 1) \\ \zeta_3(6\zeta_1 - 1) \\ 6\zeta_2\zeta_3 \end{bmatrix}, \quad \frac{\partial \mathbf{N}^T}{\partial \zeta_2} = \frac{9}{2} \begin{bmatrix} 0 \\ \frac{2}{9} - 2\zeta_2 + 3\zeta_2^2 \\ 0 \\ \zeta_1(3\zeta_1 - 1) \\ \zeta_1(6\zeta_2 - 1) \\ \zeta_3(6\zeta_2 - 1) \\ \zeta_3(3\zeta_3 - 1) \\ 0 \\ 0 \\ 6\zeta_1\zeta_3 \end{bmatrix}, \quad \frac{\partial \mathbf{N}^T}{\partial \zeta_3} = \frac{9}{2} \begin{bmatrix} 0 \\ 0 \\ \frac{2}{9} - 2\zeta_3 + 3\zeta_3^2 \\ 0 \\ 0 \\ \zeta_2(3\zeta_2 - 1) \\ \zeta_2(6\zeta_3 - 1) \\ \zeta_1(6\zeta_3 - 1) \\ \zeta_1(3\zeta_1 - 1) \\ 6\zeta_1\zeta_2 \end{bmatrix}. \quad (24.41)$$

As in the case of the 6-node triangle it is convenient to introduce the deviations from thirdpoints and centroid:  $\Delta x_4 = x_4 - \frac{1}{3}(2x_1 + x_2)$ ,  $\Delta x_5 = x_5 - \frac{1}{3}(x_1 + 2x_2)$ ,  $\Delta x_6 = x_6 - \frac{1}{3}(2x_2 + x_3)$ ,  $\Delta x_7 = x_7 - \frac{1}{3}(x_2 + 2x_3)$ ,  $\Delta x_8 = x_8 - \frac{1}{3}(2x_3 + x_1)$ ,  $\Delta x_9 = x_9 - \frac{1}{3}(x_3 + 2x_1)$ ,  $\Delta x_0 = x_0 - \frac{1}{3}(x_1 + x_2 + x_3)$ ,  $\Delta y_4 = y_4 - \frac{1}{3}(2y_1 + y_2)$ ,  $\Delta y_5 = y_5 - \frac{1}{3}(y_1 + 2y_2)$ ,  $\Delta y_6 = y_6 - \frac{1}{3}(2y_2 + y_3)$ ,  $\Delta y_7 = y_7 - \frac{1}{3}(y_2 + 2y_3)$ ,  $\Delta y_8 = y_8 - \frac{1}{3}(2y_3 + y_1)$ ,  $\Delta y_9 = y_9 - \frac{1}{3}(y_3 + 2y_1)$ , and  $\Delta y_0 = y_0 - \frac{1}{3}(y_1 + y_2 + y_3)$ .

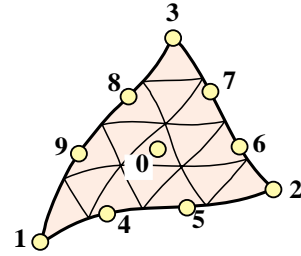


FIGURE 24.11. The 10-node cubic triangle.

Using (24.22) and (24.41), the shape function partial derivatives can be worked out to be

$$\frac{\partial \mathbf{N}^T}{\partial x} = \frac{9}{4J} \begin{bmatrix} J_{y23}(\frac{2}{9} - 2\zeta_1 + 3\zeta_1^2/2) \\ J_{y31}(\frac{2}{9} - 2\zeta_2 + 3\zeta_2^2/2) \\ J_{y12}(\frac{2}{9} - 2\zeta_3 + 3\zeta_3^2/2) \\ J_{y31}\zeta_1(3\zeta_1 - 1) + J_{y23}\zeta_2(6\zeta_1 - 1) \\ J_{y23}\zeta_2(3\zeta_2 - 1) + J_{y31}\zeta_1(6\zeta_2 - 1) \\ J_{y12}\zeta_2(3\zeta_2 - 1) + J_{y31}\zeta_3(6\zeta_2 - 1) \\ J_{y31}\zeta_3(3\zeta_3 - 1) + J_{y12}\zeta_2(6\zeta_3 - 1) \\ J_{y23}\zeta_3(3\zeta_3 - 1) + J_{y12}\zeta_1(6\zeta_3 - 1) \\ J_{y12}\zeta_1(3\zeta_1 - 1) + J_{y23}\zeta_3(6\zeta_1 - 1) \\ 6(J_{y12}\zeta_1\zeta_2 + J_{y31}\zeta_1\zeta_3 + J_{y23}\zeta_2\zeta_3) \end{bmatrix}, \quad \frac{\partial \mathbf{N}^T}{\partial y} = \frac{9}{4J} \begin{bmatrix} J_{x32}(\frac{2}{9} - 2\zeta_1 + 3\zeta_1^2/2) \\ J_{x13}(\frac{2}{9} - 2\zeta_2 + 3\zeta_2^2/2) \\ J_{x21}(\frac{2}{9} - 2\zeta_3 + 3\zeta_3^2/2) \\ J_{x13}\zeta_1(3\zeta_1 - 1) + J_{x32}\zeta_2(6\zeta_1 - 1) \\ J_{x32}\zeta_2(3\zeta_2 - 1) + J_{x13}\zeta_1(6\zeta_2 - 1) \\ J_{x21}\zeta_2(3\zeta_2 - 1) + J_{x13}\zeta_3(6\zeta_2 - 1) \\ J_{x13}\zeta_3(3\zeta_3 - 1) + J_{x21}\zeta_2(6\zeta_3 - 1) \\ J_{x32}\zeta_3(3\zeta_3 - 1) + J_{x21}\zeta_1(6\zeta_3 - 1) \\ J_{x21}\zeta_1(3\zeta_1 - 1) + J_{x32}\zeta_3(6\zeta_1 - 1) \\ 6(J_{x21}\zeta_1\zeta_2 + J_{x13}\zeta_1\zeta_3 + J_{x32}\zeta_2\zeta_3) \end{bmatrix}. \quad (24.42)$$

