

18

Shape Function Magic

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§18.1. Requirements

This Chapter explains, through a series of examples, how isoparametric shape functions can be directly constructed by geometric considerations. For a problem of variational index 1, the isoparametric shape function N_i^e associated with node i of element e must satisfy the following conditions:

- (A) *Interpolation condition.* Takes a unit value at node i , and is zero at all other nodes.
- (B) *Local support condition.* Vanishes over any element boundary (a side in 2D, a face in 3D) that does not include node i .
- (C) *Interelement compatibility condition.* Satisfies C^0 continuity between adjacent elements over any element boundary that includes node i .
- (D) *Completeness condition.* The interpolation is able to represent exactly any displacement field which is a linear polynomial in x and y ; in particular, a constant value.

Requirement (A) follows directly by interpolation from node values. Conditions (B), (C) and (D) are consequences of the *convergence* requirements discussed further in the next Chapter.¹ For the moment these three conditions may be viewed as recipes.

One can readily verify that all isoparametric shape function sets listed in Chapter 16 satisfy the first two conditions from construction. Direct verification of condition (C) is also straightforward for those examples. A statement equivalent to (C) is that the value of the shape function over a side (in 2D) or face (in 3D) common to two elements must uniquely depend only on its nodal values on that side or face.

Completeness is a property of *all* element isoparametric shape functions taken together, rather than of an individual one. If the element satisfies (B) and (C), in view of the discussion in §16.6 it is sufficient to check that the *sum of shape functions is identically one*.

§18.2. Direct Fabrication of Shape Functions

Contrary to the what the title of this Chapter implies, the isoparametric shape functions listed in Chapter 16 did not come out of a magician's hat. They can be derived systematically by a judicious inspection process. By "inspection" it is meant that the *geometric* visualization of shape functions plays a crucial role.

The method is based on the following observation. In all examples given so far the isoparametric shape functions are given as *products* of fairly simple polynomial expressions in the natural coordinates. This is no accident but a direct consequence of the definition of natural coordinates. All shape functions of Chapter 16 can be expressed as the product of m factors:

$$N_i^e = c_i L_1 L_2 \dots L_m, \quad (18.1)$$

where

$$L_j = 0, \quad j = 1, \dots, m. \quad (18.2)$$

are the homogeneous equation of lines or curves expressed as *linear* functions in the natural coordinates, and c_i is a normalization coefficient.

¹ Convergence means that the discrete FEM solution approaches the exact analytical solution as the mesh is refined.

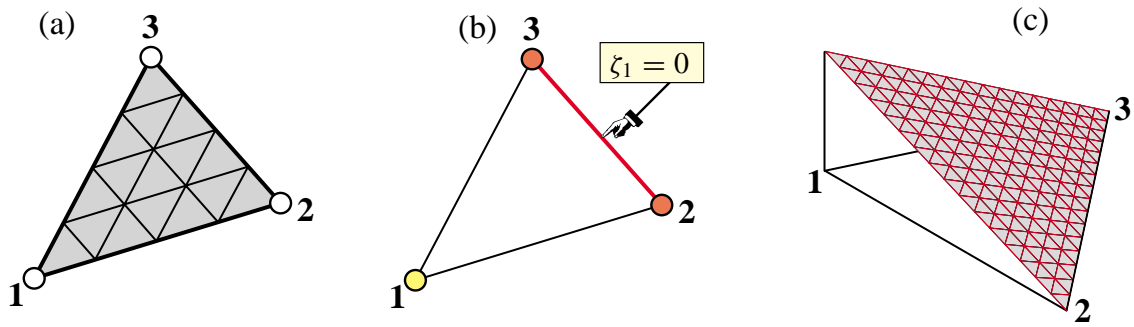


FIGURE 18.1. The three-node linear triangle: (a) element geometry; (b) equation of side opposite corner 1; (c) perspective view of the shape function $N_1 = \zeta_1$.

For two-dimensional isoparametric elements, the ingredients in (18.1) are chosen according to the following five rules.

- R1 Select the L_j as the minimal number of lines or curves linear in the natural coordinates that cross all nodes except the i^{th} node. (A *sui generis* “cross the dots” game.) Primary choices in 2D are the element sides and medians.
- R2 Set coefficient c_i so that N_i^e has the value 1 at the i^{th} node.
- R3 Check that N_i^e vanishes over all element sides that do not contain node i .
- R4 Check the polynomial order over each side that contains node i . If the order is n , there must be exactly $n + 1$ nodes on the side for compatibility to hold.
- R5 If local support (R3) and interelement compatibility (R4) are satisfied, check that the sum of shape functions is identically one.

The examples that follow show these rules in action for two-dimensional elements. Essentially the same technique is applicable to one- and three-dimensional elements.

§18.3. Triangular Element Shape Functions

This section illustrates the use of (18.1) in the construction of shape functions for the linear and the quadratic triangle. The cubic triangle is dealt with in Exercise 18.1.

§18.3.1. The Three-Node Linear Triangle

Figure 18.1 shows the three-node linear triangle that was studied in detail in Chapter 15. The three shape functions are simply the triangular coordinates: $N_i = \zeta_i$, for $i = 1, 2, 3$. Although this result follows directly from the linear interpolation formula of §15.2.4, it can be also quickly derived from the present methodology as follows.

The equation of the triangle side opposite to node i is $L_{j-k} = \zeta_i = 0$, where j and k are the cyclic permutations of i . Here symbol L_{j-k} denotes the left hand side of the homogeneous equation of the natural coordinate line that passes through node points j and k . See Figure 18.1(b) for $i = 1$, $j = 2$ and $k = 3$. Hence the obvious guess is

$$N_i^e \stackrel{\text{guess}}{=} c_i L_i. \quad (18.3)$$

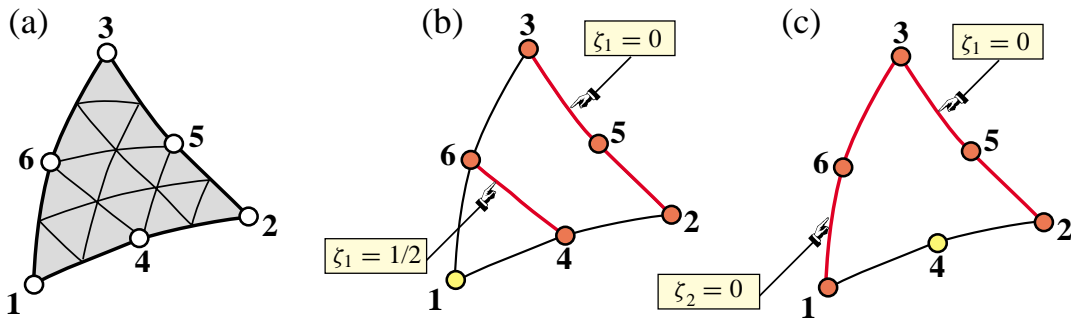


FIGURE 18.2. The six-node quadratic triangle: (a) element geometry; (b) lines (in red) whose product yields N_1^e ; (c) lines (in red) whose product yields N_4^e .

