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FEM Convergence Requirements

Convergence Requirements for Finite Element Discretization

Convergence: discrete (FEM) solution approaches the analytical (math model) solution in some sense

Convergence = Consistency + Stability

(analog of Lax-Wendroff theorem in finite differences)

Further Breakdown of Convergence Requirements

- **Consistency**

Completeness	*individual elements*
Compatibility	*element patches*

- **Stability**

Rank Sufficiency	*individual elements*
Positive Jacobian	*individual elements*

The Variational Index *m*

Bar

$$\Pi[u] = \int_0^L \left(\tfrac{1}{2}\, u' E A u' - q u \right) dx$$

$m = 1$

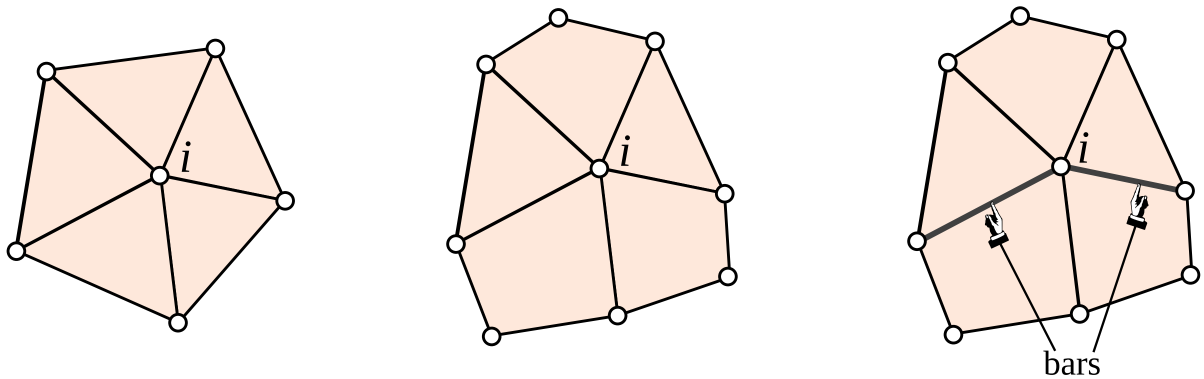
Beam

$$\Pi[v] = \int_0^L \left(\tfrac{1}{2}\, v'' E I v'' - q v \right) dx$$

$m = 2$

Element Patches

A ***patch*** is the set of all elements attached to a given node:



A finite element ***patch trial function*** is the union of shape functions activated by setting a degree of freedom at that node to unity, while all other freedoms are zero. A patch trial function "propagates" only over the patch, and is zero beyond it.

Completeness & Compatibility in Terms of *m*

Completeness

The ***element shape functions*** must represent exactly all polynomial terms of order $\leq m$ in the Cartesian coordinates. A set of shape functions that satisfies this condition is call m-complete

Compatibility

The ***patch trial functions*** must be $C^{(m-1)}$ continuous between elements, and C^{m} piecewise differentiable inside each element

Plane Stress: $m = 1$ in Two Dimensions

Completeness

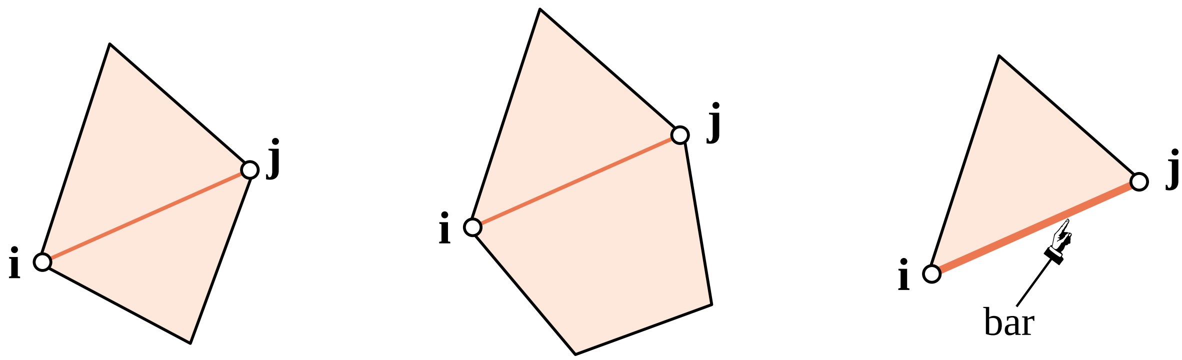
The ***element shape functions*** must represent exactly all polynomial terms of order ≤ 1 in the Cartesian coordinates. That means any *linear polynomial* in x, y with a *constant* as special case

