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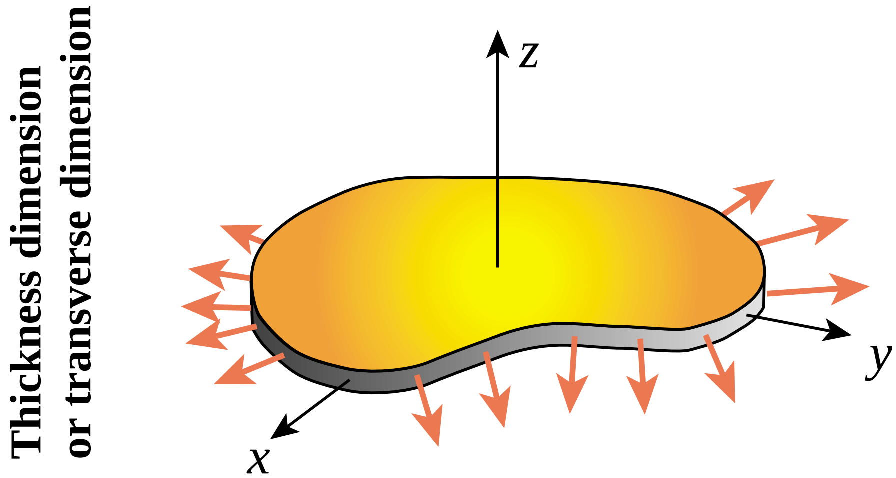
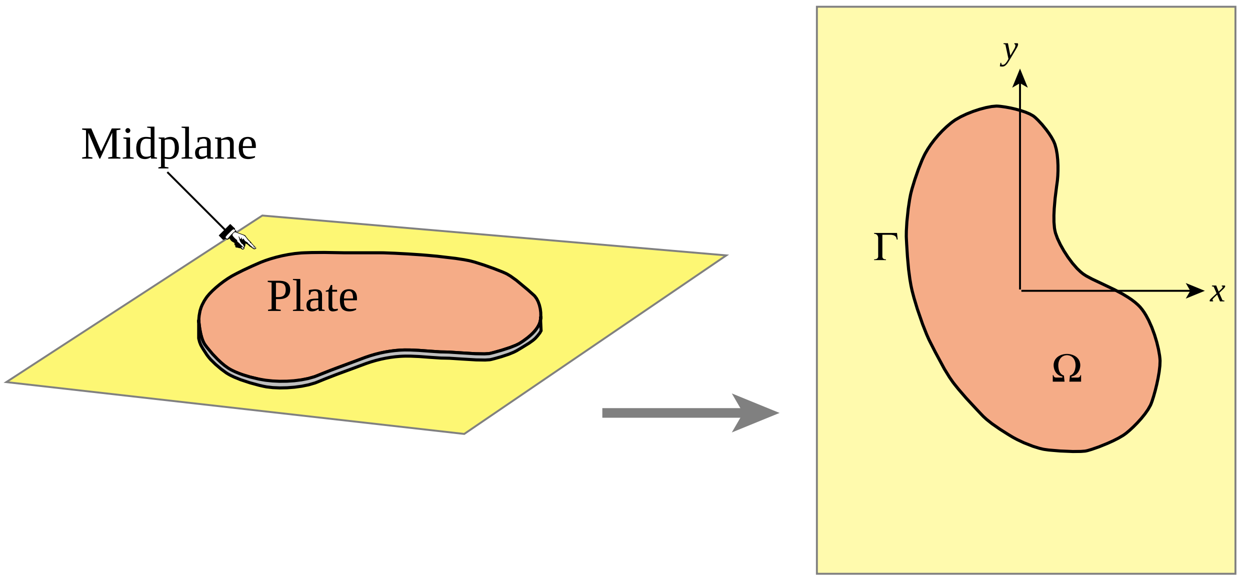
The Plane Stress Problem

Plate in Plane Stress



Inplane dimensions: in *x,y* **plane**

Mathematical Idealization as a Two Dimensional Problem



Plane Stress Physical Assumptions

Plate is flat and has a symmetry plane (the midplane)