This satisfies conditions (A) and (B) except the unit value at node i ; this holds if $c_i = 1$. The local support condition (B) follows from construction: the value of ζ_i is zero over side $j-k$. Interelement compatibility follows from R4: the variation of ζ_i along the 2 sides meeting at node i is linear and that there are two nodes on each side; cf. §15.4.2. Completeness follows since $N_1^e + N_2^e + N_3^e = \zeta_1 + \zeta_2 + \zeta_3 = 1$. Figure 18.1(c) depicts $N_1^e = \zeta_1$, drawn normal to the element in perspective view.

§18.3.2. The Six-Node Quadratic Triangle

The geometry of the six-node quadratic triangle is shown in Figure 18.2(a). Inspection reveals two types of nodes: corners (1, 2 and 3) and midside nodes (4, 5 and 6). Consequently we can expect two types of associated shape functions. We select nodes 1 and 4 as representative cases.

For both cases we try the product of *two* linear functions in the triangular coordinates because we expect the shape functions to be quadratic. These functions are illustrated in Figures 18.2(b,c) for corner node 1 and midside node 4, respectively.

For corner node 1, inspection of Figure 18.2(b) suggests trying

$$N_1^e \stackrel{\text{guess}}{=} c_1 L_{2-3} L_{4-6}, \quad (18.4)$$

Why is (18.4) expected to work? Clearly N_1^e will vanish over 2-5-3 and 4-6. This makes the function zero at nodes 2 through 6, as is obvious upon inspection of Figure 18.2(b), while being nonzero at node 1. This value can be adjusted to be unity if c_1 is appropriately chosen. The equations of the lines that appear in (18.4) are

$$L_{2-3}: \quad \zeta_1 = 0, \quad L_{4-6}: \quad \zeta_1 - \frac{1}{2} = 0. \quad (18.5)$$

Replacing into (18.3) we get

$$N_1^e = c_1 \zeta_1 (\zeta_1 - \frac{1}{2}), \quad (18.6)$$

To find c_1 , evaluate $N_1^e(\zeta_1, \zeta_2, \zeta_3)$ at node 1. The triangular coordinates of this node are $\zeta_1 = 1$, $\zeta_2 = \zeta_3 = 0$. We require that it takes a unit value there: $N_1^e(1, 0, 0) = c_1 \times 1 \times \frac{1}{2} = 1$ whence $c_1 = 2$ and finally

$$N_1^e = 2\zeta_1 (\zeta_1 - \frac{1}{2}) = \zeta_1 (2\zeta_1 - 1), \quad (18.7)$$

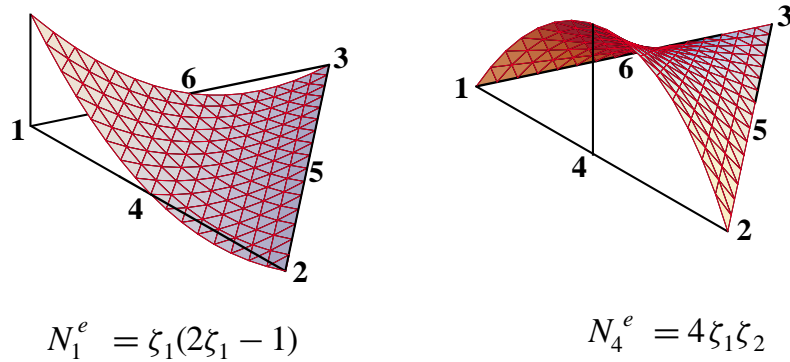


FIGURE 18.3. Perspective view of shape functions N_1^e and N_4^e for the quadratic triangle. The plot is done over a straight side triangle for programming simplicity.

as listed in §16.5.2. Figure 18.3 shows a perspective view. The other two corner shape functions follow by cyclic permutations of the corner index.

For midside node 4, inspection of Figure 18.2(c) suggests trying

$$N_4^e \stackrel{\text{guess}}{=} c_4 L_{2-3} L_{1-3} \quad (18.8)$$

Evidently (18.8) satisfies requirements (A) and (B) if c_4 is appropriately normalized. The equation of sides L_{2-3} and L_{1-3} are $\zeta_1 = 0$ and $\zeta_2 = 0$, respectively. Therefore $N_4^e(\zeta_1, \zeta_2, \zeta_3) = c_4 \zeta_1 \zeta_2$. To find c_4 , evaluate this function at node 4, the triangular coordinates of which are $\zeta_1 = \zeta_2 = \frac{1}{2}$, $\zeta_3 = 0$. We require that it takes a unit value there: $N_4^e(\frac{1}{2}, \frac{1}{2}, 0) = c_4 \times \frac{1}{2} \times \frac{1}{2} = 1$. Hence $c_4 = 4$, which gives

$$N_4^e = 4\zeta_1\zeta_2 \quad (18.9)$$

as listed in §16.5.2. Figure 18.3 shows a perspective view of this shape function. The other two midside shape functions follow by cyclic permutations of the node indices.

It remains to carry out the interelement continuity check. Consider node 1. The boundaries containing node 1 and common to adjacent elements are 1–2 and 1–3. Over each one the variation of N_1^e is quadratic in ζ_1 . Therefore the polynomial order over each side is 2. Because there are three nodes on each boundary, the compatibility condition (C) of §18.1 is verified. A similar check can be carried out for midside node shape functions. Exercise 16.1 verified that the sum of the N_i is unity. Therefore the element is complete.

§18.4. Quadrilateral Element Shape Functions

Three quadrilateral elements, with 4, 9 and 8 nodes, respectively, which are commonly used in computational mechanics serve as examples to illustrate the construction of shape functions. Elements with more nodes, such as the bicubic quadrilateral, are not treated as they are rarely used.