Compatibility

The ***patch trial functions*** must be C^0 continuous between elements, and C^1 piecewise differentiable inside each element

Interelement Continuity is the Toughest to Meet

Simplification: for ***matching meshes*** (defined in Notes) it is enough to check compatibility between a ***pair of adjacent elements***:



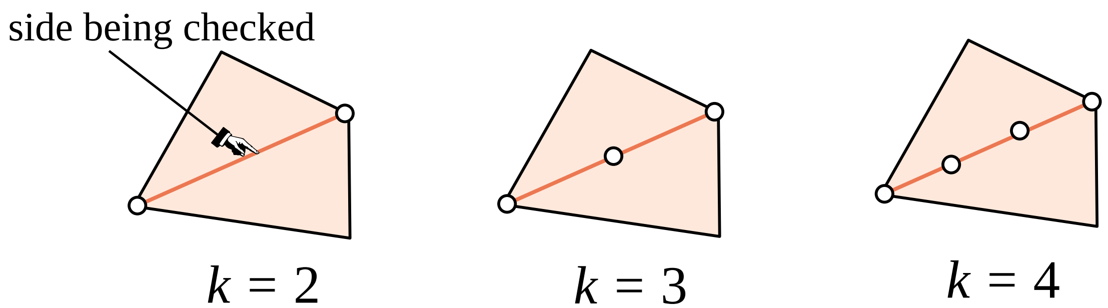
Two 3-node linear triangles

One 3-node linear triangle and one 4-node bilinear quad

One 3-node linear triangle and one 2-node bar

Side Continuity Check for Plane Stress Elements with Polynomial Shape Functions in Natural Coordinates

Let k be the number of nodes on a side:

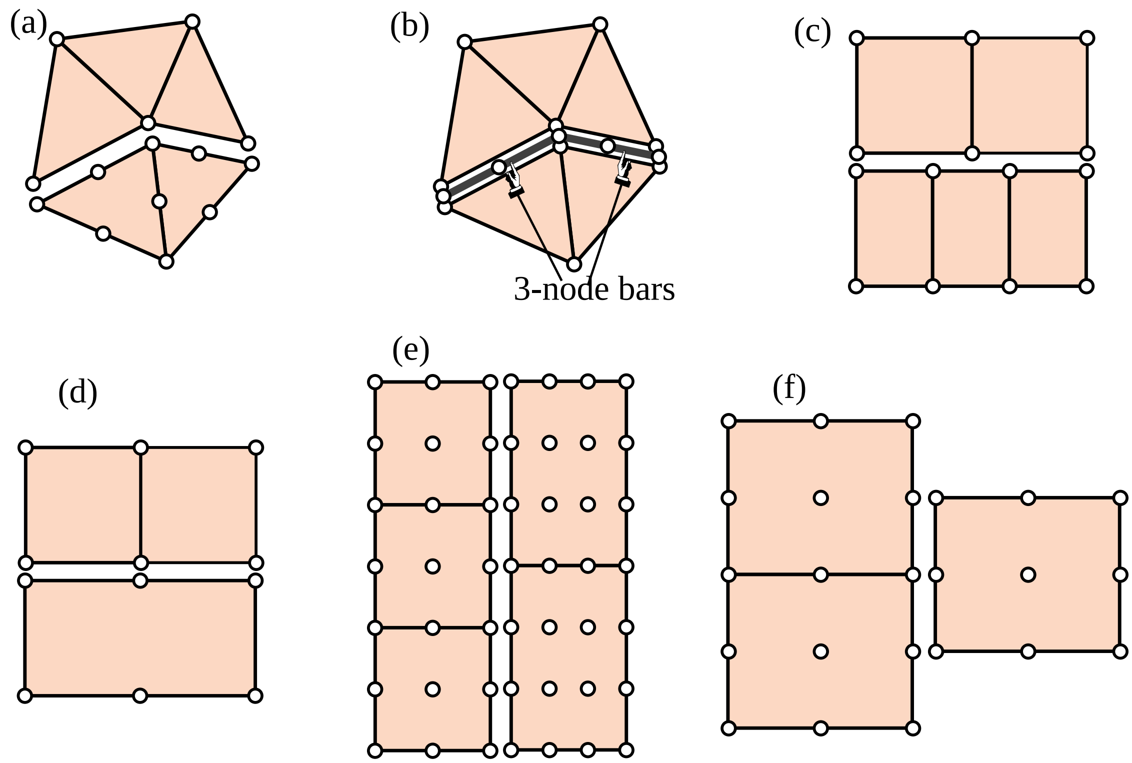


The variation of each element shape function along the side must be of polynomial order $k - 1$

