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Isoparametric Representation

Isoparametric Representation of Finite Elements

Element geometry and displacements
are represented by
same set of shape functions (iso = same)

Strong and Weak Points of Isoparametric Formulations for Structural Mechanics

Advantages

Unification: same steps for all iso-P elements
No need to distinguish straight vs. curved side elements
Quick construction of shape functions

Disadvantages

Low-order iso-P elements may be poor performers (overstiff)
Method does not extend to problems with variational index higher than 1 (e.g., plate bending and shells)

Before Isoparametric Concept was Discovered, FEM Developers Did "SuperParametric" Elems

***Element shape functions refined,
more nodes and DOFs added***

***But element geometry was kept simple
with straight sides***

For the 3 Node Linear Triangle, Recall:

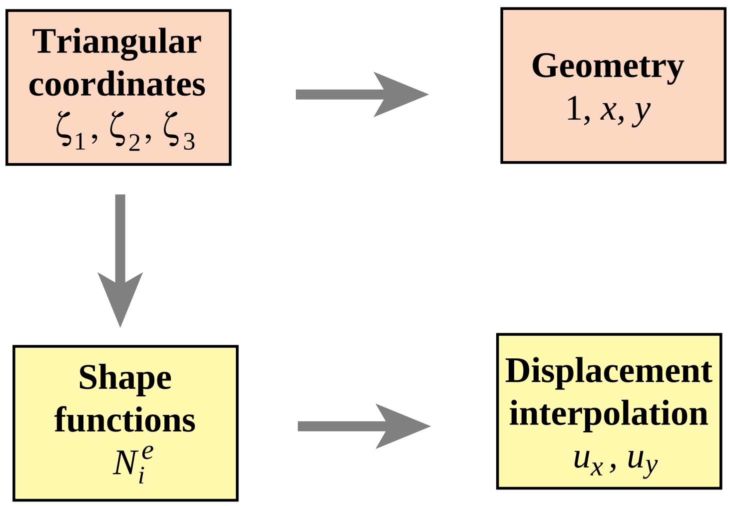
Geometric Description

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

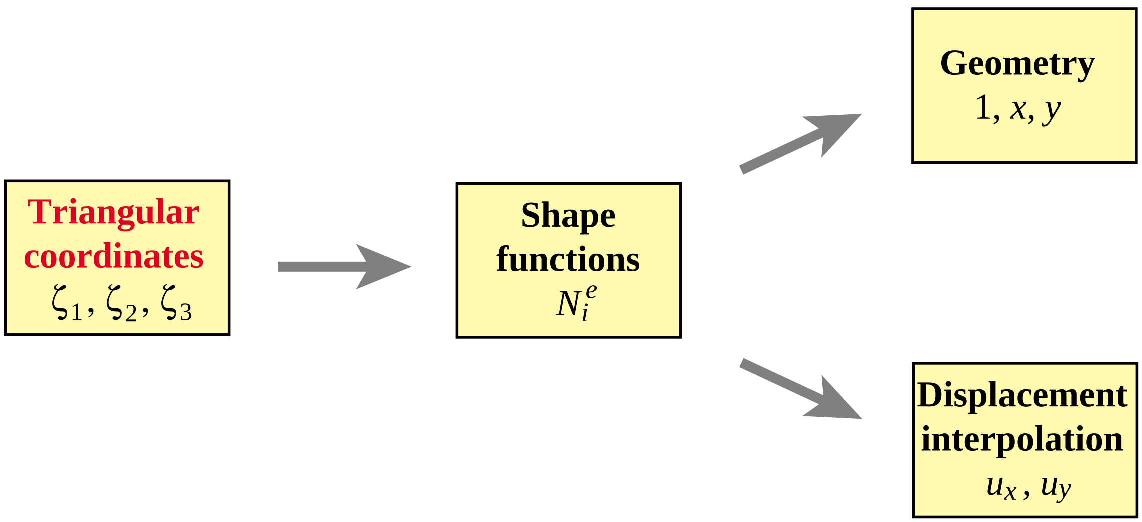
Displacement Interpolation

$$u_x = u_{x1} N_1^e + u_{x2} N_2^e + u_{x3} N_3^e = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3$$
$$u_y = u_{y1} N_1^e + u_{y2} N_2^e + u_{y3} N_3^e = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3$$