§18.4.1. The Four-Node Bilinear Quadrilateral

The element geometry and natural coordinates are shown in Figure 18.4(a). Only one type of node (corner) and associated shape function is present. Consider node 1 as typical. Inspection of

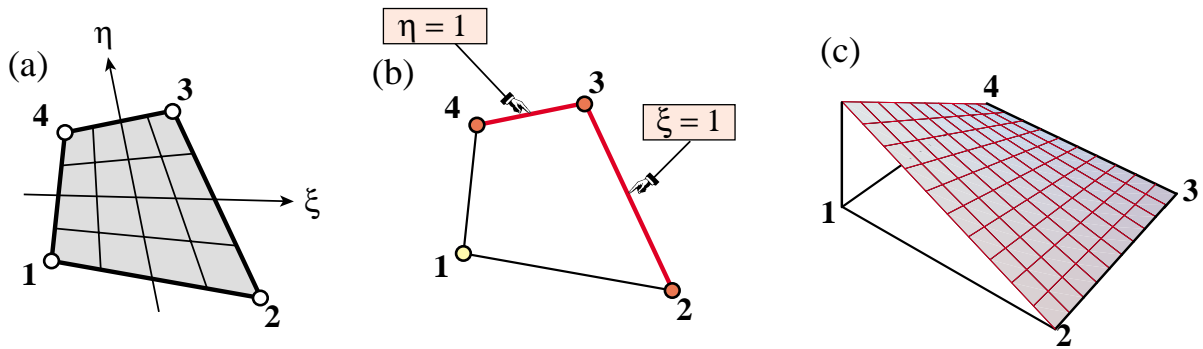


FIGURE 18.4. The four-node bilinear quadrilateral: (a) element geometry; (b) sides (in red) that do not contain corner 1; (c) perspective view of the shape function N_1^e .

Figure 18.4(b) suggests trying

$$N_1^e \stackrel{\text{guess}}{=} c_1 L_{2-3} L_{3-4} \quad (18.10)$$

This plainly vanishes over nodes 2, 3 and 4, and can be normalized to unity at node 1 by adjusting c_1 . By construction it vanishes over the sides 2–3 and 3–4 that do not belong to 1. The equation of side 2–3 is $\xi = 1$, or $\xi - 1 = 0$. The equation of side 3–4 is $\eta = 1$, or $\eta - 1 = 0$. Replacing in (18.10) yields

$$N_1^e(\xi, \eta) = c_1(\xi - 1)(\eta - 1) = c_1(1 - \xi)(1 - \eta). \quad (18.11)$$

To find c_1 , evaluate at node 1, the natural coordinates of which are $\xi = \eta = -1$:

$$N_1^e(-1, -1) = c_1 \times 2 \times 2 = 4c_1 = 1. \quad (18.12)$$

Hence $c_1 = \frac{1}{4}$ and the shape function is

$$N_1^e = \frac{1}{4}(1 - \xi)(1 - \eta), \quad (18.13)$$

as listed in §16.6.2. Figure 18.4(c) shows a perspective view.

For the other three nodes the procedure is the same, traversing the element cyclically. It can be verified that the general expression of the shape functions for this element is

$$N_i^e = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta). \quad (18.14)$$

The continuity check proceeds as follows, using N_1^e as example. Node 1 belongs to interelement boundaries 1–2 and 1–3. Over side 1–2, $\eta = -1$ is constant and N_1^e is a *linear* function of ξ . To see this, replace $\eta = -1$ in (18.13). Over side 1–3, $\xi = -1$ is constant and N_1^e is a *linear* function of η . Consequently the polynomial variation order is 1 over both sides. Because there are two nodes on each side the compatibility condition is satisfied. The sum of the shape functions is one, as shown in (16.21); thus the element is complete.

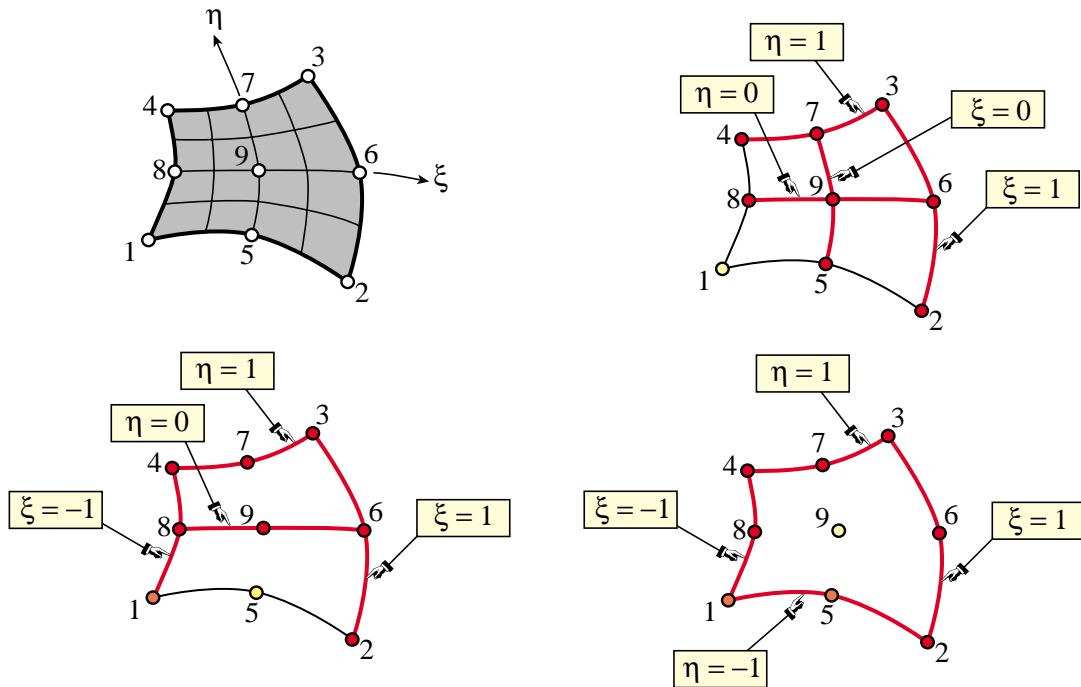


FIGURE 18.5. The nine-node biquadratic quadrilateral: (a) element geometry; (b,c,d): lines (in red) whose product makes up the shape functions N_1^e , N_5^e and N_9^e , respectively.