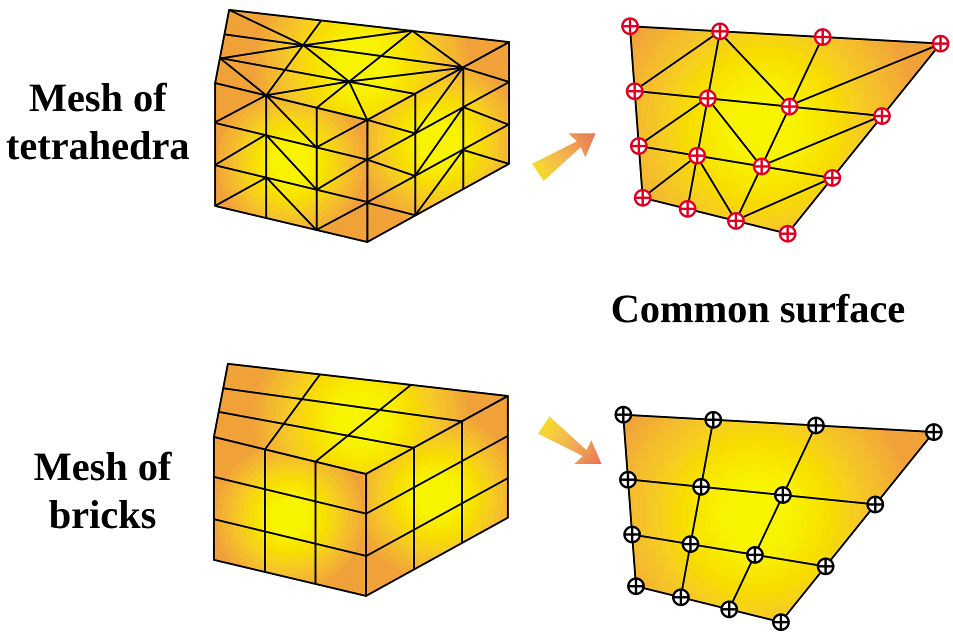
If ***more, continuity is violated***

If ***less, nodal configuration is wrong*** (too many nodes)

