

Homework Exercises for Chapter 16 - The Isoparametric Representation
Solutions

EXERCISE 16.1 Verification of this condition is necessary so that *constant functions* are interpolated correctly over the element. If the function happens to be a displacement component such as u_x or u_y , the constant value corresponds to a *rigid body motion*, which must be represented exactly. To illustrate this point suppose that for an n -node element, $c = u_{x1} = u_{x2} = \dots = u_{xn}$, which represents a rigid body motion $u_x = c$ (constant over the element). Then

$$u_x = u_{x1}N_1^e + u_{x2}N_2^e + \dots + u_{xn}N_n^e = \sum_{i=1}^n u_{xi}N_i^e = c \sum_{i=1}^n N_i^e = c \times 1 = c. \quad (\text{E16.7})$$

EXERCISE 16.2 Check for the 6-node quadratic triangle:

$$\begin{aligned} \sum_{i=1}^6 N_i^e &= 2(\zeta_1^2 + \zeta_2^2 + \zeta_3^2 + 2\zeta_1\zeta_2 + 2\zeta_2\zeta_3 + 2\zeta_3\zeta_1) - (\zeta_1 + \zeta_2 + \zeta_3) \\ &= 2(\zeta_1 + \zeta_2 + \zeta_3)^2 - (\zeta_1 + \zeta_2 + \zeta_3) = 2 \times 1 - 1 = 1 \end{aligned} \quad (\text{E16.8})$$

where the fact that $\zeta_1 + \zeta_2 + \zeta_3 = 1$ has been used.

EXERCISE 16.3 The complete shape functions of the 9-node biquadratic quadrilateral are:

$$\begin{aligned} N_1^e &= \frac{1}{4}(1-\xi)(1-\eta)\xi\eta, & N_5^e &= -\frac{1}{2}(1-\xi^2)(1-\eta)\eta, \\ N_2^e &= -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta, & N_6^e &= \frac{1}{2}(1+\xi)(1-\eta^2)\xi, \\ N_3^e &= \frac{1}{4}(1+\xi)(1+\eta)\xi\eta, & N_7^e &= \frac{1}{2}(1-\xi^2)(1+\eta)\eta, \\ N_4^e &= -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta, & N_8^e &= -\frac{1}{2}(1-\xi)(1-\eta^2)\xi, \end{aligned} \quad N_9^e = (1-\xi^2)(1-\eta^2). \quad (\text{E16.9})$$

Check that sum is unity:

$$\sum_{i=1}^4 N_i^e = \xi^2\eta^2, \quad \sum_{i=5}^8 N_i^e = \xi^2 + \eta^2 - 2\xi^2\eta^2, \quad \sum_{i=1}^8 N_i^e = \xi^2 + \eta^2 - \xi^2\eta^2. \quad (\text{E16.10})$$

Since $N_9^e = 1 - \xi^2 - \eta^2 + \xi^2\eta^2$, we see that

$$N_9^e = 1 - \sum_{i=1}^8 N_i^e, \quad \text{whence} \quad \sum_{i=1}^9 N_i^e = 1. \quad (\text{E16.11})$$

EXERCISE 16.4 To construct N_1 assume the quadratic polynomial $N_1(\xi) = a_0 + a_1\xi + a_2\xi^2$. The conditions $N_1 = 1, 0$ and 0 at $\xi = -1, 0$ and 1 (nodes 1, 3 and 2, respectively), yield $a_0 = 0, a_1 = -1/2$ and $a_2 = 1/2$, whence $N_1 = -(1/2)\xi + (1/2)\xi^2 = -\xi(1-\xi)/2$. A similar technique can be used for N_2 and N_3 . The complete set is given in (E16.2).

Unit sum check: $N_1 + N_2 + N_3 = -\frac{1}{2}\xi + \frac{1}{2}\xi^2 + \frac{1}{2}\xi + \frac{1}{2}\xi^2 + 1 - \xi^2 = 1$.

This “brute force” technique for constructing shape functions works well for this 1D element, but leads to complicated algebra in 2D and 3D. A quicker and more elegant method that directly builds the functions as product of linear factors is explained in Chapter 18.

