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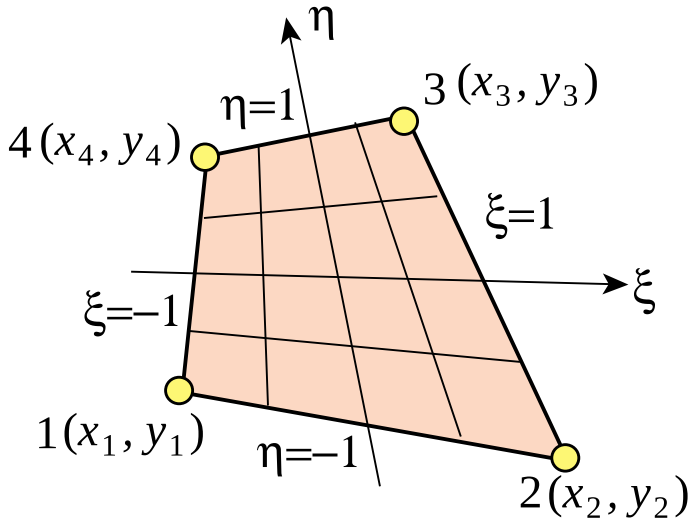
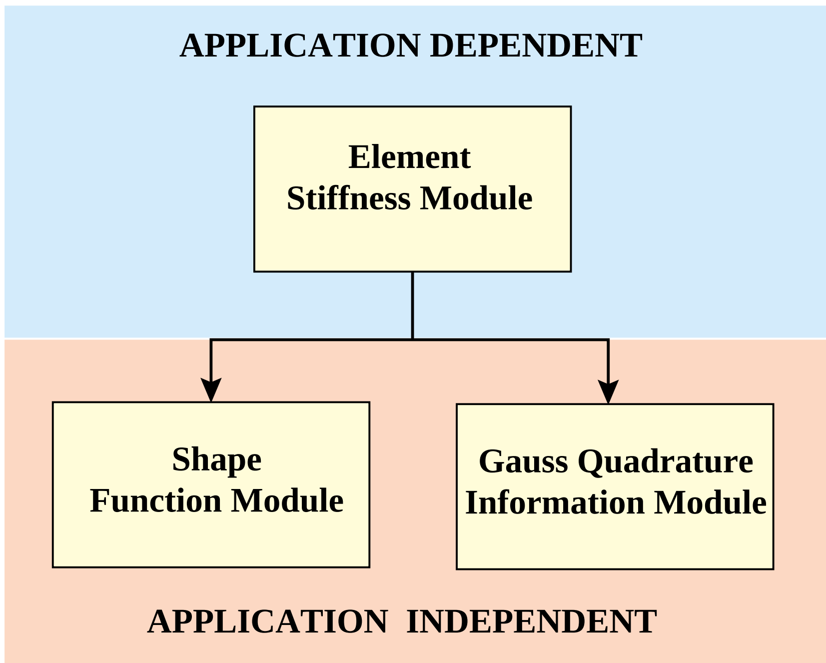
Implementation of Iso-P Quadrilateral Elements

The 4-Node Bilinear Iso-P Quad



Organization of Stiffness Computation



Quadrilateral Gauss Product Rule Module

```
QuadGaussRuleInfo[{rule_,numer_},point_]:= Module[
 {ξ,η,p1,p2,i,j,w1,w2,m,info={{Null,Null},0}},
  If [Length[rule]==2,  {p1,p2}=rule, p1=p2=rule];
  If [p1<0, Return[QuadNonProductGaussRuleInfo[
      {-p1,numer},point]]];
  If [Length[point]==2, {i,j}=point, m=point;
      j=Floor[(m-1)/p1]+1; i=m-p1*(j-1) ];
  {ξ,w1}=  LineGaussRuleInfo[{p1,numer},i];
  {η,w2}=  LineGaussRuleInfo[{p2,numer},j];
  info={{ξ,η},w1*w2};
  If [numer, Return[N[info]], Return[Simplify[info]]];
];
```

