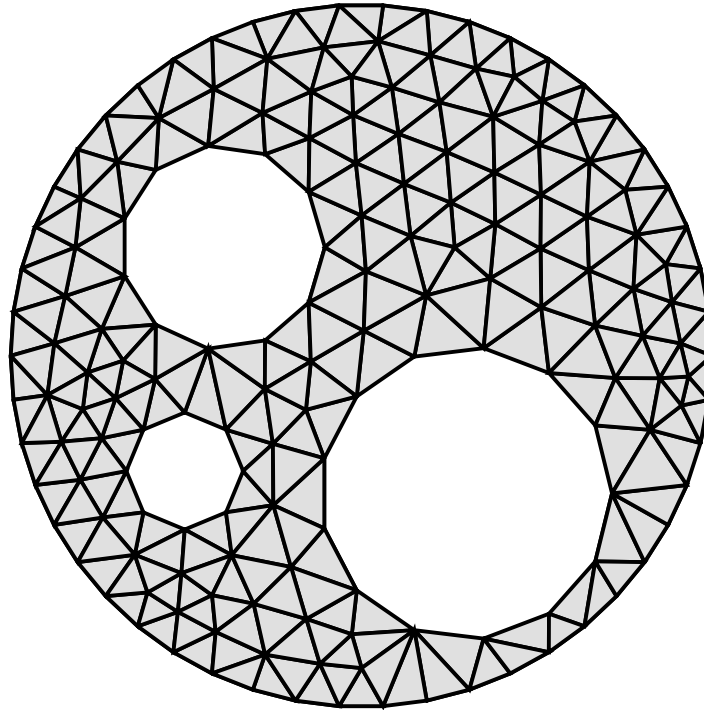


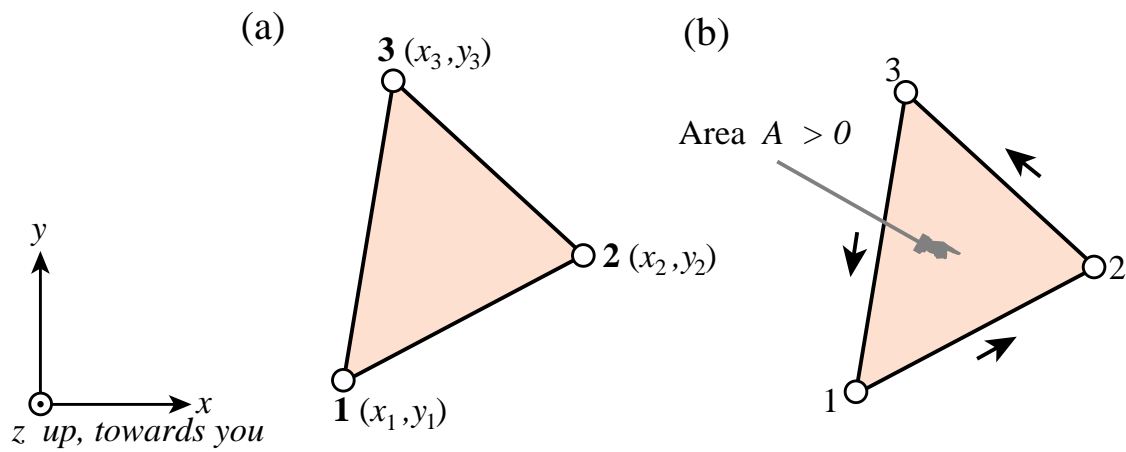
15

Three-Node Plane Stress Triangles

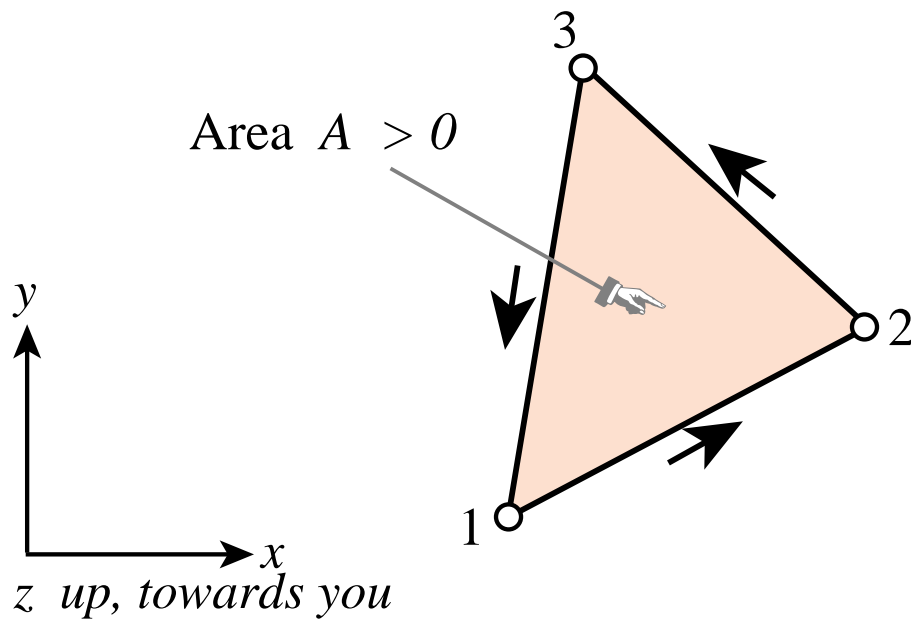
Triangles are Still Popular Because of Geometric Versatility and Ease of Automated Mesh Generation