where the expressions of the Jacobian coefficients are

$$\begin{aligned} J_{x21} &= x_{21} + \frac{9}{2} [\Delta x_4(\zeta_1(3\zeta_1 - 6\zeta_2 - 1) + \zeta_2) + \Delta x_5(\zeta_2(1 - 3\zeta_2 + 6\zeta_1) - \zeta_1) + \Delta x_6\zeta_3(6\zeta_2 - 1) \\ &\quad + \Delta x_7\zeta_3(3\zeta_3 - 1) + \Delta x_8\zeta_3(1 - 3\zeta_3) + \Delta x_9\zeta_3(1 - 6\zeta_1) + 6\Delta x_0\zeta_3(\zeta_1 - \zeta_2)], \\ J_{x32} &= x_{32} + \frac{9}{2} [\Delta x_4\zeta_1(1 - 3\zeta_1) + \Delta x_5\zeta_1(1 - 6\zeta_2) + \Delta x_6(\zeta_2(3\zeta_2 - 6\zeta_3 - 1) + \zeta_3) \\ &\quad + \Delta x_7(\zeta_3(1 - 3\zeta_3 + 6\zeta_2) - \zeta_2) + \Delta x_8\zeta_1(6\zeta_3 - 1) + \Delta x_9\zeta_1(3\zeta_1 - 1) + 6\Delta x_0\zeta_1(\zeta_2 - \zeta_3)], \\ J_{x13} &= x_{13} + \frac{9}{2} [\Delta x_4(6\zeta_1 - 1)\zeta_2 + \Delta x_5\zeta_2(3\zeta_2 - 1) + \Delta x_6\zeta_2(1 - 3\zeta_2) + \Delta x_7\zeta_2(1 - 6\zeta_3) \\ &\quad + \Delta x_8(\zeta_3(3\zeta_3 - 6\zeta_1 - 1) + \zeta_1) + \Delta x_9(\zeta_1(1 - 3\zeta_1 + 6\zeta_3) - \zeta_3) + 6\Delta x_0\zeta_2(\zeta_3 - \zeta_1)], \end{aligned} \quad (24.43)$$

```

Trig10IsoPShapeFunDer[ncoor_,tcoor_] := Module[
  {ζ1,ζ2,ζ3,x1,x2,x3,x4,x5,x6,x7,x8,x9,x0,y1,y2,y3,y4,y5,y6,y7,y8,y9,y0,
   dx4,dx5,dx6,dx7,dx8,dx9,dx0,dy4,dy5,dy6,dy7,dy8,dy9,dy0,
   Jx21,Jx32,Jx13,Jy12,Jy23,Jy31,Nf,dNx,dNy,Jdet},
  {{x1,y1},{x2,y2},{x3,y3},{x4,y4},{x5,y5},{x6,y6},{x7,y7},
   {x8,y8},{x9,y9},{x0,y0}}=ncoor; {ζ1,ζ2,ζ3}=tcoor;
  dx4=x4-(2*x1+x2)/3; dx5=x5-(x1+2*x2)/3; dx6=x6-(2*x2+x3)/3;
  dx7=x7-(x2+2*x3)/3; dx8=x8-(2*x3+x1)/3; dx9=x9-(x3+2*x1)/3;
  dy4=y4-(2*y1+y2)/3; dy5=y5-(y1+2*y2)/3; dy6=y6-(2*y2+y3)/3;
  dy7=y7-(y2+2*y3)/3; dy8=y8-(2*y3+y1)/3; dy9=y9-(y3+2*y1)/3;
  dx0=x0-(x1+x2+x3)/3; dy0=y0-(y1+y2+y3)/3;
  Nf={ζ1*(3*ζ1-1)*(3*ζ1-2),ζ2*(3*ζ2-1)*(3*ζ2-2),ζ3*(3*ζ3-1)*(3*ζ3-2),
     9*ζ1*ζ2*(3*ζ1-1),9*ζ1*ζ2*(3*ζ2-1),9*ζ2*ζ3*(3*ζ2-1),