Already covered in Chapter 17,
shown here for completeness

Shape Function Module for Bilinear Quad

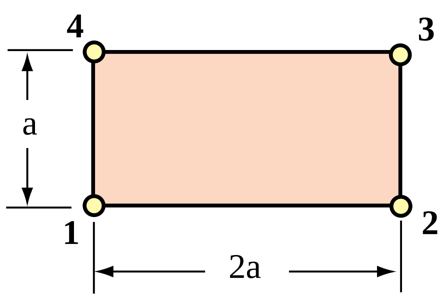
```
Quad4IsoPShapeFunDer[ncoor_,qcoor_]:= Module[
  {Nf,dNx,dNy,dNξ,dNη,i,J11,J12,J21,J22,Jdet,ξ,η,x,y},
  {ξ,η}=qcoor;
   Nf={(1-ξ)*(1-η),(1+ξ)*(1-η),(1+ξ)*(1+η),(1-ξ)*(1+η)}/4;
   dNξ ={-(1-η), (1-η),(1+η),-(1+η)}/4;
   dNη= {-(1-ξ),-(1+ξ),(1+ξ), (1-ξ)}/4;
   x=Table[ncoor[[i,1]],{i,4}]; y=Table[ncoor[[i,2]],{i,4}];
   J11=dNξ.x; J12=dNξ.y; J21=dNη.x; J22=dNη.y;
   Jdet=Simplify[J11*J22-J12*J21];
   dNx= ( J22*dNξ-J12*dNη)/Jdet;  dNx=Simplify[dNx];
   dNy= (-J21*dNξ+J11*dNη)/Jdet;  dNy=Simplify[dNy];
   Return[{Nf,dNx,dNy,Jdet}]
];
```

Already covered in Chapter 17, shown here for completeness

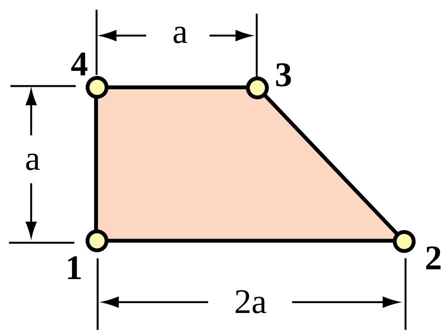
Element Stiffness Module for 4-Node Quad

```
Quad4IsoPMembraneStiffness[ncoor_,Emat_,th_,options_]:=
  Module[{i,k,p=2,numer=False,h=th,qcoor,c,w,Nf,
    dNx,dNy,Jdet,Be,Ke=Table[0,{8},{8}]},
  If [Length[options]==2,{numer,p}=options,{numer}=options];
  If [p<1||p>4, Print["p out of range"]; Return[Null]];
  For [k=1, k<=p*p, k++,
       {qcoor,w}= QuadGaussRuleInfo[{p,numer},k];
       {Nf,dNx,dNy,Jdet}=Quad4IsoPShapeFunDer[ncoor,qcoor];
        If [Length[th]>0, h=th.Nf]; c=w*Jdet*h;
        Be={Flatten[Table[{dNx[[i]],       0},{i,4}]],
            Flatten[Table[{0,       dNy[[i]]},{i,4}]],
            Flatten[Table[{dNy[[i]],dNx[[i]]},{i,4}]]};
        Ke+=Simplify[c*Transpose[Be].(Emat.Be)];
      ]; Return[Simplify[Ke]]
   ];
```