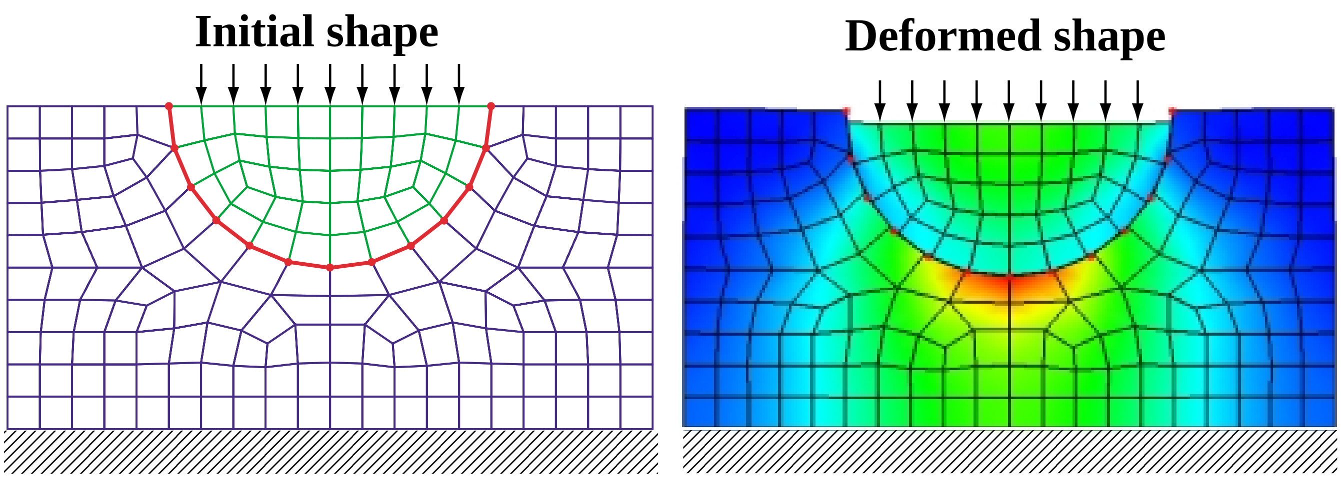
2D Nonmatching Mesh Examples



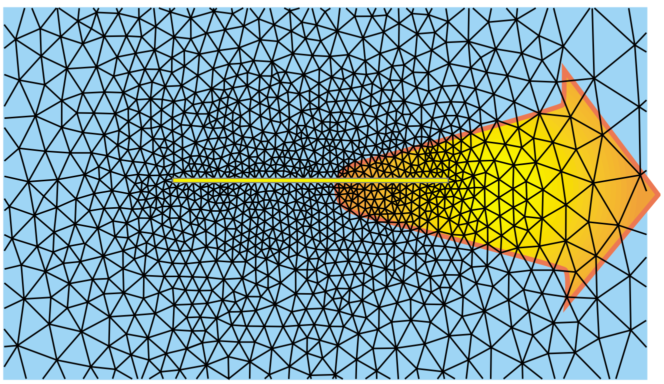
3D Nonmatching Mesh Example

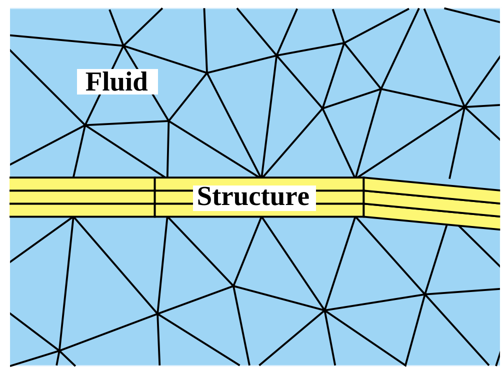


Nonmatching Meshes in Contact-With-Slip
(a Geometrically Nonlinear Problem)



Nonmatching Meshes in FSI Problem





Stability

Rank Sufficiency

The discrete model must possess the same solution uniqueness attributes of the mathematical model
For displacement finite elements:
the rigid body modes (RBMs) must be preserved
no zero-energy modes other than RBMs
Can be tested by looking at the **rank** of the stiffness matrix

Positive Jacobian Determinant

The determinant of the Jacobian matrix that relates Cartesian and natural coordinates must be everywhere ***positive*** within the element

Rank Sufficiency

The element stiffness matrix must not possess any zero-energy kinematic modes other than *rigid body modes*

This can be checked by verifing that the element stiffness matrix has the *correct rank:*

> ***correct rank = # of element DOF – # of RBMs***

A stiffness matrix that has correct rank (a.k.a. proper rank) is called *rank sufficient* and by extension, the element