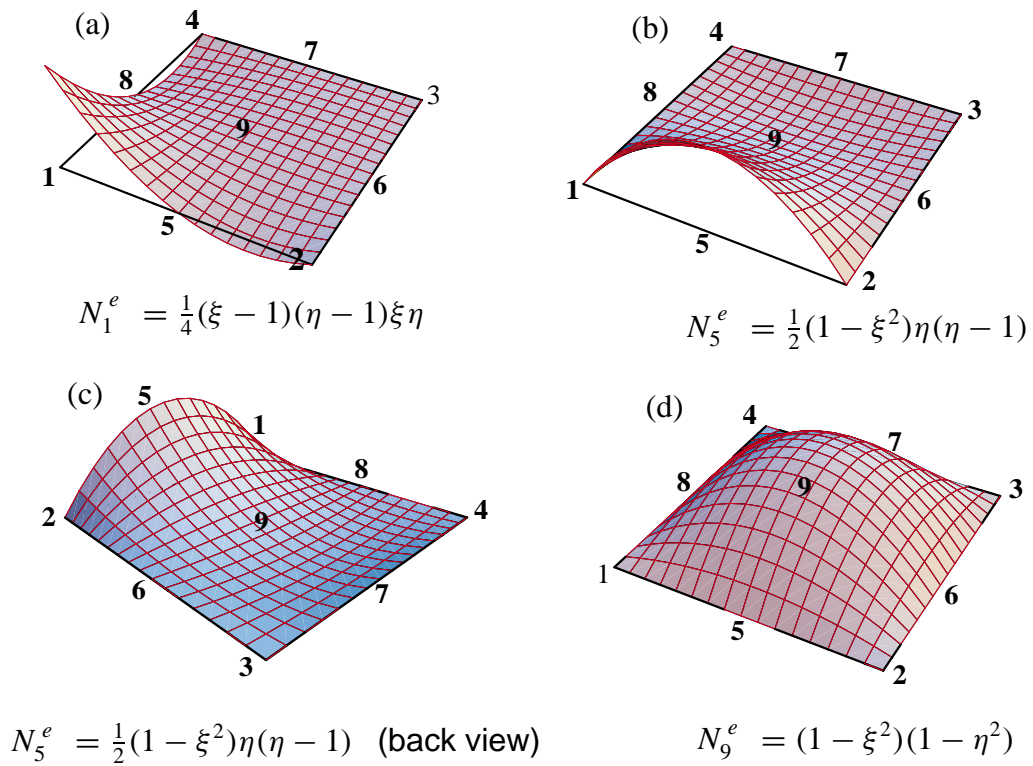


FIGURE 18.6. Perspective view of the shape functions for nodes 1, 5 and 9 of the nine-node biquadratic quadrilateral.

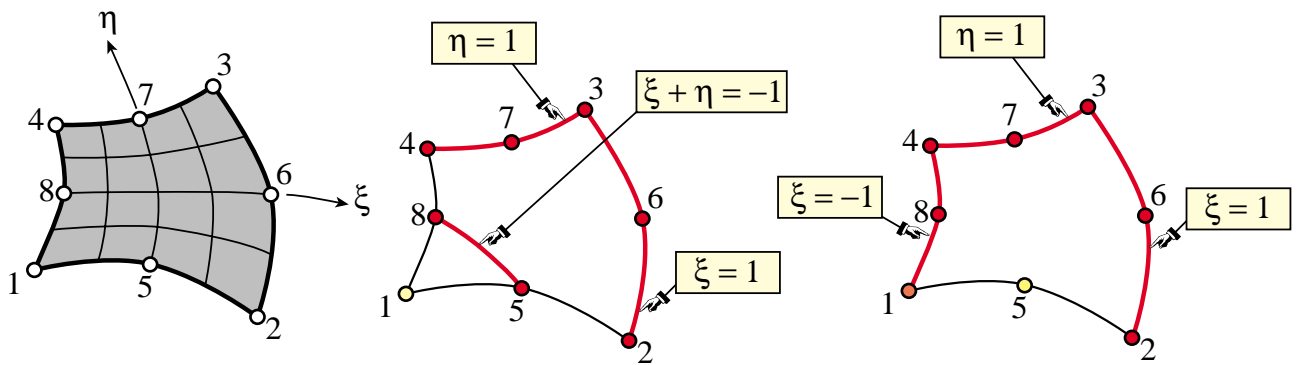


FIGURE 18.7. The eight-node serendipity quadrilateral: (a) element geometry; (b,c): lines (in red) whose product make up the shape functions N_1^e and N_5^e , respectively.

§18.4.2. The Nine-Node Biquadratic Quadrilateral

The element geometry is shown in Figure 18.5(a). This element has three types of shape functions, which are associated with corner nodes, midside nodes and center node, respectively.

The lines whose product is used to construct three types of shape functions are illustrated in Figure 18.5(b,c,d) for nodes 1, 5 and 9, respectively. The technique has been sufficiently illustrated in previous examples. Here we summarize the calculations for nodes 1, 5 and 9, which are taken as representatives of the three types:

$$N_1^e = c_1 L_{2-3} L_{3-4} L_{5-7} L_{6-8} = c_1 (\xi - 1)(\eta - 1)\xi\eta. \quad (18.15)$$

$$N_5^e = c_5 L_{2-3} L_{1-4} L_{6-8} L_{3-4} = c_5 (\xi - 1)(\xi + 1)\eta(\eta - 1) = c_5 (1 - \xi^2)\eta(1 - \eta). \quad (18.16)$$

$$N_9^e = c_9 L_{1-2} L_{2-3} L_{3-4} L_{4-1} = c_9 (\xi - 1)(\eta - 1)(\xi + 1)(\eta + 1) = c_9 (1 - \xi^2)(1 - \eta^2) \quad (18.17)$$

Imposing the normalization conditions we find

$$c_1 = \frac{1}{4}, \quad c_5 = -\frac{1}{2}, \quad c_9 = 1, \quad (18.18)$$

and we obtain the shape functions listed in §16.6.3. Perspective views are shown in Figure 18.6. The remaining N_i 's are constructed through a similar procedure.

Verification of the interelement continuity condition is immediate: the polynomial variation order of N_i^e over any side that belongs to node i is two and there are three nodes on each side. Exercise 16.2 checks that the sum of shape function is unity. Thus the element is complete.

§18.4.3. The Eight-Node “Serendipity” Quadrilateral

This is an eight-node quadrilateral element that results when the center node 9 of the biquadratic quadrilateral is eliminated by kinematic constraints. The geometry and node configuration is shown in Figure 18.7(a). This element was widely used in commercial codes during the 70s and 80s but it is gradually being phased out in favor of the 9-node quadrilateral.

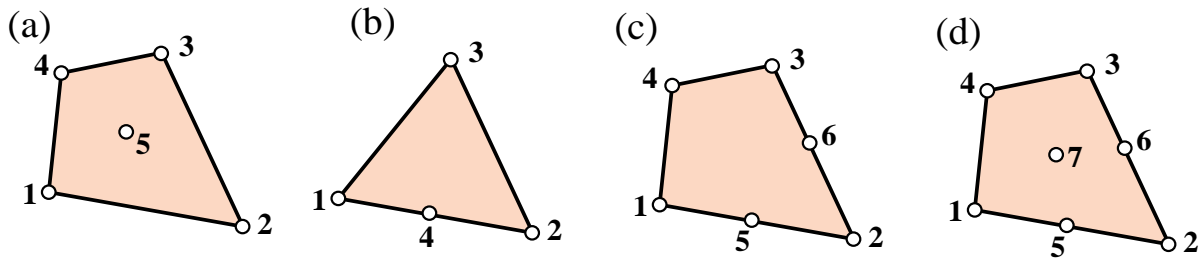


FIGURE 18.8. Node configurations for which the magic recipe does not work.