Testing the Element Stiffness Module on Two Quadrilateral Geometries



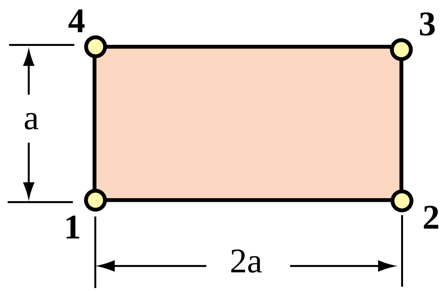
Rectangle



Right Trapezoid

Main purpose is to illustrate effect of changing Gauss integration rule

Testing the Rectangular Element




```
ClearAll[Em,nu,a,b,e,h,p,numer];  h=1;
Em=96; nu=1/3;   (* isotropic material *)
Emat=Em/(1-nu^2)*{{1,nu,0},{nu,1,0},{0,0,(1-nu)/2}};
Print["Emat=",Emat//MatrixForm];
ncoor={{0,0},{2*a,0},{2*a,a},{0,a}}; (* 2:1 rectangular geometry *)
p=2;(* 2 x  2 Gauss rule *)numer=False;(* exact symbolic arithmetic *)
Ke=Quad4IsoPMembraneStiffness[ncoor,Emat,h,{numer,p}];
Ke=Simplify[Chop[Ke]];  Print["Ke=",Ke//MatrixForm];
Print["Eigenvalues of Ke=",Chop[Eigenvalues[N[Ke]],.0000001]];
```

Uniform thickness $h = 1$, isotropic material with $E = 96$ and $\nu = 1/3$.
Rectangle dimension a cancels out in forming stiffness.

Results From Test of Rectangular Element

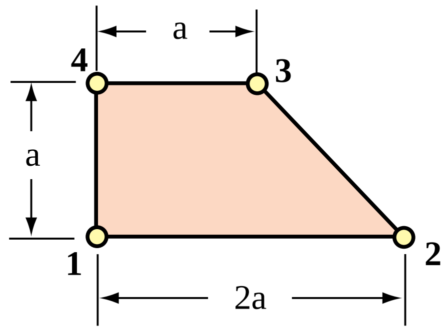
Stiffness matrix computed by *p x p* rule, *p* = 2, 3, 4, ...

$$\mathbf{K}^e = \begin{bmatrix} 42 & 18 & -6 & 0 & -21 & -18 & -15 & 0 \\ 18 & 78 & 0 & 30 & -18 & -39 & 0 & -69 \\ -6 & 0 & 42 & -18 & -15 & 0 & -21 & 18 \\ 0 & 30 & -18 & 78 & 0 & -69 & 18 & -39 \\ -21 & -18 & -15 & 0 & 42 & 18 & -6 & 0 \\ -18 & -39 & 0 & -69 & 18 & 78 & 0 & 30 \\ -15 & 0 & -21 & 18 & -6 & 0 & 42 & -18 \\ 0 & -69 & 18 & -39 & 0 & 30 & -18 & 78 \end{bmatrix}$$

Eigenvalues of stiffness matrix:

$$[\,223.64 \quad 90 \quad 78 \quad 46.3603 \quad 42 \quad 0 \quad 0 \quad 0\,]$$

Results From Test of Trapezoid Element




```
ClearAll[Em,nu,h,a,p];  h=1;
Em=48*63*13*107; nu=1/3;
Emat=Em/(1-nu^2)*{{1,nu,0},{nu,1,0},{0,0,(1-nu)/2}};
ncoor={{0,0},{2*a,0},{a,a},{0,a}};
For [p=1,p<=4,p++,
    Ke=Quad4IsoPMembraneStiffness[ncoor,Emat,h,{True,p}];
    Ke=Rationalize[Ke,0.0000001]; Print["Ke=",Ke//MatrixForm];
    Print["Eigenvalues of Ke=",Chop[Eigenvalues[N[Ke]],.0000001]]
    ];
```

Strange value of E = 48 x 63 x 13 x 107 = 4206384 is to get exact entries in the stiffness matrix computed by Gauss rules p = 1,2,3,4.

Results From Test of Trapezoid Element (cont'd)

1 x 1 rule

$$\mathbf{K}^{e}_{1\times 1} = \begin{bmatrix} 1840293 & 1051596 & -262899 & -262899 & -1840293 & -1051596 & 262899 & 262899 \\ 1051596 & 3417687 & -262899 & 1314495 & -1051596 & -3417687 & 262899 & -1314495 \\ -262899 & -262899 & 1051596 & -525798 & 262899 & 262899 & -1051596 & 525798 \\ -262899 & 1314495 & -525798 & 1051596 & 262899 & -1314495 & 525798 & -1051596 \\ -1840293 & -1051596 & 262899 & 262899 & 1840293 & 1051596 & -262899 & -262899 \\ -1051596 & -3417687 & 262899 & -1314495 & 1051596 & 3417687 & -262899 & 1314495 \\ 262899 & 262899 & -1051596 & 525798 & -262899 & -262899 & 1051596 & -525798 \\ 262899 & -1314495 & 525798 & -1051596 & -262899 & 1314495 & -525798 & 1051596 \end{bmatrix}$$