Notation for Rank Analysis of Element Stiffness

n_F number of element DOF

n_R number of independent rigid body modes

n_G number of Gauss points in integration rule for **K**

n_E order of **E** (stress-strain) matrix

r_C correct (proper) rank $n_F - n_R$

r actual rank of stiffness matrix

d rank deficiency $r_C - r$

Rank Sufficiency for Numerically Integrated Finite Elements

General case

rank of **K**: $r = \min\,(n_F - n_R,\ n_E n_G)$

rank deficiency: $d = (n_F - n_R) - r$

Plane Stress, n **nodes**

$$n_F = 2n \qquad n_R = 3 \qquad n_E = 3$$

$$r = \min\,(2n - 3,\ 3n_G)$$

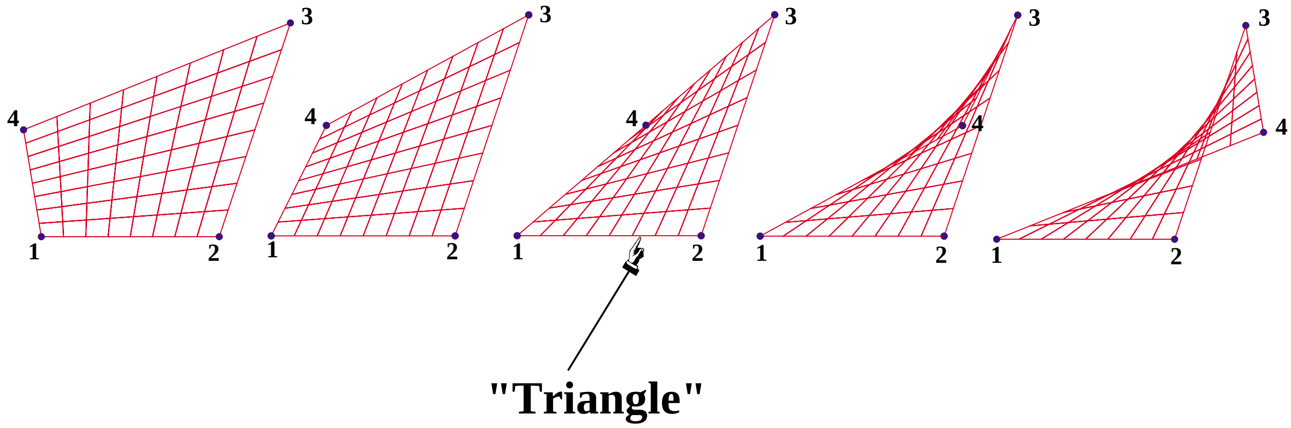
Rank Sufficiency for Some Plane Stress iso-P Elements

Element	n	n_F	$n_F - 3$	Min n_G	Recommended rule
3-node triangle	3	6	3	1	centroid*
6-node triangle	6	12	9	3	3-midpoint rule*
10-node triangle	10	20	17	6	7-point rule*
4-node quadrilateral	4	8	5	2	2 x 2
8-node quadrilateral	8	16	13	5	3 x 3
9-node quadrilateral	9	18	15	5	3 x 3
16-node quadrilateral	16	32	29	10	4 x 4

* Gauss rules for triangles are introduced in Chapter 24.