The Turner Triangle Geometry and its Nodal Configuration

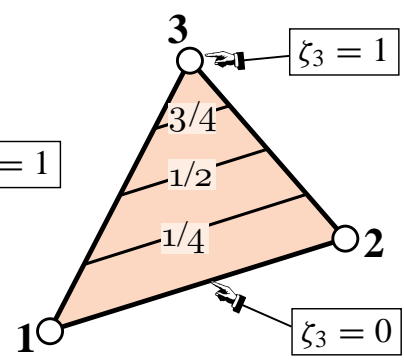
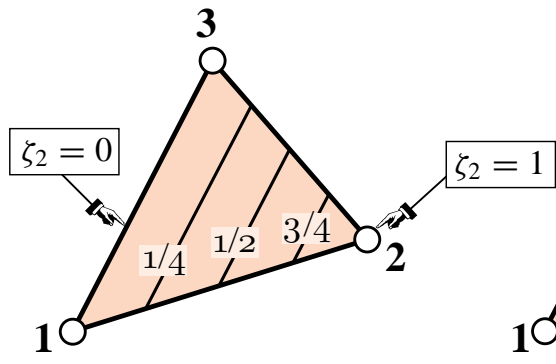
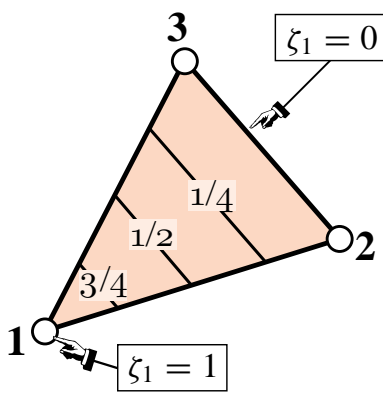


Positive Element Boundary Traversal Convention



Triangular Coordinates

ζ_1 ζ_2 ζ_3



$$\zeta_1 + \zeta_2 + \zeta_3 = 1$$

Using Triangular Coordinates to Formulate Linear Interpolation

Cartesian form

$$f(x, y) = a_0 + a_1x + a_2y$$

Variation defined by 3 corner values

$$f_1, f_2, f_3$$

Natural form

$$\begin{aligned} f(\zeta_1, \zeta_2, \zeta_3) &= f_1\zeta_1 + f_2\zeta_2 + f_3\zeta_3 = [f_1 \quad f_2 \quad f_3] \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} \\ &= [\zeta_1 \quad \zeta_2 \quad \zeta_3] \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \end{aligned}$$

Relating Triangular & Cartesian Coordinates I

Triangular to Cartesian:

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

Relating Triangular & Cartesian Coordinates II

Cartesian to Triangular:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} 2A_{23} & y_{23} & x_{32} \\ 2A_{31} & y_{31} & x_{13} \\ 2A_{12} & y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

Here $x_{jk} = x_j - x_k$ $y_{jk} = y_j - y_k$
 A_{jk} denotes the area subtended by corners j, k
and the origin of the x - y system

Partial Derivatives for Cartesian-Triangular Coordinate Transformations

$$\frac{\partial x}{\partial \zeta_i} = x_i \quad \frac{\partial y}{\partial \zeta_i} = y_i$$

$$2A \frac{\partial \zeta_i}{\partial x} = y_{jk} \quad 2A \frac{\partial \zeta_i}{\partial y} = x_{kj}$$