The 8-node quadrilateral has two types of shape functions, which are associated with corner nodes and midside nodes. Lines whose products yields the shape functions for nodes 1 and 5 are shown in Figure 18.7(b,c).

Here are the calculations for shape functions of nodes 1 and 5, which are taken again as representative cases.

$$N_1^e = c_1 L_{2-3} L_{3-4} L_{5-8} = c_1 (\xi - 1)(\eta - 1)(1 + \xi + \eta) = c_1 (1 - \xi)(1 - \eta)(1 + \xi + \eta), \quad (18.19)$$

$$N_5^e = c_5 L_{2-3} L_{3-4} L_{4-1} = c_5 (\xi - 1)(\xi + 1)(\eta - 1) = c_5 (1 - \xi^2)(1 - \eta). \quad (18.20)$$

Imposing the normalization conditions we find

$$c_1 = -\frac{1}{4}, \quad c_5 = \frac{1}{2} \quad (18.21)$$

The other shape functions follow by appropriate permutation of nodal indices. The interelement continuity and completeness verification are similar to that carried out for the nine-node element, and are relegated to exercises.

§18.5. Does the Magic Wand Always Work?

The “cross the dots” recipe (18.1)-(18.2) is not foolproof. It fails for certain node configurations although it is a reasonable way to start. It runs into difficulties, for instance, in the problem posed in Exercise 18.6, which deals with the 5-node quadrilateral depicted in Figure 18.8(a). If for node 1 one tries the product of side 2–3, side 3–4, and the diagonal 2–5–4, the shape function is easily worked out to be $N_1^e = -\frac{1}{8}(1 - \xi)(1 - \eta)(\xi + \eta)$. This satisfies conditions (A) and (B). However, it violates (C) along sides 1–2 and 4–1, because it varies quadratically over them with only two nodes per side.

§18.5.1. Hierarchical Corrections

A more robust technique relies on a *correction approach*, which employs a combination of terms such as (18.1). For example, a combination of two patterns, one with m factors and one with n factors, is

$$N_i^e = c_i L_1^c L_2^c \dots L_m^c + d_i L_1^d L_2^d \dots L_n^d, \quad (18.22)$$

Here two normalization coefficients: c_i and d_i , appear. In practice trying forms such as (18.22) from scratch becomes cumbersome. The development is best done *hierarchically*. The first term is

taken to be that of a lower order element, called the *parent element*, for which the one-shot approach works. The second term is then a corrective shape function that vanishes at the nodes of the parent element. If this is insufficient one more corrective term is added, and so on.

The technique is best explained through examples. Exercise 18.6 illustrates the procedure for the element of Figure 18.8(a). The next subsection works out the element of Figure 18.8(b).

§18.5.2. Transition Element Example

The hierarchical correction technique is useful for *transition elements*, which have corner nodes but midnodes only over certain sides. Three examples are pictured in Figure 18.8(b,c,d). Shape functions that work can be derived with one, two and three hierarchical corrections, respectively.

As an example, let us construct the shape function N_1^e for the 4-node transition triangle shown in Figure 18.8(b). Candidate lines for the recipe (18.1) are obviously the side 2–3: $\zeta_1 = 0$, and the median 3–4: $\zeta_1 = \zeta_2$. Accordingly we try

$$N_1^e \stackrel{\text{guess}}{=} c_1 \zeta_1 (\zeta_1 - \zeta_2), \quad N_1(1, 0, 0) = 1 = c_1. \quad (18.23)$$

This function $N_1^e = \zeta_1(\zeta_1 - \zeta_2)$ satisfies conditions (A) and (B) but fails compatibility: over side 1–3 of equation $\zeta_2 = 0$, because $N_1^e(\zeta_1, 0, \zeta_3) = \zeta_1^2$. This varies quadratically but there are only 2 nodes on that side. Thus (18.23) is no good.

To proceed hierarchically we start from the shape function for the 3-node linear triangle: $N_1^e = \zeta_1$. This will not vanish at node 4, so apply a correction that vanishes at all nodes but 4. From knowledge of the quadratic triangle midpoint functions, that is obviously $\zeta_1 \zeta_2$ times a coefficient to be determined. The new guess is

$$N_1^e \stackrel{\text{guess}}{=} \zeta_1 + c_1 \zeta_1 \zeta_2. \quad (18.24)$$

Coefficient c_1 is determined by requiring that N_1^e vanish at 4: $N_1^e(\frac{1}{2}, \frac{1}{2}, 0) = \frac{1}{2} + c_1 \frac{1}{4} = 0$, whence $c_1 = -2$ and the shape function is

$$N_1^e = \zeta_1 - 2\zeta_1 \zeta_2. \quad (18.25)$$

This is easily checked to satisfy compatibility on all sides. The verification of completeness is left to Exercise 18.8.

Note that since $N_1^e = \zeta_1(1 - 2\zeta_2)$, (18.25) can be constructed as the normalized product of lines $\zeta_1 = 0$ and $\zeta_2 = 1/2$. The latter passes through 4 and is parallel to 1–3. As part of the opening moves in the shape function game this would be a lucky guess indeed. If one goes to a more complicated element no obvious factorization is possible.