     9*ζ2*ζ3*(3*ζ3-1),9*ζ3*ζ1*(3*ζ3-1),9*ζ3*ζ1*(3*ζ1-1),54*ζ1*ζ2*ζ3}/2;
  Jx21=x2-x1+(9/2)*(dx4*(ζ1*(3*ζ1-6*ζ2-1)+ζ2)+
    dx5*(ζ2*(1-3*ζ2+6*ζ1)-ζ1)+dx6*ζ3*(6*ζ2-1)+dx7*ζ3*(3*ζ3-1)+
    dx8*ζ3*(1-3*ζ3)+dx9*ζ3*(1-6*ζ1)+6*dx0*ζ3*(ζ1-ζ2));
  Jx32=x3-x2+(9/2)*(dx4*ζ1*(1-3*ζ1)+dx5*ζ1*(1-6*ζ2)+
    dx6*(ζ2*(3*ζ2-6*ζ3-1)+ζ3)+dx7*(ζ3*(1-3*ζ3+6*ζ2)-ζ2)+
    dx8*ζ1*(6*ζ3-1)+dx9*ζ1*(3*ζ1-1)+6*dx0*ζ1*(ζ2-ζ3));
  Jx13=x1-x3+(9/2)*(dx4*(6*ζ1-1)*ζ2+dx5*ζ2*(3*ζ2-1)+
    dx6*ζ2*(1-3*ζ2)+dx7*ζ2*(1-6*ζ3)+dx8*(ζ3*(3*ζ3-6*ζ1-1)+ζ1)+
    dx9*(ζ1*(1-3*ζ1+6*ζ3)-ζ3)+6*dx0*ζ2*(ζ3-ζ1));
  Jy12=y1-y2-(9/2)*(dy4*(ζ1*(3*ζ1-6*ζ2-1)+ζ2)+
    dy5*(ζ2*(1-3*ζ2+6*ζ1)-ζ1)+dy6*ζ3*(6*ζ2-1)+dy7*ζ3*(3*ζ3-1)+
    dy8*ζ3*(1-3*ζ3)+dy9*ζ3*(1-6*ζ1)+6*dy0*ζ3*(ζ1-ζ2));
  Jy23=y2-y3-(9/2)*(dy4*ζ1*(1-3*ζ1)+dy5*ζ1*(1-6*ζ2)+
    dy6*(ζ2*(3*ζ2-6*ζ3-1)+ζ3)+dy7*(ζ3*(1-3*ζ3+6*ζ2)-ζ2)+
    dy8*ζ1*(6*ζ3-1)+dy9*ζ1*(3*ζ1-1)+6*dy0*ζ1*(ζ2-ζ3));
  Jy31=y3-y1-(9/2)*(dy4*(6*ζ1-1)*ζ2+dy5*ζ2*(3*ζ2-1)+
    dy6*ζ2*(1-3*ζ2)+dy7*ζ2*(1-6*ζ3)+dy8*(ζ3*(3*ζ3-6*ζ1-1)+ζ1)+
    dy9*(ζ1*(1-3*ζ1+6*ζ3)-ζ3)+6*dy0*ζ2*(ζ3-ζ1));
  Jdet = Jx21*Jy31-Jy12*Jx13;
  dNx={Jy23*(2/9-2*ζ1+3*ζ1^2),Jy31*(2/9-2*ζ2+3*ζ2^2),Jy12*(2/9-2*ζ3+3*ζ3^2),
     Jy31*ζ1*(3*ζ1-1)+Jy23*ζ2*(6*ζ1-1),Jy23*ζ2*(3*ζ2-1)+Jy31*ζ1*(6*ζ2-1),
     Jy12*ζ2*(3*ζ2-1)+Jy31*ζ3*(6*ζ2-1),Jy31*ζ3*(3*ζ3-1)+Jy12*ζ2*(6*ζ3-1),
     Jy23*ζ3*(3*ζ3-1)+Jy12*ζ1*(6*ζ3-1),Jy12*ζ1*(3*ζ1-1)+Jy23*ζ3*(6*ζ1-1),
     6*(Jy12*ζ1*ζ2+Jy31*ζ1*ζ3+Jy23*ζ2*ζ3)}/(2*Jdet/9);
  dNy={Jx32*(2/9-2*ζ1+3*ζ1^2),Jx13*(2/9-2*ζ2+3*ζ2^2),Jx21*(2/9-2*ζ3+3*ζ3^2),
     Jx13*ζ1*(3*ζ1-1)+Jx32*ζ2*(6*ζ1-1),Jx32*ζ2*(3*ζ2-1)+Jx13*ζ1*(6*ζ2-1),
     Jx21*ζ2*(3*ζ2-1)+Jx13*ζ3*(6*ζ2-1),Jx13*ζ3*(3*ζ3-1)+Jx21*ζ2*(6*ζ3-1),
     Jx32*ζ3*(3*ζ3-1)+Jx21*ζ1*(6*ζ3-1),Jx21*ζ1*(3*ζ1-1)+Jx32*ζ3*(6*ζ1-1),
     6*(Jx21*ζ1*ζ2+Jx13*ζ1*ζ3+Jx32*ζ2*ζ3)}/(2*Jdet/9);
  Return[Simplify[{Nf,dNx,dNy,Jdet}]]
];