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ClearAll[L,α,ξ,EA];
Ne={-ξ*(1-ξ)/2,ξ*(1+ξ)/2,1-ξ^2}; J=(L-4*L*α*ξ)/2;
B=Simplify[D[Ne,ξ]/J]; BTBJ=Transpose[{B}].{B}*J; w1=w2=1;
Ke=Simplify[EA*(w1*BTBJ/.ξ->-Sqrt[1/3])+(w2*BTBJ/.ξ->Sqrt[1/3])];
Print["Ke=",Ke//MatrixForm];

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FIGURE E16.4. *Mathematica* script for item (c) of Exercise 16.5.**EXERCISE 16.5**

- (a) From the definition of element geometry in the second row of (E16.3):

$$x = x_1 N_1^e + x_2 N_2^e + x_3 N_3^e = \ell \left[\frac{1}{2} \xi (\xi + 1) + \left(\frac{1}{2} + \alpha \right) (1 - \xi^2) \right], \quad (\text{E16.12})$$

we get

$$J = \frac{\partial x}{\partial \xi} = \left(\frac{1}{2} - 2\xi\alpha \right) \ell. \quad (\text{E16.13})$$

(In 1D, the Jacobian is a scalar and is the same as its determinant.) J is linear in ξ . Consequently its max/min values over the element occur at the end nodes:

$$J_1 = J|_{\xi=-1} = \left(\frac{1}{2} - 2\alpha \right) \ell, \quad J_2 = J|_{\xi=1} = \left(\frac{1}{2} + 2\alpha \right) \ell \quad (\text{E16.14})$$

Both J_1 and J_2 are obviously positive if $-\frac{1}{4} < \alpha < \frac{1}{4}$, and if so $J > 0$ inside the element. If $\alpha = 0$, $J = \frac{1}{2}\ell$ is constant over the element.

- (b) The strain displacement matrix linking
- $e = \mathbf{B} \mathbf{u}^e$
- is given by

$$\mathbf{B} = \frac{d\mathbf{N}^e}{dx} = \frac{d\mathbf{N}^e}{d\xi} \frac{d\xi}{dx} = J^{-1} \left[\frac{\partial N_1^e}{\partial \xi} \quad \frac{\partial N_2^e}{\partial \xi} \quad \frac{\partial N_3^e}{\partial \xi} \right] = \frac{1}{\ell \left(\frac{1}{2} - 2\alpha\xi \right)} \left[\xi - \frac{1}{2} \quad \xi + \frac{1}{2} \quad -2\xi \right] \quad (\text{E16.15})$$

- (c) From Chapter 12 the element strain energy is (12.18). Replacing the
- $e = \mathbf{B} \mathbf{u}^e$
- of item (b) gives
- $U^e = \frac{1}{2} (\mathbf{u}^e)^T \int_0^\ell \mathbf{B}^T \mathbf{E} \mathbf{B} dx \mathbf{u}^e \stackrel{\text{def}}{=} \frac{1}{2} (\mathbf{u}^e)^T \mathbf{K}^e \mathbf{u}^e$
- , whence

$$\mathbf{K}^e = \int_0^\ell \mathbf{B}^T \mathbf{E} \mathbf{B} dx = \int_{-1}^1 \mathbf{B}^T \mathbf{E} \mathbf{B} \frac{dx}{d\xi} d\xi = \int_{-1}^1 \mathbf{B}^T \mathbf{E} \mathbf{B} J d\xi. \quad (\text{E16.16})$$

For the two-point Gauss rule the *Mathematica* script of Figure E16.4 gives

$$\mathbf{K}^e = \frac{EA}{\ell(3-16\alpha^2)} \begin{bmatrix} 7-16\alpha & 1 & -8(1-2\alpha) \\ 1 & 7+16\alpha & -8(1+2\alpha) \\ -8(1-2\alpha) & -8(1+2\alpha) & 16 \end{bmatrix} \quad (\text{E16.17})$$

If $\alpha = 0$ this reduces to

$$\mathbf{K}^e = \frac{EA}{3\ell} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \quad (\text{E16.18})$$