Cell 18.1 *Mathematica* Module to Draw a Function over a Triangle Region

```

PlotTriangleShapeFunction[xytrig_, f_, Nsub_, aspect_] := Module[
  {Ni, line3D={}, poly3D={}, zc1, zc2, zc3, xyf1, xyf2, xyf3,
   xc, yc, x1, x2, x3, y1, y2, y3, z1, z2, z3, iz1, iz2, iz3, d},
  {{x1, y1, z1}, {x2, y2, z2}, {x3, y3, z3}} = Take[xytrig, 3];
  xc = {x1, x2, x3}; yc = {y1, y2, y3}; Ni = Nsub*3;
  Do [ Do [iz3 = Ni - iz1 - iz2; If [iz3 <= 0, Continue[]]; d = 0;
    If [Mod[iz1 + 2, 3] == 0 && Mod[iz2 - 1, 3] == 0, d = 1];
    If [Mod[iz1 - 2, 3] == 0 && Mod[iz2 + 1, 3] == 0, d = -1];
    If [d == 0, Continue[]];
    zc1 = N[{iz1 + d + d, iz2 - d, iz3 - d} / Ni];
    zc2 = N[{iz1 - d, iz2 + d + d, iz3 - d} / Ni];
    zc3 = N[{iz1 - d, iz2 - d, iz3 + d + d} / Ni];
    xyf1 = {xc.zc1, yc.zc1, f[zc1[[1]], zc1[[2]], zc1[[3]]]};
    xyf2 = {xc.zc2, yc.zc2, f[zc2[[1]], zc2[[2]], zc2[[3]]]};
    xyf3 = {xc.zc3, yc.zc3, f[zc3[[1]], zc3[[2]], zc3[[3]]]};
    AppendTo[poly3D, Polygon[{xyf1, xyf2, xyf3}]];
    AppendTo[line3D, Line[{xyf1, xyf2, xyf3, xyf1}],
      {iz2, 1, Ni - iz1}], {iz1, 1, Ni}];
  Show[ Graphics3D[RGBColor[1, 0, 0]], Graphics3D[poly3D],
    Graphics3D[Thickness[.002]], Graphics3D[line3D],
    Graphics3D[RGBColor[0, 0, 0]], Graphics3D[Thickness[.005]],
    Graphics3D[Line[xytrig]], PlotRange -> All,
    BoxRatios -> {1, 1, aspect}, Boxed -> False]
];
ClearAll[f1, f4];
xyc1 = {0, 0, 0}; xyc2 = {3, 0, 0}; xyc3 = {Sqrt[3], 3/2, 0};
xytrig = N[{xyc1, xyc2, xyc3, xyc1}]; Nsub = 16;
f1[zeta1_, zeta2_, zeta3_] := zeta1*(2*zeta1 - 1);
f4[zeta1_, zeta2_, zeta3_] := 4*zeta1*zeta2;
PlotTriangleShapeFunction[xytrig, f1, Nsub, 1/2];
PlotTriangleShapeFunction[xytrig, f4, Nsub, 1/2.5];

```

§18.6. *Mathematica Modules to Plot Shape Functions

A *Mathematica* module called `PlotTriangleShape Functions`, listed in Cell 18.1, has been developed to draw perspective plots of shape functions $N_i(\zeta_1, \zeta_2, \zeta_3)$ over a triangular region. The region is assumed to have straight sides to simplify the logic. The test statements that follow the module produce the shape function plots shown in Figure 18.3 for the 6-node quadratic triangle. Argument `Nsub` controls the plot resolution while `aspect` controls the xyz box aspect ratio. The remaining arguments are self explanatory.

Another *Mathematica* module called `PlotQuadrilateralShape Functions`, listed in Cell 18.2, has been developed to produce perspective plots of shape functions $N_i(\xi, \eta)$ over a quadrilateral region. The region is assumed to have straight sides to simplify the logic. The test statements that follow the module produce

Cell 18.2 Mathematica Module to Draw a Function over a Quadrilateral Region

```

PlotQuadrilateralShapeFunction[xyquad_,f_,Nsub_,aspect_]:=Module[
  {Ne,Nev,line3D={},poly3D={},xyf1,xyf2,xyf3,i,j,n,ixi,ieta,
  xi,eta,x1,x2,x3,x4,y1,y2,y3,y4,z1,z2,z3,z4,xc,yc},
  {{x1,y1,z1},{x2,y2,z2},{x3,y3,z3},{x4,y4,z4}}=Take[xyquad,4];
  xc={x1,x2,x3,x4}; yc={y1,y2,y3,y4};
  Ne[xi_,eta_]:=N[{(1-xi)*(1-eta),(1+xi)*(1-eta),
  (1+xi)*(1+eta),(1-xi)*(1+eta)}/4]; n=Nsub;
  Do [ Do [ ixi=(2*i-n-1)/n; ieta=(2*j-n-1)/n;
  {xi,eta}=N[{ixi-1/n,ieta-1/n}]; Nev=Ne[xi,eta];
  xyf1={xc.Nev,yc.Nev,f[xi,eta]};
  {xi,eta}=N[{ixi+1/n,ieta-1/n}]; Nev=Ne[xi,eta];
  xyf2={xc.Nev,yc.Nev,f[xi,eta]};
  {xi,eta}=N[{ixi+1/n,ieta+1/n}]; Nev=Ne[xi,eta];
  xyf3={xc.Nev,yc.Nev,f[xi,eta]};
  {xi,eta}=N[{ixi-1/n,ieta+1/n}]; Nev=Ne[xi,eta];
  xyf4={xc.Nev,yc.Nev,f[xi,eta]};
  AppendTo[poly3D,Polygon[{xyf1,xyf2,xyf3,xyf4}]];
  AppendTo[line3D,Line[{xyf1,xyf2,xyf3,xyf4,xyf1}]],
  {i,1,Nsub}],{j,1,Nsub}];
  Show[ Graphics3D[RGBColor[1,0,0]],Graphics3D[poly3D],
  Graphics3D[Thickness[.002]],Graphics3D[line3D],
  Graphics3D[RGBColor[0,0,0]],Graphics3D[Thickness[.005]],
  Graphics3D[Line[xyquad]], PlotRange->All,
  BoxRatios->{1,1,aspect},Boxed->False]
];
ClearAll[f1,f5,f9];
xyc1={0,0,0}; xyc2={3,0,0}; xyc3={3,3,0}; xyc4={0,3,0};
xyquad=N[{xyc1,xyc2,xyc3,xyc4,xyc1}]; Nsub=16;
f1[xi_,eta_]:= (1/2)*(xi-1)*(eta-1)*xi*eta;
f5[xi_,eta_]:= (1/2)*(1-xi^2)*eta*(eta-1);
f9[xi_,eta_]:= (1-xi^2)*(1-eta^2);
PlotQuadrilateralShapeFunction[xyquad,f1,Nsub,1/2];
PlotQuadrilateralShapeFunction[xyquad,f5,Nsub,1/2.5];
PlotQuadrilateralShapeFunction[xyquad,f9,Nsub,1/3];

```

the shape function plots shown in Figure 18.6(a,b,d) for the 9-node biquadratic quadrilateral. Argument *Nsub* controls the plot resolution while *aspect* controls the *xyz* box aspect ratio. The remaining arguments are self explanatory.

Notes and Bibliography

The name “shape functions” for interpolation functions directly expressed in terms of physical coordinates (the node displacements in the case of isoparametric elements) was coined by Irons. The earliest published reference seems to be the paper [18]. This was presented in 1965 at the first Wright-Patterson conference, the first all-FEM meeting that strongly influenced the development of computational mechanics in Generation 2. The key connection to numerical integration was presented in [143], although it is mentioned in prior internal reports. A comprehensive exposition is given in the textbook by Irons and Ahmad [147].