```

FIGURE 24.12. Shape function module for 10-node cubic triangle.

and

$$\begin{aligned}
J_{y12} &= y_{12} - \frac{9}{2} \left[ \Delta y_4 (\zeta_1 (3\zeta_1 - 6\zeta_2 - 1) + \zeta_2) + \Delta y_5 (\zeta_2 (1 - 3\zeta_2 + 6\zeta_1) - \zeta_1) + \Delta y_6 \zeta_3 (6\zeta_2 - 1) \right. \\
&\quad \left. + \Delta y_7 \zeta_3 (3\zeta_3 - 1) + \Delta y_8 \zeta_3 (1 - 3\zeta_3) + \Delta y_9 \zeta_3 (1 - 6\zeta_1) + 6\Delta y_0 \zeta_3 (\zeta_1 - \zeta_2) \right], \\
J_{y23} &= y_{23} - \frac{9}{2} \left[ \Delta y_4 \zeta_1 (1 - 3\zeta_1) + \Delta y_5 \zeta_1 (1 - 6\zeta_2) + \Delta y_6 (\zeta_2 (3\zeta_2 - 6\zeta_3 - 1) + \zeta_3) \right. \\
&\quad \left. + \Delta y_7 (\zeta_3 (1 - 3\zeta_3 + 6\zeta_2) - \zeta_2) + \Delta y_8 \zeta_1 (6\zeta_3 - 1) + \Delta y_9 \zeta_1 (3\zeta_1 - 1) + 6\Delta y_0 \zeta_1 (\zeta_2 - \zeta_3) \right], \\
J_{y31} &= y_{31} - \frac{9}{2} \left[ \Delta y_4 (6\zeta_1 - 1) \zeta_2 + \Delta y_5 \zeta_2 (3\zeta_2 - 1) + \Delta y_6 \zeta_2 (1 - 3\zeta_2) + \Delta y_7 \zeta_2 (1 - 6\zeta_3) \right. \\
&\quad \left. + \Delta y_8 (\zeta_3 (3\zeta_3 - 6\zeta_1 - 1) + \zeta_1) + \Delta y_9 (\zeta_1 (1 - 3\zeta_1 + 6\zeta_3) - \zeta_3) + 6\Delta y_0 \zeta_2 (\zeta_3 - \zeta_1) \right].
\end{aligned} \tag{24.44}$$

The Jacobian determinant is  $J = \frac{1}{2}(J_{x21}J_{y31} - J_{y12}J_{x13})$ . The shape function module Trig10IsoPShapeFunDer that implements these expressions is listed in Figure 24.12. This has the same arguments as Trig6IsoPShapeFunDer. As there Jdet denotes  $2J$ .

```

Trig10IsoPMembraneStiffness[ncoor_,Emat_,th_,options_]:=
Module[{i,k,l,p=6,numer=False,h=th,tcoor,w,c,
Nf,dNx,dNy,Jdet,Be,Ke=Table[0,{20},{20}]},
If [Length[options]>=1, numer=options[[1]]];
If [Length[options]>=2, p= options[[2]]];
If [!MemberQ[{1,3,-3,6,7},p], Print["Illegal p"]; Return[Null]];
For [k=1, k<=Abs[p], k++,
{tcoor,w}= TrigGaussRuleInfo[{p,numer},k];
{Nf,dNx,dNy,Jdet}= Trig10IsoPShapeFunDer[ncoor,tcoor];
If [numer, {Nf,dNx,dNy,Jdet}=N[{Nf,dNx,dNy,Jdet}]];
If [Length[th]==10, h=th.Nf]; c=w*Jdet*h/2;
Be= {Flatten[Table[{dNx[[i]],0},{i,10}]},
Flatten[Table[{0,dNy[[i]]},{i,10}]},
Flatten[Table[{dNy[[i]],dNx[[i]]},{i,10}]]};
Ke+=c*Transpose[Be].(Emat.Be);
]; If[!numer,Ke=Simplify[Ke]]; Return[Ke]
];

```

FIGURE 24.13. Stiffness module for 10-node cubic triangle.

### §24.5.2. \*Stiffness Module

Module `Trig10IsoPMembraneStiffness`, listed in Figure 24.13, implements the computation of the element stiffness matrix of the 10-node cubic plane stress triangle. The module is invoked as

$$Ke = \text{Trig10IsoPMembraneStiffness}[ncoor,Emat,th,options] \quad (24.45)$$

this has the same arguments of the quadratic triangle module invoked in (24.32), with the following changes. The node coordinate list `ncoor` contains 10 entries:  $\{\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \dots, \{x_9, y_9\}, \{x_0, y_0\}\}$ . If the plate thickness varies, `th` is a list of 10 entries:  $\{h_1, h_2, h_3, h_4, \dots, h_9, h_0\}$ . The triangle Gauss rule specified in options as  $\{\text{numer}, \text{rule}\}$  should be 6 or 7 to produce a rank sufficient stiffness. If omitted `rule = 6` is assumed. The module returns `Ke` as an  $20 \times 20$  symmetric matrix pertaining to the following arrangement of nodal displacements:

$$\mathbf{u}^e = [u_{x1} \ u_{y1} \ u_{x2} \ u_{y2} \ u_{x3} \ u_{y3} \ u_{x4} \ u_{y4} \ \dots \ u_{x9} \ u_{y9} \ u_{x0} \ u_{y0}]^T. \quad (24.46)$$

**Remark 24.3.** For symbolic work with this element the 7-point rule should be preferred because the exact expressions of the abscissas and weights at sample points are simpler than for the 6-point rule, as can be observed in Figure 24.3. This speeds up algebraic simplification. For numerical work the 6-point rule is slightly faster.

### §24.5.3. \*Test Element

The stiffness module is tested on the superparametric triangle geometry shown in Figure 24.14, which has been already used for the quadratic triangle.

The corner nodes are placed at  $(0, 0)$ ,  $(6, 2)$  and  $(4, 4)$ . The 6 side nodes are located at the thirdpoints of the sides and the interior node at the centroid. The plate has unit thickness. The material is isotropic with  $E = 1920$  and  $\nu = 0$ .

