Homework Exercises for Chapter 15. The Linear Plane Stress Triangle

Solutions

EXERCISE 15.1 From (15.21):

$$\mathbf{K}^e = \mathbf{B}^T\mathbf{E}\mathbf{B}\int_{\Omega^e}(h_1\zeta_1 + h_2\zeta_2 + h_3\zeta_3)\,d\Omega^e. \tag{E15.6}$$

From (15.27):

$$\int_{\Omega^e}\zeta_1\,d\Omega^e = \int_{\Omega^e}\zeta_2\,d\Omega^e = \int_{\Omega^e}\zeta_3\,d\Omega^e = \frac{A}{3}. \tag{E15.7}$$

Replacing we get

$$\mathbf{K}^e = A h_m\,\mathbf{B}^T\mathbf{E}\mathbf{B}, \tag{E15.8}$$

where $h_m = (h_1 + h_2 + h_3)/3$ is the mean thickness. This also happens to be the thickness at the centroid.

EXERCISE 15.2 Tabulation of the right-hand side of (15.27) for $i + j + k \le 3$:

i	j	k	$\dfrac{i!\,j!\,k!}{(i+j+k+2)!}$
0	0	0	1/2
1	0	0	1/6
2	0	0	1/12
1	1	0	1/24
3	0	0	1/20
2	1	0	1/60
1	1	1	1/120

EXERCISE 15.3 Use (15.27). The integral with monomial exponents $i = 3$, $j = k = 0$ evaluates to $6A/60$; that with $i = 2$, $j = 1$, $k = 0$ to $2A/60$; and that with $i = j = k = 1$ to $A/60$. The x forces vanish: $f_{xi} = 0$, $i = 1, 2, 3$. For the y forces:

$$f_{y1} = \frac{A}{60}\Big[b_{y1}(6h_1 + 2h_2 + 2h_3) + b_{y2}(2h_1 + 2h_2 + h_3) + b_{y3}(2h_1 + h_2 + 2h_3)\Big] \tag{E15.9}$$

f_{y2} and f_{y3} follow by 3-cyclic permutation. Verification for $b_{y1} = b_{y2} = b_{y3} = b_y$ and $h_1 = h_2 = h_3 = h$ gives $hAb_y/3$ at the three corners, which is the element-by-element lumping.

EXERCISE 15.4

$$\begin{aligned} f_{x1} &= \tfrac{1}{6}h\,L_{21}(2q_{x1} + q_{x2}), \quad f_{x2} = \tfrac{1}{6}h\,L_{21}(q_{x1} + 2q_{x2}),\\ f_{y1} &= \tfrac{1}{6}h\,L_{21}(2q_{y1} + q_{y2}), \quad f_{y2} = \tfrac{1}{6}h\,L_{21}(q_{y1} + 2q_{y2}), \quad f_{x3} = f_{y3} = 0. \end{aligned} \tag{E15.10}$$

Here $L_{21} = \sqrt{x_{21}^2 + y_{21}^2}$ is the length of side 1-2.

EXERCISE 15.5

$$\mathbf{K}^e = \begin{bmatrix} 18.75 & 9.375 & -12.5 & -6.25 & -6.25 & -3.125\\ 9.375 & 18.75 & 6.25 & 12.5 & -15.625 & -31.25\\ -12.5 & 6.25 & 75.0 & -37.5 & -62.5 & 31.25\\ -6.25 & 12.5 & -37.5 & 75.0 & 43.75 & -87.5\\ -6.25 & -15.625 & -62.5 & 43.75 & 68.75 & -28.125\\ -3.125 & -31.25 & 31.25 & -87.5 & -28.125 & 118.75 \end{bmatrix} \tag{E15.11}$$

EXERCISE 15.6 Set $u_{x1} = u_{x2} = u_{x3} = 1$ and $u_{y1} = u_{y2} = u_{y3} = 0$. This is a translational rigid body motion along x and consequently the element node forces obtained from $\mathbf{f}^e = \mathbf{K}^e\mathbf{u}^e$ must be exactly zero. These forces are the sum of columns 1, 3 and 5. A similar result is obtained for columns 2, 4 and 6 on switching x and y. Because the stiffness matrix is symmetric, the same result holds on replacing "columns" by "rows." (The result extends to any mechanical finite element.)

EXERCISE 15.7 Not assigned.

EXERCISE 15.8 Not assigned.

EXERCISE 15.9 Not assigned.

EXERCISE 15.10 Not assigned.

EXERCISE 15.11 Not assigned

EXERCISE 15.12 The answer should be a hierarchical diagram such as

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Main program in Cell 8 - drives the FEM analysis
   GenerateNodes - generates node coordinates of regular mesh
   GenerateTriangles - generate element node lists of regular mesh
   GenerateEndAxialForces - generates end axial forces on cantilever
   GenerateEndMomentForces - generates end moment forces on cantilever
   GenerateEndShearForces - generates end-shear loads on cantilever
   Plot2DMesh -  plots a 2D mesh
   LinearSolutionOfPlaneStressModel - drives solution of FEM problem
      AssembleMasterStiffOfPlaneStressModel - assembles master stiffness
          StiffnessOf3NodePlaneStressTriangle - forms element stiffness
          MergeElemIntoMasterStiff - merges element stiffness into master
                                     stiffness
      ModifyNodeForcesForDBC - modifies node forces for displacement BC
      ModifyMasterStiff - modifies master stiffness for displacement BC
      (Linear solution: done by Mathematica built-in Inverse function)
      StressesInPlaneStressModel - recovers element stresses
   ContourPlotNodeFuncOver2DMesh - Contourplots a node-defined
                                   function over 2D mesh
      PlotFunctionOverTriangle - plot function over triangle
         ContourPolyColor - picks polygon display color
      PlotFunctionOverQuadrilateral - like title says
         ContourPolyColor - picks polygon display color
   ContourPlotElemFuncOver2DMesh - Contourplots an element-defined
                                   function over 2D mesh
      PlotFunctionOverTriangle - plot function over triangle
         ContourPolyColor - picks polygon display color
      PlotFunctionOverQuadrilateral - like title says
         ContourPolyColor - picks polygon display color
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