All loads and support conditions are midplane symmetric

Thickness dimension is much smaller than inplane dimensions

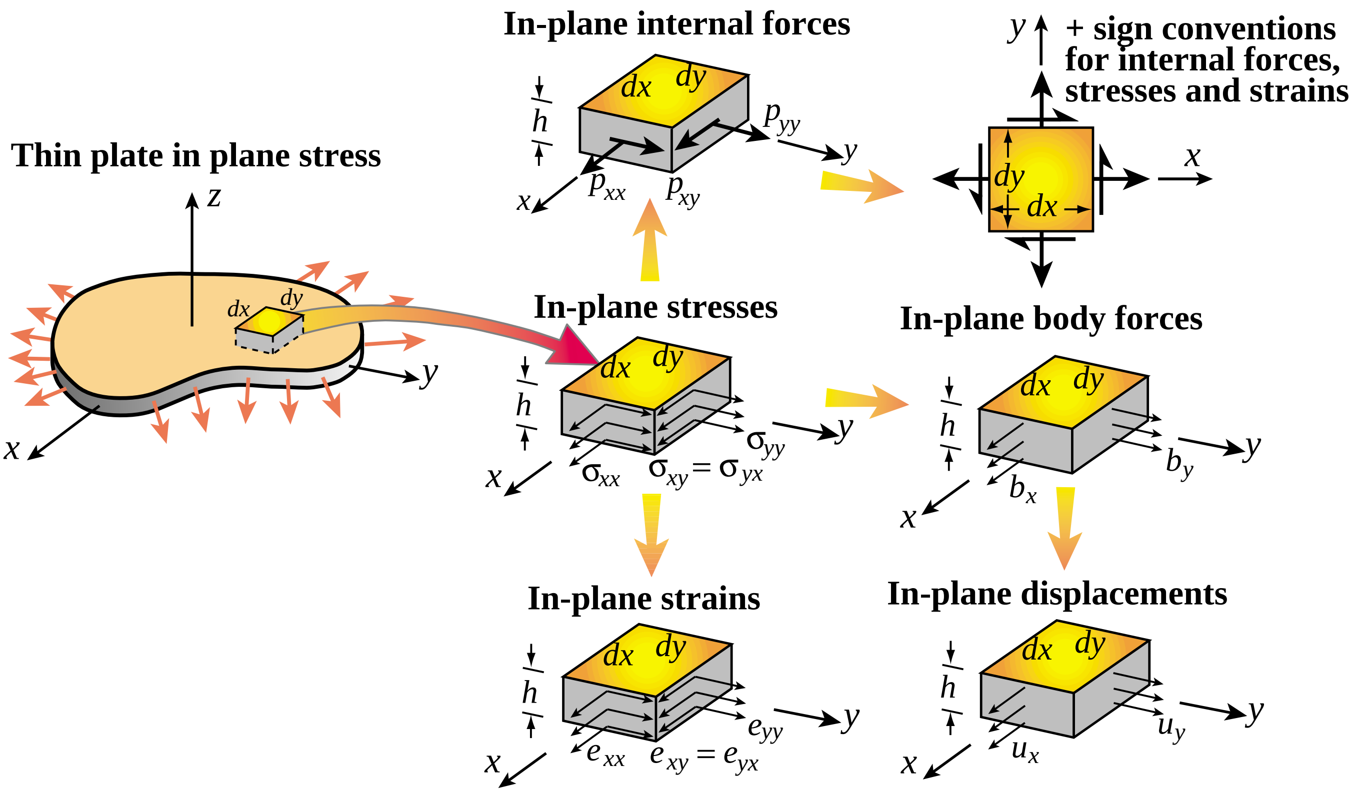
Inplane displacements, strains and stresses uniform through thickness