The script of Figure 24.15 computes  $\mathbf{K}^e$  using the 7-point Gauss rule and exact arithmetic.

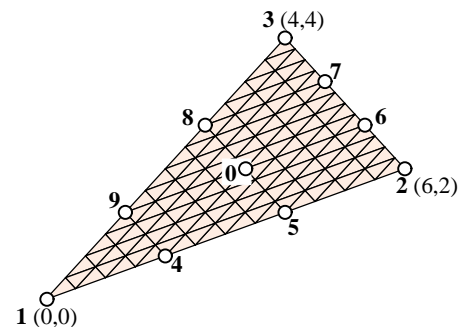


FIGURE 24.14. Test 10-node triangle.

The returned stiffness matrix using either 6- or 7-point integration is the same, since for a superparametric element the integrand is quartic in the triangular coordinates. For the given combination of inputs the entries are exact integers:

$$\mathbf{K}^e = \begin{bmatrix}
 306 & 102 & -42 & -42 & -21 & 21 & -315 & -333 & 171 & 153 \\
 102 & 306 & 42 & 42 & -63 & -105 & 351 & 369 & -135 & -117 \\
 -42 & 42 & 1224 & -408 & -210 & 42 & 252 & -180 & -234 & 306 \\
 -42 & 42 & -408 & 1224 & 126 & -294 & 108 & -36 & -378 & 450 \\
 -21 & -63 & -210 & 126 & 1122 & -306 & -99 & 27 & -99 & 27 \\
 21 & -105 & 42 & -294 & -306 & 1938 & 27 & -171 & 27 & -171 \\
 -315 & 351 & 252 & 108 & -99 & 27 & 5265 & -1215 & -1539 & -243 \\
 -333 & 369 & -180 & -36 & 27 & -171 & -1215 & 6885 & 729 & -891 \\
 171 & -135 & -234 & -378 & -99 & 27 & -1539 & 729 & 5265 & -1215 \\
 153 & -117 & 306 & 450 & 27 & -171 & -243 & -891 & -1215 & 6885 \\
 -27 & -9 & -1602 & 990 & 819 & -171 & 81 & 81 & -405 & -405 \\
 -9 & -27 & 306 & -2286 & -459 & 1179 & 81 & 405 & -405 & -2025 \\
 -27 & -9 & 828 & -468 & -1611 & 315 & 81 & 81 & 81 & 81 \\
 -9 & -27 & -180 & 1116 & 999 & -2223 & 81 & 405 & 81 & 405 \\
 99 & 225 & -108 & 36 & -72 & 144 & 810 & -324 & 810 & -324 \\
 -63 & 387 & 36 & -108 & -540 & -684 & -324 & 1134 & -324 & 1134 \\
 -144 & -504 & -108 & 36 & 171 & -99 & -4050 & 1620 & 810 & -324 \\
 180 & -828 & 36 & -108 & 189 & 531 & 1620 & -5670 & -324 & 1134 \\
 0 & 0 & 0 & 0 & 0 & 0 & -486 & -486 & -4860 & 1944 \\
 0 & 0 & 0 & 0 & 0 & 0 & -486 & -2430 & 1944 & -6804 \\
 \\
 -27 & -9 & -27 & -9 & 99 & -63 & -144 & 180 & 0 & 0 \\
 -9 & -27 & -9 & -27 & 225 & 387 & -504 & -828 & 0 & 0 \\
 -1602 & 306 & 828 & -180 & -108 & 36 & -108 & 36 & 0 & 0 \\
 990 & -2286 & -468 & 1116 & 36 & -108 & 36 & -108 & 0 & 0 \\
 819 & -459 & -1611 & 999 & -72 & -540 & 171 & 189 & 0 & 0 \\
 -171 & 1179 & 315 & -2223 & 144 & -684 & -99 & 531 & 0 & 0 \\
 81 & 81 & 81 & 81 & 810 & -324 & -4050 & 1620 & -486 & -486 \\
 81 & 405 & 81 & 405 & -324 & 1134 & 1620 & -5670 & -486 & -2430 \\
 -405 & -405 & 81 & 81 & 810 & -324 & 810 & -324 & -4860 & 1944 \\
 -405 & -2025 & 81 & 405 & -324 & 1134 & -324 & 1134 & 1944 & -6804 \\
 5265 & -1215 & -3483 & 729 & 162 & 0 & 162 & 0 & -972 & 0 \\
 -1215 & 6885 & 1701 & -4779 & 0 & -162 & 0 & -162 & 0 & 972 \\
 -3483 & 1701 & 5265 & -1215 & -810 & 0 & 162 & 0 & -486 & -486 \\
 729 & -4779 & -1215 & 6885 & 0 & 810 & 0 & -162 & -486 & -2430 \\
 162 & 0 & -810 & 0 & 5265 & -1215 & -1296 & -486 & -4860 & 1944 \\
 0 & -162 & 0 & 810 & -1215 & 6885 & 486 & -2592 & 1944 & -6804 \\
 162 & 0 & 162 & 0 & -1296 & 486 & 5265 & -1215 & -972 & 0 \\
 0 & -162 & 0 & -162 & -486 & -2592 & -1215 & 6885 & 0 & 972 \\
 -972 & 0 & -486 & -486 & -4860 & 1944 & -972 & 0 & 12636 & -2916 \\
 0 & 972 & -486 & -2430 & 1944 & -6804 & 0 & 972 & -2916 & 16524
 \end{bmatrix} \quad (24.47)$$