The quick way of developing shape functions presented here was used in the writer’s 1966 thesis [67] for triangular elements. The qualifier “magic” arose from the timing for covering this Chapter in a Fall Semester course: the lecture falls near Halloween.

References

Referenced items have been moved to Appendix R.

Homework Exercises for Chapter 18
Shape Function Magic

EXERCISE 18.1 [A/C:10+10] The complete cubic triangle for plane stress has 10 nodes located as shown in Figure E18.1, with their triangular coordinates listed in parentheses.

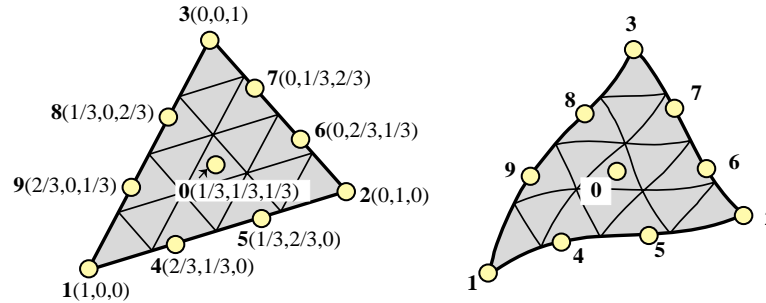


FIGURE E18.1. Ten-node cubic triangle for Exercise 18.1. The left picture shows the superparametric element whereas the right one shows the isoparametric version with curved sides.

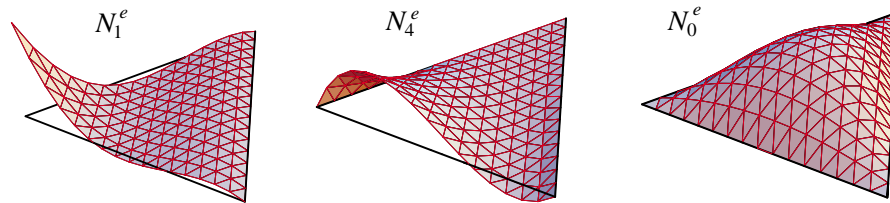


FIGURE E18.2. Perspective plots of the shape functions N_1^e , N_4^e and N_0^e for the 10-node cubic triangle.

- (a) Construct the cubic shape functions N_1^e , N_4^e and N_0^e for nodes 1, 4, and 0 (the interior node is labeled as zero, not 10) using the line-product technique. [Hint: each shape function is the product of 3 and only 3 lines.] Perspective plots of those 3 functions are shown in Figure E18.2.
- (b) Construct the missing 7 shape functions by appropriate node number permutations, and verify that the sum of the 10 functions is identically one. For the unit sum check use the fact that $\zeta_1 + \zeta_2 + \zeta_3 = 1$.

EXERCISE 18.2 [A:15] Find an alternative shape function N_1^e for corner node 1 of the 9-node quadrilateral of Figure 18.5(a) by using the diagonal lines 5–8 and 2–9–4 in addition to the sides 2–3 and 3–4. Show that the resulting shape function violates the compatibility condition (C) stated in §18.1.

EXERCISE 18.3 [A/C:15] Complete the above exercise for all nine nodes. Add the shape functions (use a CAS and simplify) and verify whether their sum is unity.

EXERCISE 18.4 [A/C:20] Verify that the shape functions N_1^e and N_5^e of the eight-node serendipity quadrilateral discussed in §18.4.3 satisfy the interelement compatibility condition (C) stated in §18.1. Obtain all 8 shape functions and verify that their sum is unity.

EXERCISE 18.5 [C:15] Plot the shape functions N_1^e and N_5^e of the eight-node serendipity quadrilateral studied in §18.4.3 using the module `PlotQuadrilateralShapeFunction` listed in Cell 18.2.

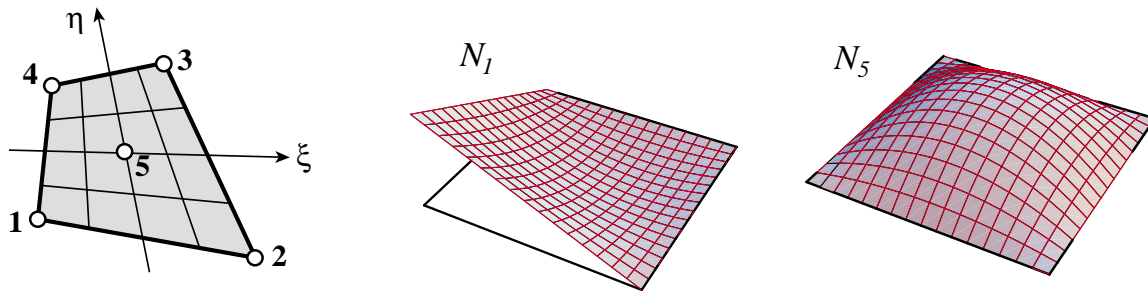


FIGURE E18.3. Five node quadrilateral element for Exercise 18.6.

EXERCISE 18.6 [A:15]. A five node quadrilateral element has the nodal configuration shown in Figure E18.3. Perspective views of N_1^e and N_5^e are shown in that Figure.² Find five shape functions $N_i^e, i = 1, 2, 3, 4, 5$ that satisfy compatibility, and also verify that their sum is unity.

Hint: develop $N_5(\xi, \eta)$ first for the 5-node quad using the line-product method; then the corner shape functions $\bar{N}_i(\xi, \eta)$ ($i = 1, 2, 3, 4$) for the 4-node quad (already given in the Notes); finally combine $N_i = \bar{N}_i + \alpha N_5$, determining α so that all N_i vanish at node 5. Check that $N_1 + N_2 + N_3 + N_4 + N_5 = 1$ identically.

EXERCISE 18.7 [A:15]. An eight-node “brick” finite element for three dimensional analysis has three isoparametric natural coordinates called ξ, η and μ . These coordinates vary from -1 at one face to $+1$ at the opposite face, as sketched in Figure E18.4.

Construct the (trilinear) shape function for node 1 (follow the node numbering of the figure). The equations of the brick faces are:

1485 : $\xi = -1$	2376 : $\xi = +1$
1265 : $\eta = -1$	4378 : $\eta = +1$
1234 : $\mu = -1$	5678 : $\mu = +1$

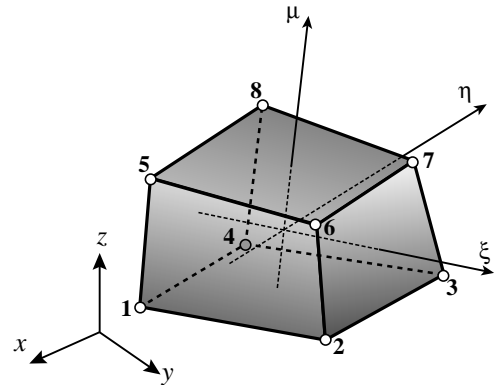


FIGURE E18.4. Eight-node isoparametric “brick” element for Exercise 18.7.

