Homework Exercises for Chapter 14 - The Plane Stress Problem

Solutions

EXERCISE 14.1 Here is the solution to go from plane strain to plane stress:

```
ClearAll[Em,ν,Emstar,νstar]; Efac=Em*(1-ν)/((1+ν)*(1-2ν));
Emat=Efac*{{1,ν/(1-ν),0},{ν/(1-ν),1,0},{0,0,(1-2*ν)/(2*(1-ν))}};
Print["Plane strain E matrix:", Emat//MatrixForm];
EEmat=Simplify[Emat/.{Em->Emstar,ν->νstar}];
sol=Solve[{EEmat[[1,1]]==Em/(1-ν^2),EEmat[[1,2]]==Em*ν/(1-ν^2)},
    {Emstar,νstar}]; sol=Simplify[sol];
Print["replacement rule:",sol];
Print["Check - this should be the plane stress E matrix:",
       Simplify[EEmat/.sol[[1]]]//MatrixForm];
```

Plane strain E matrix: $\begin{pmatrix} \frac{\mathrm{Em}\,(1-\nu)}{(1-2\,\nu)\,(1+\nu)} & \frac{\mathrm{Em}\,\nu}{(1-2\,\nu)\,(1+\nu)} & 0 \\ \frac{\mathrm{Em}\,\nu}{(1-2\,\nu)\,(1+\nu)} & \frac{\mathrm{Em}\,(1-\nu)}{(1-2\,\nu)\,(1+\nu)} & 0 \\ 0 & 0 & \frac{\mathrm{Em}}{2\,(1+\nu)} \end{pmatrix}$

replacement rule: $\left\{\left\{\mathrm{Emstar} \Rightarrow \frac{\mathrm{Em}\,(1+2\,\nu)}{(1+\nu)^2},\ \nu\mathrm{star} \Rightarrow \frac{\nu}{1+\nu}\right\}\right\}$

Check – this should be the plane stress E matrix: $\begin{pmatrix} \frac{\mathrm{Em}}{1-\nu^2} & \frac{\mathrm{Em}\,\nu}{1-\nu^2} & 0 \\ \frac{\mathrm{Em}\,\nu}{1-\nu^2} & \frac{\mathrm{Em}}{1-\nu^2} & 0 \\ 0 & 0 & \frac{\mathrm{Em}}{2+2\,\nu} \end{pmatrix}$

FIGURE E14.2. Solution for Exercise 14.1. Plane strain to plane stress.

It gives $E^* = E\,\dfrac{(1+2\nu)}{(1+\nu)^2}$ and $\nu^* = \dfrac{\nu}{1+\nu}$.

Here is to go from plane stress to plane strain:

```
ClearAll[Em,ν,Emstar,νstar]; Efac=Em/(1-ν^2);
Emat=Simplify[Efac*{{1,ν,0},{ν,1,0},{0,0,(1-ν)/2}}];
Print["Plane stress E matrix:", Emat//MatrixForm];
EEmat=Simplify[Emat/.{Em->Emstar,ν->νstar}];
sol=Solve[{EEmat[[1,1]]==Em*(1-ν)/((1+ν)*(1-2ν)),
           EEmat[[1,2]]==Em*ν/((1+ν)*(1-2ν))},
           {Emstar,νstar}]; sol=Simplify[sol];
Print["replacement rule:",sol];
Print["Check - this should be the plane strain E matrix:",
       Simplify[EEmat/.sol[[1]]]//MatrixForm];
```

Plane stress E matrix: $\begin{pmatrix} \frac{\mathrm{Em}}{1-\nu^2} & \frac{\mathrm{Em}\,\nu}{1-\nu^2} & 0 \\ \frac{\mathrm{Em}\,\nu}{1-\nu^2} & \frac{\mathrm{Em}}{1-\nu^2} & 0 \\ 0 & 0 & \frac{\mathrm{Em}}{2+2\,\nu} \end{pmatrix}$

replacement rule: $\left\{\left\{\mathrm{Emstar} \Rightarrow \frac{\mathrm{Em}}{1-\nu^2},\ \nu\mathrm{star} \Rightarrow -\frac{\nu}{-1+\nu}\right\}\right\}$

Check – this should be the plane strain E matrix: $\begin{pmatrix} \frac{\mathrm{Em}\,(-1+\nu)}{-1+\nu+2\,\nu^2} & -\frac{\mathrm{Em}\,\nu}{-1+\nu+2\,\nu^2} & 0 \\ -\frac{\mathrm{Em}\,\nu}{-1+\nu+2\,\nu^2} & \frac{\mathrm{Em}\,(-1+\nu)}{-1+\nu+2\,\nu^2} & 0 \\ 0 & 0 & \frac{\mathrm{Em}}{2+2\,\nu} \end{pmatrix}$

FIGURE E14.3. Solution for Exercise 14.1. Plane stress to plane strain.

It gives $E^* = \dfrac{E}{1-\nu^2}$ and $\nu^* = \dfrac{\nu}{1-\nu}$. Credit is given for doing it either way.

EXERCISE 14.2 For plane strain:

$$
\begin{aligned}
\mathbf{E} &= \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}_\mu + \mathbf{E}_\lambda .
\end{aligned}
\tag{E14.10}
$$

For plane stress:

$$
\begin{aligned}
\mathbf{E} &= \begin{bmatrix} \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu} & \dfrac{2\lambda\mu}{\lambda+2\mu} & 0 \\ \dfrac{2\lambda\mu}{\lambda+2\mu} & \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu} & 0 \\ 0 & 0 & \mu \end{bmatrix} = \begin{bmatrix} 2\mu-\bar\lambda & 0 & 0 \\ 0 & 2\mu-\bar\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} + \bar\lambda \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \bar\lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{E}{2(1+\nu)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{E\nu}{1-\nu^2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{E}_\mu + \mathbf{E}_\lambda .
\end{aligned}
\tag{E14.11}
$$

(The second step above is actually unnecessary, one could go directly to the third expression.) Here $\bar\lambda = 2\lambda\mu/(\lambda+2\mu) = E\nu/(1-\nu^2)$ is a modified Lamé constant; the inverse relation being $\lambda = 2\bar\lambda\mu/(2\mu-\bar\lambda)$.