The eigenvalues are

$$[26397. \quad 16597. \quad 14937. \quad 12285. \quad 8900.7 \quad 7626.2 \quad 5417.8 \quad 4088.8 \quad 3466.8 \quad 3046.4 \\
 1751.3 \quad 1721.4 \quad 797.70 \quad 551.82 \quad 313.22 \quad 254.00 \quad 28.019 \quad 0 \quad 0 \quad 0] \quad (24.48)$$

The 3 zero eigenvalues pertain to the three independent rigid-body modes in two dimensions. The 17 other eigenvalues are positive. Consequently the computed  $\mathbf{K}^e$  has the correct rank of 17.

```

ClearAll[Em,ν,a,b,e,h]; Em=1920; ν=0; h=1;
ncoor={{0,0},{6,2},{4,4}};
x4=(2*x1+x2)/3; x5=(x1+2*x2)/3; y4=(2*y1+y2)/3; y5=(y1+2*y2)/3;
x6=(2*x2+x3)/3; x7=(x2+2*x3)/3; y6=(2*y2+y3)/3; y7=(y2+2*y3)/3;
x8=(2*x3+x1)/3; x9=(x3+2*x1)/3; y8=(2*y3+y1)/3; y9=(y3+2*y1)/3;
x0=(x1+x2+x3)/3; y0=(y1+y2+y3)/3;
ncoor={{x1,y1},{x2,y2},{x3,y3},{x4,y4},{x5,y5},{x6,y6},{x7,y7},
{x8,y8},{x9,y9},{x0,y0}};
Emat=Em/(1-ν^2)*{{1,ν,0},{ν,1,0},{0,0,(1-ν)/2}};
Ke=Trig10IsoPMembraneStiffness[ncoor,Emat,h,{False,7}];
Ke=Simplify[Ke]; Print[Ke//MatrixForm]; ev=Chop[Eigenvalues[N[Ke]]];
Print["eigs of Ke=",ev];

```

FIGURE 24.15. Script for testing cubic triangle of Figure 24.14.

### Notes and Bibliography

The 3-node, 6-node and 10-node plane stress triangular elements are generated by complete polynomials in two-dimensions. In order of historical appearance:

1. The three-node linear triangle, also known as *Constant Strain Triangle* (CST) and Turner triangle, was developed as “triangular skin panel” by Turner, Clough and Martin in 1951–53 [46,47] using interelement flux assumptions, and published in 1956 [251]. It is not clear when the assumed-displacement derivation, which yields the same stiffness matrix, was done first. The displacement derivation is mentioned in passing by Clough in [38] and worked out in the theses of Melosh [174] and Wilson [264].
2. The six-node quadratic triangle, also known as *Linear Strain Triangle* (LST) and Veubeke triangle, was developed by B. M. Fraeijs de Veubeke in 1962–63 [277]; published 1965 [99].
3. The ten-node cubic triangle, also known as *Quadratic Strain Triangle* (QST), was developed by the writer in 1965; published 1966 [65]. Shape functions for the cubic triangle were presented there but used for plate bending instead of plane stress.

A version of the cubic triangle with freedoms migrated to corners and recombined to produce a “drilling rotation” at corners, was used in static and dynamic shell analysis in Carr’s thesis under Ray Clough [35,42]. The drilling-freedom idea was independently exploited for rectangular and quadrilateral elements, respectively, in the theses of Abu-Ghazaleh [2] and Willam [263], both under Alex Scordelis. A variant of the Willam quadrilateral, developed by Bo Almroth at Lockheed, has survived in the nonlinear shell analysis code STAGS as element 410 [205].

Numerical integration came into FEM by the mid 1960s. Five triangle integration rules were tabulated in the writer’s thesis [65, pp. 38–39]. These were gathered from three sources: two papers by Hammer and Stroud [121,122] and the 1964 Handbook of Mathematical Functions [1, §25.4]. They were adapted to FEM by converting Cartesian abscissas to triangle coordinates. The table has been reproduced in Zienkiewicz’ book since the second edition [274, Table 8.2] and, with corrections and additions, in the monograph of Strang and Fix [224, p. 184].

The monograph by Stroud [227] contains a comprehensive collection of quadrature formulas for multiple integrals. That book gathers most of the formulas known by 1970, as well as references until that year. (Only a small fraction of Stroud’s tabulated rules, however, are suitable for FEM work.) The collection has been periodically kept up to date by Cools [49–51], who also maintains a dedicated web site: <http://www.cs.kuleuven.ac.be/~nines/research/ecf/ecf.html> This site provides rule information in 16- and 32-digit accuracy for many geometries and dimensionalities as well as a linked “index card” of references to source publications.

### References

Referenced items moved to Appendix R.

### Homework Exercises for Chapter 24 Implementation of Iso-P Triangular Elements

**EXERCISE 24.1** [C:20] Write an element stiffness module and a shape function module for the 4-node “transition” iso-P triangular element with only one side node: 4, which is located between 1 and 2. See Figure E24.1. The shape functions are  $N_1 = \zeta_1 - 2\zeta_1\zeta_2$ ,  $N_2 = \zeta_2 - 2\zeta_1\zeta_2$ ,  $N_3 = \zeta_3$  and  $N_4 = 4\zeta_1\zeta_2$ . Use the three interior point integration rule=3.