- (d) If
- $\alpha = 0$
- , from item (a)
- $J = \frac{1}{2}\ell$
- is constant whereas
- \mathbf{B}
- is linear in
- ξ
- . Because
- EA
- is constant,
- $EA \mathbf{B}^T \mathbf{B} J$
- is a
- quadratic*
- polynomial in
- ξ
- . This is exactly integrated by a Gauss rule with 2 or more points.

```

ClearAll[L,α,ξ,EA];
Ne={-ξ*(1-ξ)/2,ξ*(1+ξ)/2,1-ξ^2}; J=(L-4*L*α*ξ)/2;
B=Simplify[{D[Ne,ξ]/J}]; Print["B=",B];
Ke = Simplify[Integrate[EA*Transpose[B].B*J, {ξ, -1, 1},
Assumptions -> EA>0 && L>0 && α > 0 && α < 1/2]];
Print["Ke exact:",Ke];
Print["Ke exact for α=0:",(Ke/.α->0)//MatrixForm];
Keseries=Normal[Series[Ke,{α,0,2}]];
Print["Keseries=",Keseries//MatrixForm];
Print["Ke series for α=0:",
Simplify[Keseries/.α->0]//MatrixForm];

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FIGURE E16.5. *Mathematica* script for Exercise 16.6.

$$B = \left\{ \left\{ \frac{1-2\xi}{-L+4L\alpha\xi}, \frac{1+2\xi}{L-4L\alpha\xi}, -\frac{4\xi}{L-4L\alpha\xi} \right\} \right\}$$

$$\text{Ke exact: } \left\{ \left\{ \frac{EA(i\pi - 8\alpha - 4i\pi\alpha + 32\alpha^2 + 4i\pi\alpha^2 - (1-2\alpha)^2 \text{Log}[-1+4\alpha] + (1-2\alpha)^2 \text{Log}[1+4\alpha])}{32L\alpha^3}, \right. \right.$$

$$\left. \frac{EA(-8\alpha + (-1+4\alpha^2) \text{Log}[1-4\alpha] + (1-4\alpha^2) \text{Log}[1+4\alpha])}{32L\alpha^3}, \right.$$

$$\left. \left. - \frac{EA(-1+2\alpha)(8\alpha + \text{Log}[1-4\alpha] - \text{Log}[1+4\alpha])}{16L\alpha^3} \right\}, \right.$$

$$\left\{ \frac{EA(-8\alpha + (-1+4\alpha^2) \text{Log}[1-4\alpha] + (1-4\alpha^2) \text{Log}[1+4\alpha])}{32L\alpha^3}, \right.$$

$$\frac{EA(i(\pi(1+2\alpha)^2 + 8i\alpha(1+4\alpha)) - (1+2\alpha)^2 \text{Log}[-1+4\alpha] + (1+2\alpha)^2 \text{Log}[1+4\alpha])}{32L\alpha^3},$$

$$\frac{EA(1+2\alpha)(8\alpha + \text{Log}[1-4\alpha] - \text{Log}[1+4\alpha])}{16L\alpha^3} \left. \right\}, \left\{ -\frac{EA(-1+2\alpha)(8\alpha + \text{Log}[1-4\alpha] - \text{Log}[1+4\alpha])}{16L\alpha^3}, \right.$$

$$\left. \frac{EA(1+2\alpha)(8\alpha + \text{Log}[1-4\alpha] - \text{Log}[1+4\alpha])}{16L\alpha^3}, -\frac{EA(8\alpha + \text{Log}[1-4\alpha] - \text{Log}[1+4\alpha])}{8L\alpha^3} \right\}$$

(bunch of error messages deleted to save space)

$$\text{Ke exact for } \alpha=0: \begin{pmatrix} \text{Indeterminate} & \text{Indeterminate} & \text{Indeterminate} \\ \text{Indeterminate} & \text{Indeterminate} & \text{Indeterminate} \\ \text{Indeterminate} & \text{Indeterminate} & \text{Indeterminate} \end{pmatrix}$$

$$\text{Keseries} = \begin{pmatrix} \frac{7EA}{3L} - \frac{16EA\alpha}{3L} + \frac{272EA\alpha^2}{15L} & \frac{EA}{3L} + \frac{112EA\alpha^2}{15L} & -\frac{8EA}{3L} + \frac{16EA\alpha}{3L} - \frac{128EA\alpha^2}{5L} \\ \frac{EA}{3L} + \frac{112EA\alpha^2}{15L} & \frac{7EA}{3L} + \frac{16EA\alpha}{3L} + \frac{272EA\alpha^2}{15L} & -\frac{8EA}{3L} - \frac{16EA\alpha}{3L} - \frac{128EA\alpha^2}{5L} \\ -\frac{8EA}{3L} + \frac{16EA\alpha}{3L} - \frac{128EA\alpha^2}{5L} & -\frac{8EA}{3L} - \frac{16EA\alpha}{3L} - \frac{128EA\alpha^2}{5L} & \frac{16EA}{3L} + \frac{256EA\alpha^2}{5L} \end{pmatrix}$$

$$\text{Ke series for } \alpha=0: \begin{pmatrix} \frac{7EA}{3L} & \frac{EA}{3L} & -\frac{8EA}{3L} \\ \frac{EA}{3L} & \frac{7EA}{3L} & -\frac{8EA}{3L} \\ -\frac{8EA}{3L} & -\frac{8EA}{3L} & \frac{16EA}{3L} \end{pmatrix}$$

FIGURE E16.6. Results of running the script of Figure E16.5 under *Mathematica* 4.2.

EXERCISE 16.6 The script used for this item is shown in Figure E16.5. Results of running the script under *Mathematica* 4.2 are given in Figure E16.6.

The stiffness entries given by exact integration are typified by $K_{11} = (EA/(32\alpha^3\ell))(i\pi - 8\alpha - 4i\pi\alpha + 32\alpha^2 + 4i\pi\alpha^2 - (1-2\alpha)^2 \log(4\alpha - 1) + (1-2\alpha)^2 \log(1+4\alpha))$, $K_{12} = (EA/(32\alpha^3\ell))(8(\alpha + (4\alpha^2 - 1) \log(1-4\alpha) + (1-4\alpha^2) \log(1+4\alpha))$, etc. The entries are complex expressions because of the particular choice of path integration made by *Mathematica* to integrate rational functions in ξ . Evaluation at $\alpha = 0$ fails because entries become 0/0.

These indeterminate limits can be resolved by expanding \mathbf{K}^e in Taylor series about $\alpha = 0$, and then evaluating the series at $\alpha = 0$. This reproduces the result (E16.18). All these numerical difficulties and gyrations are bypassed with 2-point Gauss quadrature.