EXERCISE 18.8 [A:15]. Consider the 4-node transition triangular element of Figure 18.8(b). The shape function for node 1, $N_1 = \zeta_1 - 2\zeta_1\zeta_2$ was derived in §18.5.2 by the correction method. Show that the others are $N_2 = \zeta_2 - 2\zeta_1\zeta_2, N_3 = \zeta_3$ and $N_4 = 4\zeta_1\zeta_2$. Check that compatibility and completeness are verified.

EXERCISE 18.9 [A:15]. Construct the six shape functions for the 6-node transition quadrilateral element of Figure 18.8(c). Hint: for the corner nodes, use two corrections to the shape functions of the 4-node bilinear quadrilateral. Check compatibility and completeness. Partial result: $N_1 = \frac{1}{4}(1 - \xi)(1 - \eta) - \frac{1}{4}(1 - \xi^2)(1 - \eta)$.

EXERCISE 18.10 [A:20]. Consider a 5-node transition triangle in which midnode 6 on side 1-3 is missing. Show that $N_1^e = \zeta_1 - 2\zeta_1\zeta_2 - 2\zeta_2\zeta_3$. Can this be expressed as a line product like (18.1)?

² Although this N_1^e resembles the N_1^e of the 4-node quadrilateral depicted in Figure 18.4, they are not the same. That in Figure E18.3 must vanish at node 5 ($\xi = \eta = 0$). On the other hand, the N_1^e of Figure 18.4 takes the value $\frac{1}{4}$ there.

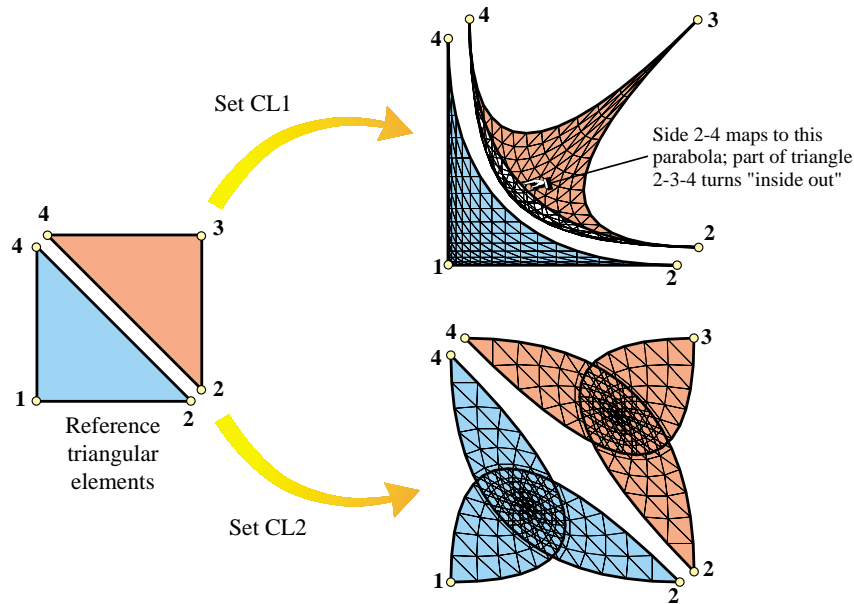


FIGURE E18.5. Mapping of reference triangles under sets (E18.1) and (E18.2). Triangles are slightly separated at the diagonal 2–4 for visualization convenience.

EXERCISE 18.11 [A:30]. The three-node linear triangle is known to be a poor performer for stress analysis. In an effort to improve it, Dr. I. M. Clueless proposes two sets of quadratic shape functions:

$$\text{CL1: } N_1 = \zeta_1^2, \quad N_2 = \zeta_2^2, \quad N_3 = \zeta_3^2. \tag{E18.1}$$

$$\text{CL2: } N_1 = \zeta_1^2 + 2\zeta_2\zeta_3, \quad N_2 = \zeta_2^2 + 2\zeta_3\zeta_1, \quad N_3 = \zeta_3^2 + 2\zeta_1\zeta_2. \tag{E18.2}$$

Dr. C. writes a learned paper claiming that both sets satisfy the interpolation condition, that set CL1 will work because it is conforming and that set CL2 will work because $N_1 + N_2 + N_3 = 1$. He provides no numerical examples. You get the paper for review. Show that the claims are false, and both sets are worthless. Hint: study §16.6 and Figure E18.5.

EXERCISE 18.12 [A:25]. Another way of constructing shape functions for “incomplete” elements is through kinematic multifreedom constraints (MFCs) applied to a “parent” element that contains the one to be derived. Suppose that the 9-node biquadratic quadrilateral is chosen as parent, with shape functions called N_i^P , $i = 1, \dots, 9$ given in §18.4.2. To construct the shape functions of the 8-node serendipity quadrilateral, the motions of node 9 are expressed in terms of the motions of the corner and midside nodes by the interpolation formulas

$$\begin{aligned} u_{x9} &= \alpha(u_{x1} + u_{x2} + u_{x3} + u_{x4}) + \beta(u_{x5} + u_{x6} + u_{x7} + u_{x8}), \\ u_{y9} &= \alpha(u_{y1} + u_{y2} + u_{y3} + u_{y4}) + \beta(u_{y5} + u_{y6} + u_{y7} + u_{y8}), \end{aligned} \tag{E18.3}$$

where α and β are scalars to be determined. (In the terminology of Chapter 9, u_{x9} and u_{y9} are slaves while boundary DOFs are masters.) Show that the shape functions of the 8-node quadrilateral are then $N_i = N_i^P + \alpha N_9^P$ for $i = 1, \dots, 4$ and $N_i = N_i^P + \beta N_9^P$ for $i = 5, \dots, 8$. Furthermore, show that α and β can be determined by two conditions:

1. The unit sum condition: $\sum_{i=1}^8 N_i = 1$, leads to $4\alpha + 4\beta = 1$.
2. Exactness of displacement interpolation for ξ^2 and η^2 leads to $2\alpha + \beta = 0$.

Solve these two equations for α and β , and verify that the serendipity shape functions given in §18.4.3 result.

EXERCISE 18.13 [A:25] Construct the 16 shape functions of the bicubic quadrilateral.