Transverse stresses σ_{zz}, σ_{xz} **and** σ_{yz} **negligible, set to 0**

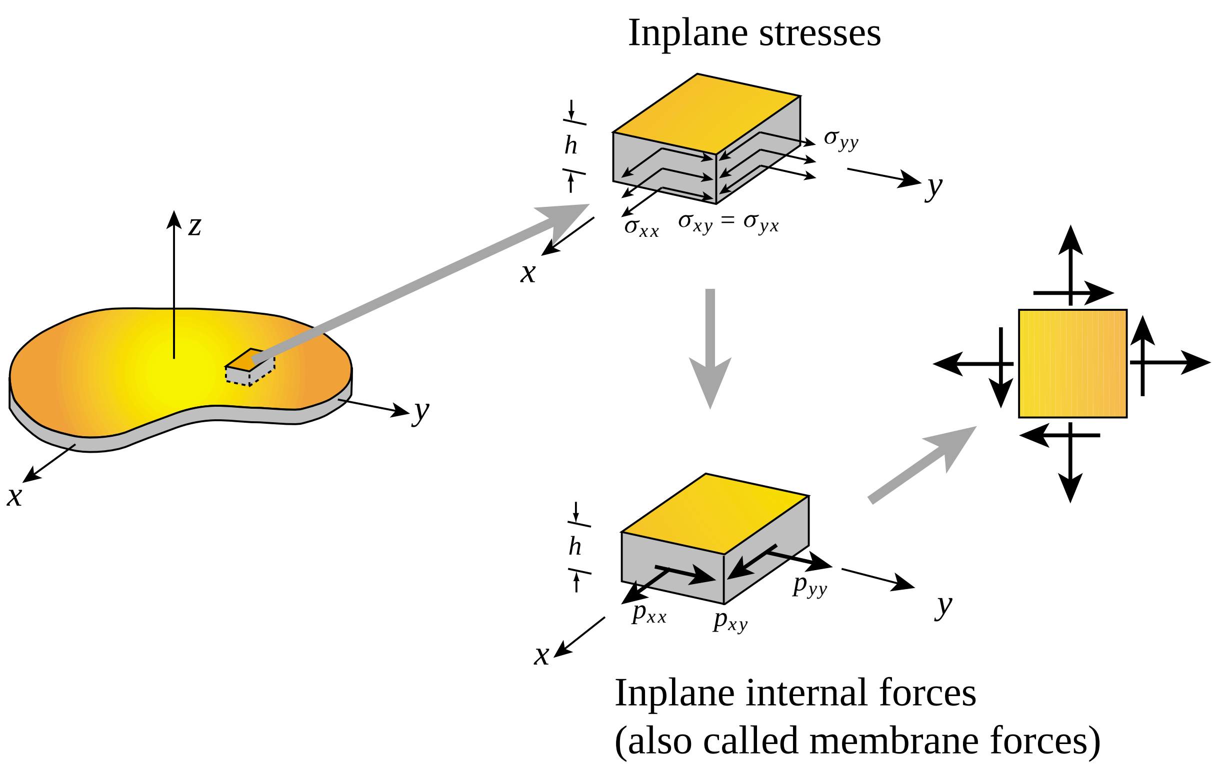
Unessential but used in this course:

Plate fabricated of homogeneous material through thickness

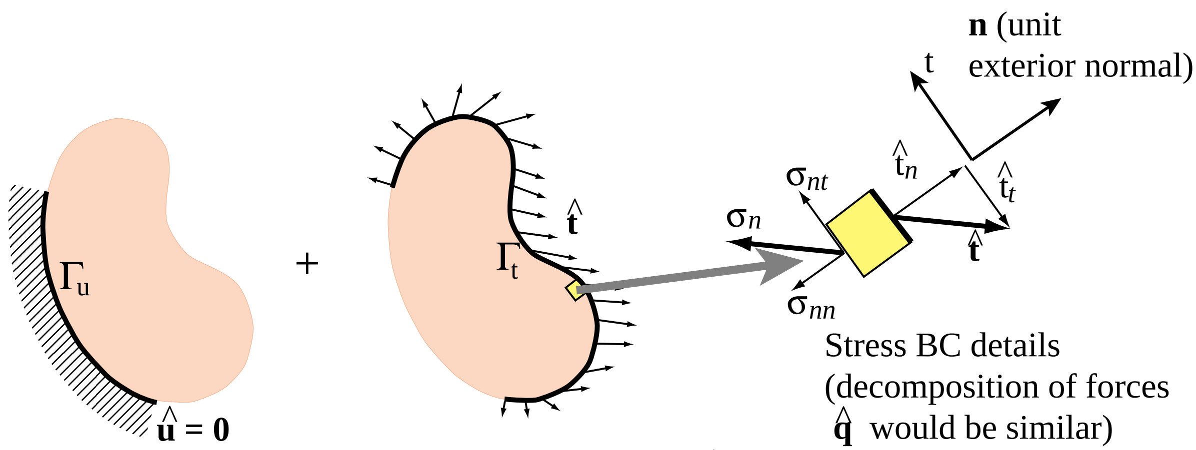
Notation for Stresses, Strains, Forces, Displacements