$$i = 1,2,3 \quad j = 2,3,1 \quad k = 3,1,2$$

Cartesian Partial Derivatives of a General Triangular-Coordinates Function

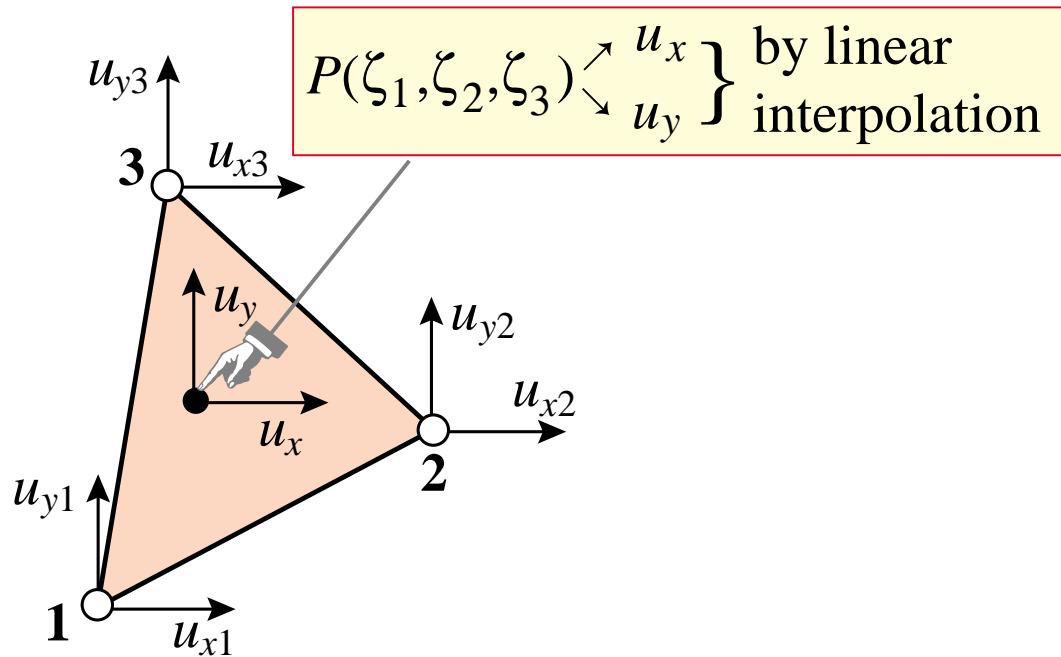
$$f = f(\zeta_1, \zeta_2, \zeta_3)$$

$$\frac{\partial f}{\partial x} = \frac{1}{2A} \left(\frac{\partial f}{\partial \zeta_1} y_{23} + \frac{\partial f}{\partial \zeta_2} y_{31} + \frac{\partial f}{\partial \zeta_3} y_{12} \right)$$

$$\frac{\partial f}{\partial y} = \frac{1}{2A} \left(\frac{\partial f}{\partial \zeta_1} x_{32} + \frac{\partial f}{\partial \zeta_2} x_{13} + \frac{\partial f}{\partial \zeta_3} x_{21} \right)$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \zeta_1} \\ \frac{\partial f}{\partial \zeta_2} \\ \frac{\partial f}{\partial \zeta_3} \end{bmatrix}$$

Displacement Interpolation over Turner Triangle



Displacement Interpolation (Cont'd)

$$u_x = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3$$

$$u_y = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3$$

In matrix form

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{N} \mathbf{u}^e$$

The shape functions are $N_i = \zeta_i$, $i = 1, 2, 3$

Strain-Displacement Relations

$$\mathbf{e} = \mathbf{D} \mathbf{N} \mathbf{u}^e = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{B} \mathbf{u}^e$$

Stress-Strain Relations

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{E} \mathbf{e}$$

Element Stiffness Matrix of Turner Triangle

$$\mathbf{K}^e = \int_{\Omega^e} h \mathbf{B}^T \mathbf{E} \mathbf{B} d\Omega$$

Because \mathbf{B} and \mathbf{E} are constant over the triangle area:

$$\begin{aligned} \mathbf{K}^e &= \mathbf{B}^T \mathbf{E} \mathbf{B} \int_{\Omega^e} h d\Omega \\ &= \frac{1}{4A^2} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \int_{\Omega^e} h d\Omega \end{aligned}$$

If the Plate Thickness h is Constant

$$\mathbf{K}^e = \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Consistent Node Force Vector for Body Forces

$$\mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

$$\mathbf{f}^e = \int_{\Omega^e} h (\mathbf{N}^e)^T \mathbf{b} d\Omega = \int_{\Omega^e} h \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_1 \\ \zeta_2 & 0 \\ 0 & \zeta_2 \\ \zeta_3 & 0 \\ 0 & \zeta_3 \end{bmatrix} \mathbf{b} d\Omega$$

If Body Forces are Constant over Element

Using *

$$\int_{\Omega^e} \zeta_1 d\Omega = \int_{\Omega^e} \zeta_2 d\Omega = \int_{\Omega^e} \zeta_3 d\Omega = \frac{1}{3}A$$

we get

$$\mathbf{f}^e = \frac{Ah}{3} \begin{bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{bmatrix}$$

same result as
straightforward
"load lumping"

* These area integrals are instances of a general formula given on the next slide

Integrating Triangular Coordinate Monomials

Integrals

$$\int_{\Omega^e} \zeta_1 d\Omega = \int_{\Omega^e} \zeta_2 d\Omega = \int_{\Omega^e} \zeta_3 d\Omega = \frac{1}{3}A \quad (*)$$