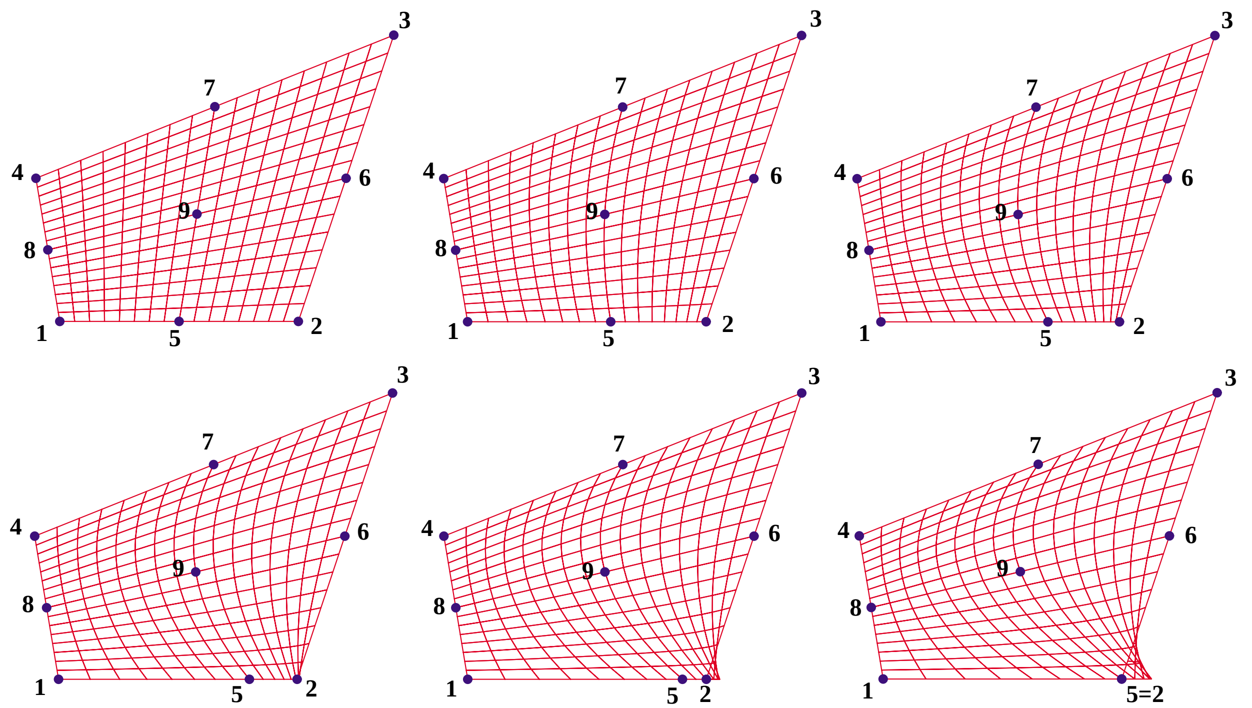
Positive Jacobian Requirement

Displacing a Corner Node of 4-Node Quad



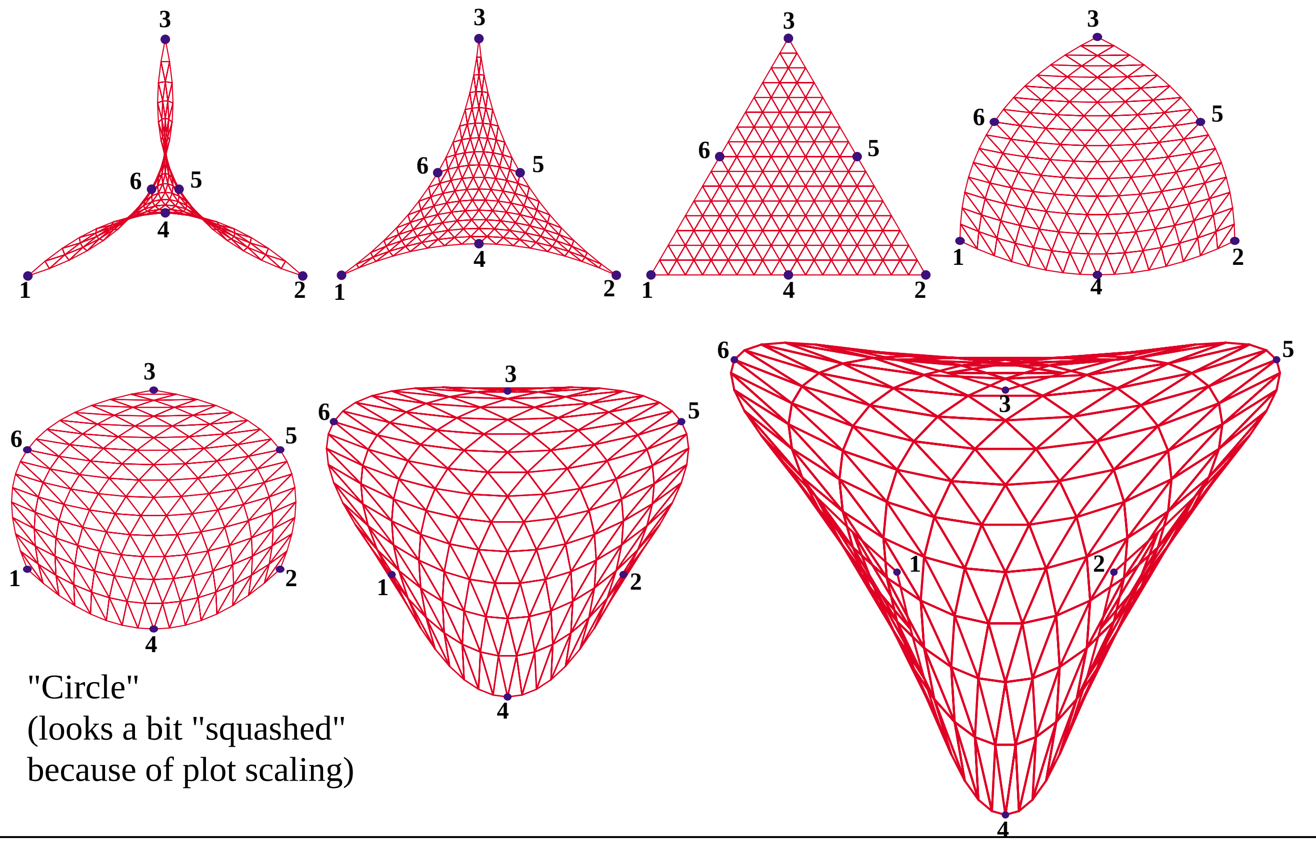
Positive Jacobian (cont'd)

Displacing a Midside Node of 9-Node Quad



Positive Jacobian (cont'd)

Displacing Midside Nodes of 6-Node EquilateralTriangle



"Circle"
(looks a bit "squashed"
because of plot scaling)