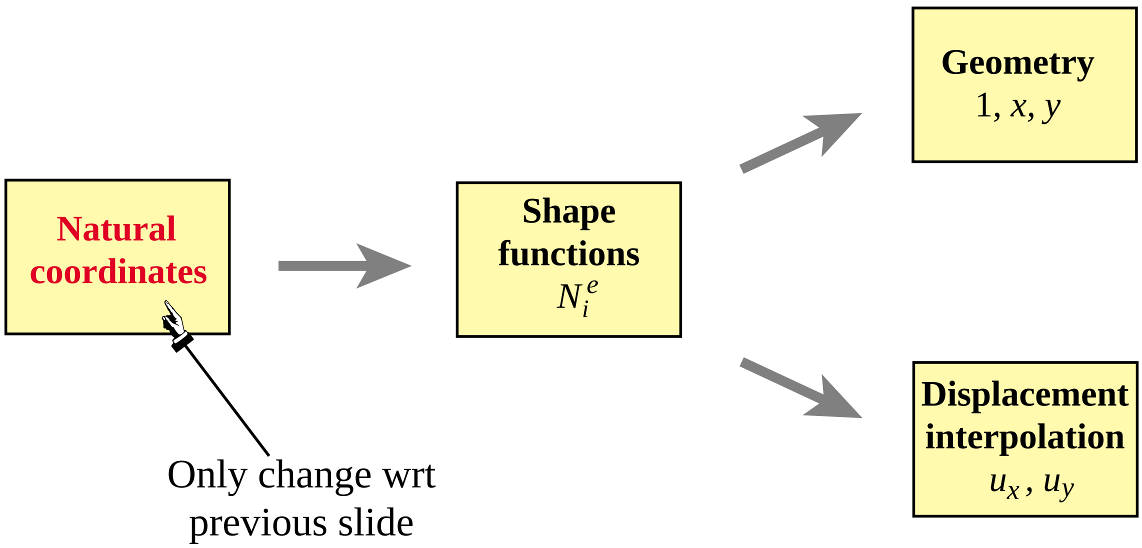
SuperParametric Representation (Triangles)



Isoparametric Representation (Iso=Equal) for Triangular Elements



Isoparametric Representation for *any* 2D Element



In 3D: $1,x,y$ **becomes** $1,x,y,z$ **etc**

Iso-P Representation of 2D Plane Stress Elements with *n* Nodes

Element Geometry:

$$1 = \sum_{i=1}^{n} N_i^e \ , \qquad x = \sum_{i=1}^{n} x_i N_i^e \ , \qquad y = \sum_{i=1}^{n} y_i N_i^e$$

Displacement Interpolation

$$u_x = \sum_{i=1}^{n} u_{xi} N_i^e \ , \qquad u_y = \sum_{i=1}^{n} u_{yi} N_i^e$$

Matrix Form of Above

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \\ u_{x1} & u_{x2} & \ldots & u_{xn} \\ u_{y1} & u_{y2} & \ldots & u_{yn} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}$$

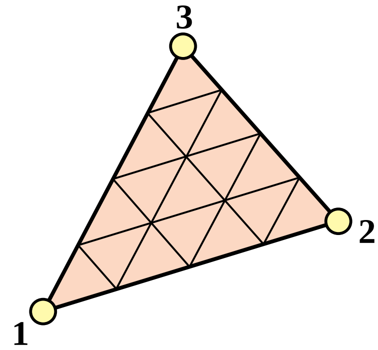
More Rows May be Added to Interpolate other Quantities from Node Values

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \\ h \\ T \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \\ h_1 & h_2 & \dots & h_n \\ T_1 & T_2 & \dots & T_n \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}$$

thickness h

temperature T

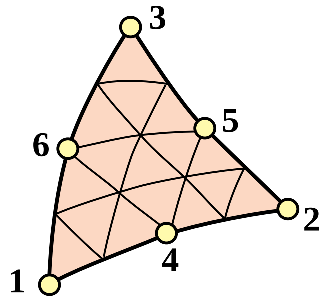
The Linear Triangle



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

$$N_1^e = \zeta_1, \qquad N_2^e = \zeta_2, \qquad N_3^e = \zeta_3$$

The Quadratic Triangle



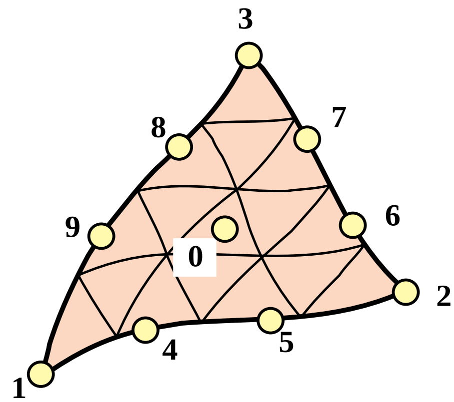
$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ N_5^e \\ N_6^e \end{bmatrix}$$