Inplane Forces are Obtained by Stress Integration Through Thickness



Plane Stress Boundary Conditions



Boundary displacements $\hat{\mathbf{u}}$ are prescribed on Γ_u (figure depicts fixity condition)

Boundary tractions $\hat{\mathbf{t}}$ or boundary forces $\hat{\mathbf{q}}$ are prescribed on Γ_t

The Plane Stress Problem

Given:

geometry
material properties
wall fabrication (thickness only for homogeneous plates)
applied body forces
boundary conditions:
- ***prescribed boundary forces or tractions***
- ***prescribed displacements***

Find:

inplane displacements
inplane strains
inplane stresses and/or internal forces

Matrix Notation for Internal Fields

$$\mathbf{u}(x, y) = \begin{bmatrix} u_x(x, y) \\ u_y(x, y) \end{bmatrix}$$

displacements

$$\mathbf{e}(x, y) = \begin{bmatrix} e_{xx}(x, y) \\ e_{yy}(x, y) \\ 2e_{xy}(x, y) \end{bmatrix}$$

strains (factor of 2 in e_{xy} simplifies "energy dot products")

$$\boldsymbol{\sigma}(x, y) = \begin{bmatrix} \sigma_{xx}(x, y) \\ \sigma_{yy}(x, y) \\ \sigma_{xy}(x, y) \end{bmatrix}$$

stresses

Governing Plane Stress Elasticity Equations in Matrix Form

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

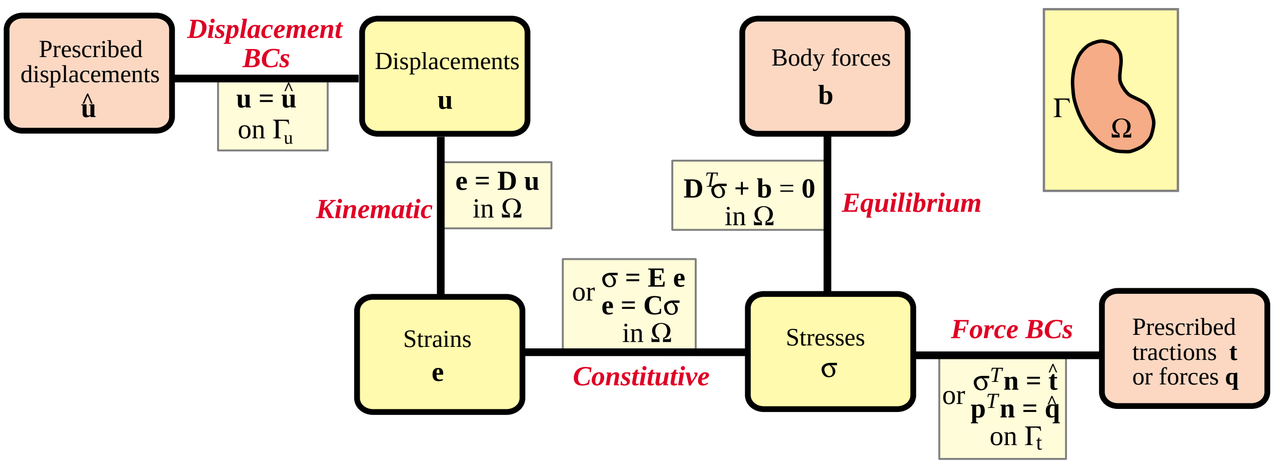
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