Test the element for the geometry of the triangle depicted in Figure 24.7, removing nodes 5 and 6, and with  $E = 2880$ ,  $\nu = 1/3$  and  $h = 1$ . Report results for the stiffness matrix  $\mathbf{K}^e$  and its 8 eigenvalues. Note: Notebook Trig6Stiffness.nb for the 6-node triangle, posted on the index of Chapter 24, may be used as “template” in support of this Exercise. Partial results:  $K_{11} = 1980$ ,  $K_{18} = 1440$ .

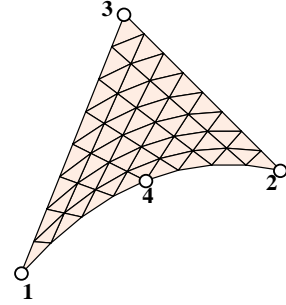


FIGURE E24.1. The 4-node transition triangle for Exercise 24.1.

**EXERCISE 24.2** [A/C:5+20] As in the foregoing Exercise, but now write the module for a five-node triangular “transition” element that lacks midnode 6 opposite corner 2. Begin by deriving the five shape functions.

**EXERCISE 24.3** [A+C:25] Consider the superparametric straight-sided 6-node triangle where side nodes are located at the midpoints. By setting  $x_4 = \frac{1}{2}(x_1 + x_2)$ ,  $x_5 = \frac{1}{2}(x_2 + x_3)$ ,  $x_3 = \frac{1}{2}(x_3 + x_1)$ ,  $y_4 = \frac{1}{2}(y_1 + y_2)$ ,  $y_5 = \frac{1}{2}(y_2 + y_3)$ ,  $y_3 = \frac{1}{2}(y_3 + y_1)$  in (24.26), deduce that

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 \\ 3x_1 & 2x_1 + x_2 & 2x_1 + x_3 \\ 3y_1 & 2y_1 + y_2 & 2y_1 + y_3 \end{bmatrix} \zeta_1 + \begin{bmatrix} 1 & 1 & 1 \\ x_1 + 2x_2 & 3x_2 & 2x_2 + x_3 \\ y_1 + 2y_2 & 3y_2 & 2y_2 + y_3 \end{bmatrix} \zeta_2 + \begin{bmatrix} 1 & 1 & 1 \\ x_1 + 2x_3 & x_2 + 2x_3 & 3x_3 \\ y_1 + 2y_3 & y_2 + 2y_3 & 3y_3 \end{bmatrix} \zeta_3. \quad (\text{E24.1})$$

This contradicts publications that, by mistakenly assuming that the results for the linear triangle (Chapter 15) can be extended by analogy, take

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \quad (\text{E24.2})$$

Show, however, that

$$2J = 2A = \det \mathbf{J} = x_3 y_{12} + x_1 y_{23} + x_2 y_{31}, \quad \mathbf{P} = \frac{1}{2J} \begin{bmatrix} y_{23} & y_{32} \\ y_{31} & x_{13} \\ y_{12} & x_{21} \end{bmatrix} \quad (\text{E24.3})$$

are the same for both (E24.1) and (E24.2). Thus the mistake has no effect on the computation of derivatives.

**EXERCISE 24.4** [A/C:15+15] Consider the superparametric straight-sided 6-node triangle where side nodes are at the midpoints of the sides and the thickness  $h$  is constant. Using the 3-midpoint quadrature rule show that the element stiffness can be expressed in a closed form obtained in 1966 [65]:

$$\mathbf{K}^e = \frac{1}{3} Ah (\mathbf{B}_1^T \mathbf{E} \mathbf{B}_1 + \mathbf{B}_2^T \mathbf{E} \mathbf{B}_2 + \mathbf{B}_3^T \mathbf{E} \mathbf{B}_3) \quad (\text{E24.4})$$

in which  $A$  is the triangle area and

$$\begin{aligned} \mathbf{B}_1 &= \frac{1}{2A} \begin{bmatrix} y_{32} & 0 & y_{31} & 0 & y_{12} & 0 & 2y_{23} & 0 & 2y_{32} & 0 & 2y_{23} & 0 \\ 0 & x_{23} & 0 & x_{13} & 0 & x_{21} & 0 & 2x_{32} & 0 & 2x_{23} & 0 & 2x_{32} \\ x_{23} & y_{32} & x_{13} & y_{31} & x_{21} & y_{12} & 2x_{32} & 2y_{23} & 2x_{23} & 2y_{32} & 2x_{32} & 2y_{23} \end{bmatrix} \\ \mathbf{B}_2 &= \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{13} & 0 & y_{12} & 0 & 2y_{31} & 0 & 2y_{31} & 0 & 2y_{13} & 0 \\ 0 & x_{32} & 0 & x_{31} & 0 & x_{21} & 0 & 2x_{13} & 0 & 2x_{13} & 0 & 2x_{31} \\ x_{32} & y_{23} & x_{31} & y_{13} & x_{21} & y_{12} & 2x_{13} & 2y_{31} & 2x_{13} & 2y_{31} & 2x_{31} & 2y_{13} \end{bmatrix} \\ \mathbf{B}_3 &= \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{21} & 0 & 2y_{21} & 0 & 2y_{12} & 0 & 2y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{12} & 0 & 2x_{12} & 0 & 2x_{21} & 0 & 2x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{12} & y_{21} & 2x_{12} & 2y_{21} & 2x_{21} & 2y_{12} & 2x_{21} & 2y_{12} \end{bmatrix} \end{aligned} \quad (\text{E24.5})$$

With this form  $\mathbf{K}^e$  can be computed in approximately 1000 floating-point operations.