2 x 2 rule

$$\mathbf{K}^{e}_{2\times 2} = \begin{bmatrix} 2062746 & 1092042 & -485352 & -303345 & -1395387 & -970704 & -182007 & 182007 \\ 1092042 & 3761478 & -303345 & 970704 & -970704 & -2730105 & 182007 & -2002077 \\ -485352 & -303345 & 1274049 & -485352 & -182007 & 182007 & -606690 & 606690 \\ -303345 & 970704 & -485352 & 1395387 & 182007 & -2002077 & 606690 & -364014 \\ -1395387 & -970704 & -182007 & 182007 & 2730105 & 1213380 & -1152711 & -424683 \\ -970704 & -2730105 & 182007 & -2002077 & 1213380 & 4792851 & -424683 & -60669 \\ -182007 & 182007 & -606690 & 606690 & -1152711 & -424683 & 1941408 & -364014 \\ 182007 & -2002077 & 606690 & -364014 & -424683 & -60669 & -364014 & 2426760 \end{bmatrix}$$

3 x 3 rule

$$\mathbf{K}^{e}_{3\times 3} = \begin{bmatrix} 2067026 & 1093326 & -489632 & -304629 & -1386827 & -968136 & -190567 & 179439 \\ 1093326 & 3764046 & -304629 & 968136 & -968136 & -2724969 & 179439 & -2007213 \\ -489632 & -304629 & 1278329 & -484068 & -190567 & 179439 & -598130 & 609258 \\ -304629 & 968136 & -484068 & 1397955 & 179439 & -2007213 & 609258 & -358878 \\ -1386827 & -968136 & -190567 & 179439 & 2747225 & 1218516 & -1169831 & -429819 \\ -968136 & -2724969 & 179439 & -2007213 & 1218516 & 4803123 & -429819 & -70941 \\ -190567 & 179439 & -598130 & 609258 & -1169831 & -429819 & 1958528 & -358878 \\ 179439 & -2007213 & 609258 & -358878 & -429819 & -70941 & -358878 & 2437032 \end{bmatrix}$$

4 x 4 rule

$$\mathbf{K}^{e}_{4\times 4} = \begin{bmatrix} 2067156 & 1093365 & -489762 & -304668 & -1386567 & -968058 & -190827 & 179361 \\ 1093365 & 3764124 & -304668 & 968058 & -968058 & -2724813 & 179361 & -2007369 \\ -489762 & -304668 & 1278459 & -484029 & -190827 & 179361 & -597870 & 609336 \\ -304668 & 968058 & -484029 & 1398033 & 179361 & -2007369 & 609336 & -358722 \\ -1386567 & -968058 & -190827 & 179361 & 2747745 & 1218672 & -1170351 & -429975 \\ -968058 & -2724813 & 179361 & -2007369 & 1218672 & 4803435 & -429975 & -71253 \\ -190827 & 179361 & -597870 & 609336 & -1170351 & -429975 & 1959048 & -358722 \\ 179361 & -2007369 & 609336 & -358722 & -429975 & -71253 & -358722 & 2437344 \end{bmatrix}$$

Results From Test of Trapezoid Element (cont'd)

Eigenvalues of stiffness matrices obtained by different Gauss rules:

Rule	Eigenvalues (scaled by 10^{-6}) of $\mathbf{K}^e$							
1×1	8.77276	3.68059	2.26900	0	0	0	0	0
2×2	8.90944	4.09769	3.18565	2.64521	1.54678	0	0	0
3×3	8.91237	4.11571	3.19925	2.66438	1.56155	0	0	0
4×4	8.91246	4.11627	3.19966	2.66496	1.56199	0	0	0

Rank deficient by two