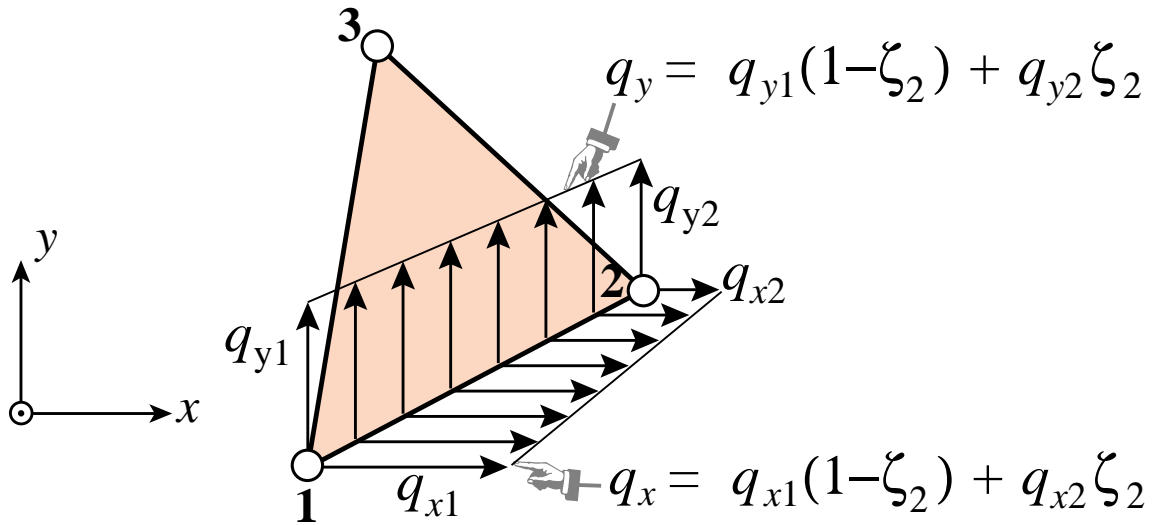
used in the previous slide are instances of the general formula

$$\frac{1}{2A} \int_{\Omega^e} \zeta_1^i \zeta_2^j \zeta_3^k d\Omega = \frac{i! j! k!}{(i + j + k + 2)!}$$

where exponents i, j, k are nonnegative integers: $i \geq 0, j \geq 0, k \geq 0$.
To get (*) set $i = 1, j = k = 0$, then permute cyclically.

The general formula is valid for triangles with *straight* sides and *constant* metric. It does not extend to curved-side triangles, a restriction not mentioned in FEM textbooks.

Line Forces on Turner Triangle Side



Lumping to node forces given as Exercise 15.4

Implementation of Turner Triangle Stiffness Computation as *Mathematica* Module

```
Trig3TurnerMembraneStiffness[ncoor_,Emat_,h_,numer_]:=Module[{
  x1,x2,x3,y1,y2,y3,x21,x32,x13,y12,y23,y31,A,Be,Ke},
  {{x1,y1},{x2,y2},{x3,y3}}=ncoor;
  A=Simplify[(x2*y3-x3*y2+(x3*y1-x1*y3)+(x1*y2-x2*y1))/2];
  {x21,x32,x13,y12,y23,y31}={x2-x1,x3-x2,x1-x3,y1-y2,y2-y3,y3-y1};
  Be={{y23,0,y31,0,y12,0},{0,x32,0,x13,0,x21},
      {x32,y23,x13,y31,x21,y12}}/(2*A);
  If [numer, Be=N[Be]]; Ke=A*h*Transpose[Be].Emat.Be;
  Return[Ke];
```

Can be downloaded from web site

Testing the Turner Triangle Stiffness Module

```
ncoor={{0,0},{3,1},{2,2}}; Emat=8*{{4,1,0},{1,4,0},{0,0,2}};
Ke=Trig3TurnerMembraneStiffness[ncoor,Emat,1,False];
Print["Ke=",Ke//MatrixForm];
Print["eigs of Ke=",Chop[Eigenvalues[N[Ke]]]];
Show[Graphics[RGBColor[1,0,0]],Graphics[AbsoluteThickness[2]],
Graphics[Polygon[ncoor]],Axes->True];
```

$$Ke = \begin{pmatrix} 6 & 3 & -4 & -2 & -2 & -1 \\ 3 & 6 & 2 & 4 & -5 & -10 \\ -4 & 2 & 24 & -12 & -20 & 10 \\ -2 & 4 & -12 & 24 & 14 & -28 \\ -2 & -5 & -20 & 14 & 22 & -9 \\ -1 & -10 & 10 & -28 & -9 & 38 \end{pmatrix}$$

eigs of Ke = {75.5334, 32.8648, 11.6017, 0, 0, 0}