$$N_1^e = \zeta_1(2\zeta_1 - 1) \qquad N_4^e = 4\zeta_1\zeta_2$$

$$N_2^e = \zeta_2(2\zeta_2 - 1) \qquad N_5^e = 4\zeta_2\zeta_3$$

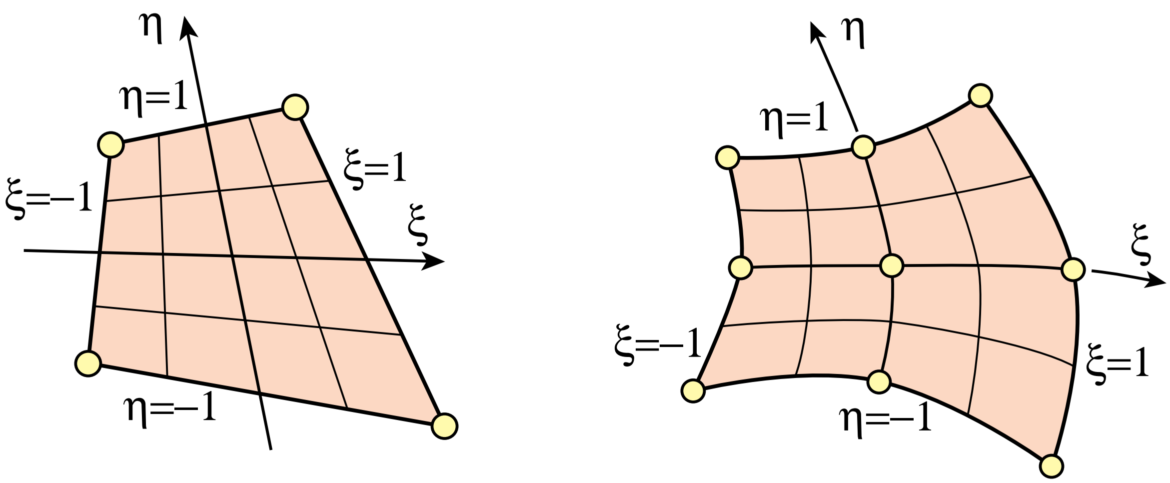
$$N_3^e = \zeta_3(2\zeta_3 - 1) \qquad N_6^e = 4\zeta_3\zeta_1$$

The Cubic Triangle

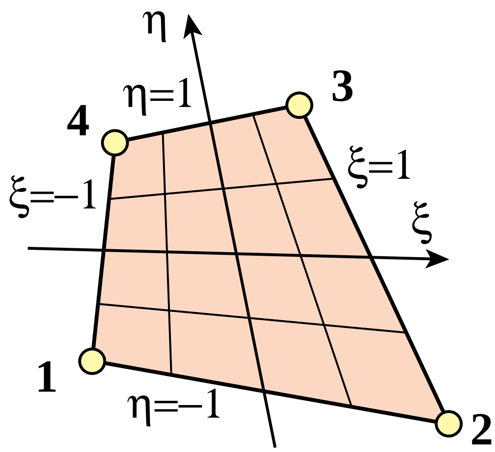


Shape functions developed as Exercise in Chapter 18

Quadrilateral Coordinates ξ, η



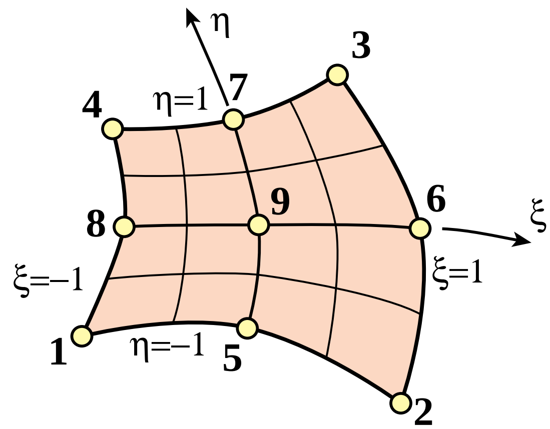
4-Node Bilinear Quadrilateral



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \end{bmatrix}$$

$$\begin{aligned} N_1^e &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ N_2^e &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ N_3^e &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N_4^e &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}$$

9 Node Biquadratic Quadrilateral



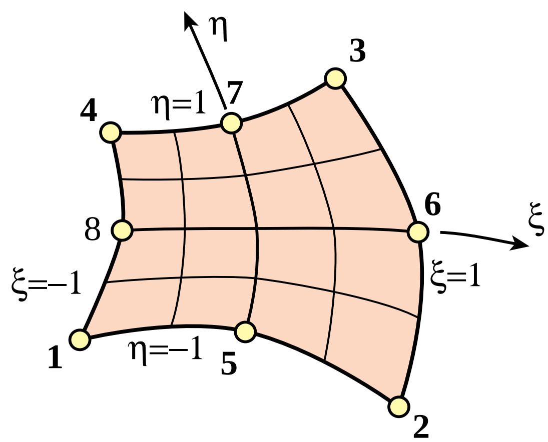
$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} & u_{x9} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} & u_{y9} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_9^e \end{bmatrix}$$

$$N_1^e = \tfrac{1}{4}(1-\xi)(1-\eta)\xi\eta \qquad N_5^e = -\tfrac{1}{2}(1-\xi^2)(1-\eta)\eta$$

$$N_2^e = -\tfrac{1}{4}(1+\xi)(1-\eta)\xi\eta \qquad N_6^e = \tfrac{1}{2}(1+\xi)(1-\eta^2)\xi \qquad N_9^e = (1-\xi^2)(1-\eta^2)$$

... ...

8 Node "Serendipity" Quadrilateral



Derivation of shape functions is an Exercise in Chapter 18