```
ClearAll[L,α,ξL,ξR];
Ne={-ξ*(1-ξ)/2,ξ*(1+ξ)/2,1-ξ^2}; J=(L-4*L*α*ξ)/2;
fe=FullSimplify[Integrate[q*Ne*J,{ξ,ξL,ξR}]];
Print["fe:", fe];
Print["fe for full bar with α=0:",Simplify[fe/.{ξL->-1,ξR->1,α->0}]];
```

FIGURE E16.7. *Mathematica* script for Exercise 16.6.

EXERCISE 16.7 The *Mathematica* script of Figure E16.7 gives the complicated result:

$$\mathbf{f}^e = \frac{qL}{24} \begin{bmatrix} 3\xi_L^2 - 2(1 + 4\alpha)\xi_L^3 + 6\alpha\xi_L^4 + \xi_R^2(-3 + 2(1 + \alpha(4 - 3\xi_R))\xi_R) \\ -3\xi_L^2 + (-2 + 8\alpha)\xi_L^3 + 6\alpha\xi_L^4 + \xi_R^2(3 - 2\xi_R(-1 + \alpha(4 + 3\xi_R))) \\ -4(\xi_L(3 - \xi_L^2 + 3\alpha\xi_L(-2 + \xi_L^2)) - 3\xi_R + 6\alpha\xi_R^2 + \xi_R^3 - 3\alpha\xi_R^4) \end{bmatrix} \quad (\text{E16.19})$$

If $\alpha = 0$, $\xi_L = -1$ and $\xi_R = 1$ so the load extends over the whole bar, this expression simplifies to

$$\mathbf{f}^e = \frac{qL}{6} [1 \quad 1 \quad 4]^T. \quad (\text{E16.20})$$

This is nothing but Simpson's rule for integration.