Using next the 3-interior-point quadrature rule, show that the element stiffness can be expressed again as (E24.4) but with

$$\begin{aligned} \mathbf{B}_1 &= \frac{1}{6A} \begin{bmatrix} 5y_{23} & 0 & y_{13} & 0 & y_{21} & 0 & 2y_{21} + 6y_{31} & 0 & 2y_{32} & 0 & 6y_{12} + 2y_{13} & 0 \\ 0 & 5x_{32} & 0 & x_{31} & 0 & x_{12} & 0 & 2x_{12} + 6x_{13} & 0 & 2x_{23} & 0 & 6x_{21} + 2x_{31} \\ 5x_{32} & 5y_{23} & x_{31} & y_{13} & x_{12} & y_{21} & 2x_{12} + 6x_{13} & 2y_{21} + 6y_{31} & 2x_{23} & 2y_{32} & 6x_{21} + 2x_{31} & 6y_{12} + 2y_{13} \end{bmatrix} \\ \mathbf{B}_2 &= \frac{1}{6A} \begin{bmatrix} y_{32} & 0 & 5y_{31} & 0 & y_{21} & 0 & 2y_{21} + 6y_{23} & 0 & 6y_{12} + 2y_{32} & 0 & 2y_{13} & 0 \\ 0 & x_{23} & 0 & 5x_{13} & 0 & x_{12} & 0 & 2x_{12} + 6x_{32} & 0 & 6x_{21} + 2x_{23} & 0 & 2x_{31} \\ x_{23} & y_{32} & 5x_{13} & 5y_{31} & x_{12} & y_{21} & 2x_{12} + 6x_{32} & 2y_{21} + 6y_{23} & 6x_{21} + 2x_{23} & 6y_{12} + 2y_{32} & 2x_{31} & 2y_{13} \end{bmatrix} \\ \mathbf{B}_3 &= \frac{1}{6A} \begin{bmatrix} y_{32} & 0 & y_{13} & 0 & 5y_{12} & 0 & 2y_{21} & 0 & 6y_{31} + 2y_{32} & 0 & 2y_{13} + 6y_{23} & 0 \\ 0 & x_{23} & 0 & x_{31} & 0 & 5x_{21} & 0 & 2x_{12} & 0 & 6x_{13} + 2x_{23} & 0 & 2x_{31} + 6x_{32} \\ x_{23} & y_{32} & x_{31} & y_{13} & 5x_{21} & 5y_{12} & 2x_{12} & 2y_{21} & 6x_{13} + 2x_{23} & 6y_{31} + 2y_{32} & 2x_{31} + 6x_{32} & 2y_{13} + 6y_{23} \end{bmatrix} \end{aligned} \quad (\text{E24.6})$$

The fact that two very different expressions yield the same  $\mathbf{K}^e$  explains why sometimes authors rediscover the same element derived with different methods.

Yet another set that produces the correct stiffness is

$$\begin{aligned} \mathbf{B}_1 &= \frac{1}{6A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{21} & 0 & 2y_{21} & 0 & 2y_{12} & 0 & 2y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{12} & 0 & 2x_{12} & 0 & 2x_{21} & 0 & 2x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{12} & y_{21} & 2x_{12} & 2y_{21} & 2x_{21} & 2y_{12} & 2x_{21} & 2y_{12} \end{bmatrix} \\ \mathbf{B}_2 &= \frac{1}{6A} \begin{bmatrix} y_{32} & 0 & y_{31} & 0 & y_{12} & 0 & 2y_{23} & 0 & 2y_{32} & 0 & 2y_{23} & 0 \\ 0 & x_{23} & 0 & x_{13} & 0 & x_{21} & 0 & 2x_{32} & 0 & 2x_{23} & 0 & 2x_{32} \\ x_{23} & y_{32} & x_{13} & y_{31} & x_{21} & y_{12} & 2x_{32} & 2y_{23} & 2x_{23} & 2y_{32} & 2x_{32} & 2y_{23} \end{bmatrix} \\ \mathbf{B}_3 &= \frac{1}{6A} \begin{bmatrix} y_{23} & 0 & y_{13} & 0 & y_{12} & 0 & 2y_{31} & 0 & 2y_{31} & 0 & 2y_{13} & 0 \\ 0 & x_{32} & 0 & x_{31} & 0 & x_{21} & 0 & 2x_{13} & 0 & 2x_{13} & 0 & 2x_{31} \\ x_{32} & y_{23} & x_{31} & y_{13} & x_{21} & y_{12} & 2x_{13} & 2y_{31} & 2x_{13} & 2y_{31} & 2x_{31} & 2y_{13} \end{bmatrix} \end{aligned} \quad (\text{E24.7})$$

**EXERCISE 24.5** [A/C:40] (Research paper level) Characterize the most general form of the  $\mathbf{B}_i$  ( $i = 1, 2, 3$ ) matrices that produce the same stiffness matrix  $\mathbf{K}^e$  in (E24.4). (Entails solving an algebraic Riccati equation.)

**EXERCISE 24.6** [A/C:25]. Derive the 4-node transition triangle by forming the 6-node stiffness and applying MFCs to eliminate 5 and 6. Prove that this technique only works if sides 1–3 and 2–3 are straight with nodes 5 and 6 initially at the midpoint of those sides.