EXERCISE 14.3 A solution using *Mathematica* is shown in Figure E14.4. In this script `gxy` stands for $2e_{xy}$.

```
ClearAll[Em,ν,α,Gm,exx,eyy,gxy,sxx,syy,sxy,ΔT]; Gm=Em/(2*(1+ν));
s={sxx,syy,sxy}; e={exx,eyy,gxy};
eqs={exx==(sxx-ν*syy)/Em+α*ΔT,eyy==(syy-ν*sxx)/Em+α*ΔT,gxy==sxy/Gm};
s=Simplify[s/.Solve[eqs,s][[1]]];
Emat=Simplify[Table[Coefficient[s[[i]],e[[j]]],{i,1,3},{j,1,3}]];
Tmat=Simplify[Table[Coefficient[s[[i]],ΔT],{i,1,3}]];
Print["Emat=",Emat//MatrixForm,",  Tmat=",Tmat];
```

$$
\text{Emat}=\begin{pmatrix} -\frac{\text{Em}}{-1+\nu^2} & -\frac{\text{Em}\,\nu}{-1+\nu^2} & 0 \\ -\frac{\text{Em}\,\nu}{-1+\nu^2} & -\frac{\text{Em}}{-1+\nu^2} & 0 \\ 0 & 0 & \frac{\text{Em}}{2+2\,\nu} \end{pmatrix}, \quad \text{Tmat}=\left\{\frac{\text{Em}\,\alpha}{-1+\nu}, \frac{\text{Em}\,\alpha}{-1+\nu}, 0\right\}
$$

FIGURE E14.4. Solution for Exercise 14.3.

Transcribing the answer:

$$
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} - \frac{E\alpha}{1-\nu} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Delta T,
\tag{E14.12}
$$

The only difference with respect to the first of (E14.1) is the thermal stress vector, which vanishes if $\Delta T = 0$. This form may be found in any book on elasticity.

EXERCISE 14.4

Equilibrium along x and y give $t_x\,ds = \sigma_{xx}\,dx + \sigma_{yx}\,dy$ and $t_y\,ds = \sigma_{xy}\,dx + \sigma_{yy}\,dy$, respectively. Dividing through by ds and setting $dx/ds = n_x$, $dy/ds = n_y$, $\sigma_{yx} = \sigma_{xy}$, yields (14.10).

EXERCISE 14.5

The verification of (a) and (b) is immediate on expanding the quadratic forms. Item (c) is more difficult. A brute force solution using *Mathematica* for an arbitrary material matrix **E** is shown in Figure E14.5.

```
ClearAll[exx,eyy,gxy,sxx,syy,sxy,E11,E22,E33,E12,E13,E23];
Emat={{E11,E12,E13},{E12,E22,E23},{E13,E23,E33}};
s={sxx,syy,sxy}; e={exx,eyy,gxy}; m={exx,eyy,sxy};
eqs={sxx==E11*exx+E12*eyy+E13*gxy,syy==E12*exx+E22*eyy+E23*gxy,
     sxy==E13*exx+E23*eyy+E33*gxy};
sol=Simplify[Simplify[Solve[eqs,{sxx,syy,gxy}]]];
Print[sol]; U=Simplify[(e.Emat.e/2)/.sol[[1]]];
fac[i_,j_]:=If[i==j,1,1/2];
A=Table[fac[i,j]*Coefficient[U,m[[i]]*m[[j]]],{i,1,3},{j,1,3}];
Print["A=",A//MatrixForm];
```

$$\Big\{\Big\{\text{sxx} \Rightarrow \text{E11 exx} + \text{E12 eyy} - \frac{\text{E13 (E13 exx + E23 eyy - sxy)}}{\text{E33}},\; \text{syy} \Rightarrow \text{E12 exx} + \text{E22 eyy} - \frac{\text{E23 (E13 exx + E23 eyy - sxy)}}{\text{E33}},\; \text{gxy} \Rightarrow \frac{-\text{E13 exx} - \text{E23 eyy} + \text{sxy}}{\text{E33}}\Big\}\Big\}$$

$$\text{A}=\begin{pmatrix} \frac{-\text{E13}^2+\text{E11 E33}}{\text{2 E33}} & \frac{-\text{2 E13 E23}+\text{2 E12 E33}}{\text{4 E33}} & 0 \\ \frac{-\text{2 E13 E23}+\text{2 E12 E33}}{\text{4 E33}} & \frac{-\text{E23}^2+\text{E22 E33}}{\text{2 E33}} & 0 \\ 0 & 0 & \frac{1}{\text{2 E33}} \end{pmatrix}$$

FIGURE E14.5. Solution for Exercise 14.5.

Transcribing the result, the matrix for item (c) is

$$\mathbf{A} = \begin{bmatrix} \dfrac{E_{11}E_{33} - E_{13}^2}{2E_{33}} & \dfrac{E_{12}E_{33} - 2E_{13}E_{23}}{4E_{33}} & 0 \\ & \dfrac{E_{22}E_{33} - E_{23}^2}{2E_{33}} & 0 \\ symm & & \dfrac{1}{2E_{33}} \end{bmatrix} \tag{E14.13}$$