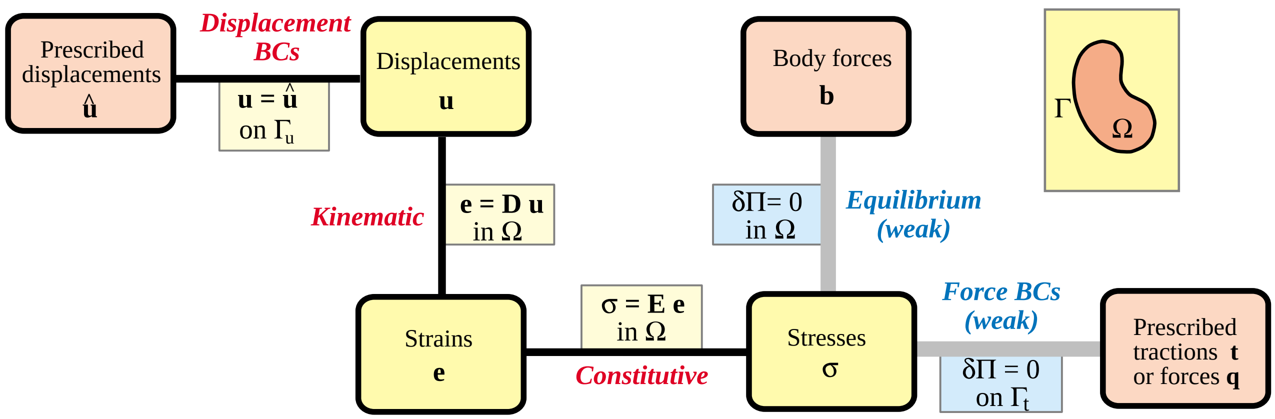
or

$$\mathbf{e} = \mathbf{D}\mathbf{u} \qquad \boldsymbol{\sigma} = \mathbf{E}\mathbf{e} \qquad \mathbf{D}^T\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$$

Strong-Form Tonti Diagram of Plane Stress Governing Equations



TPE-Based Weak Form Diagram of Plane Stress Governing Equations



Total Potential Energy of Plate in Plane Stress

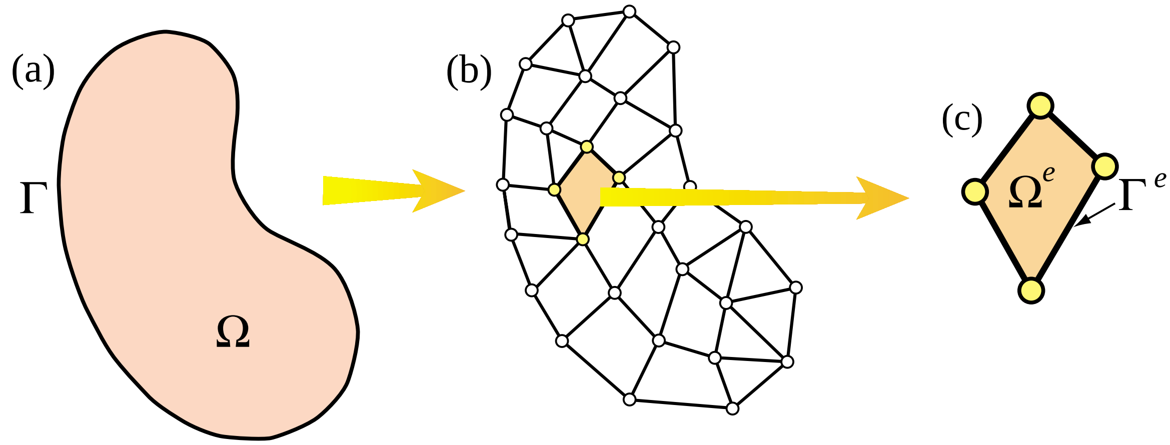
$$\Pi = U - W$$

$$U = \tfrac{1}{2}\int_\Omega h\,\boldsymbol{\sigma}^T \mathbf{e}\,d\Omega = \tfrac{1}{2}\int_\Omega h\,\mathbf{e}^T \mathbf{E}\mathbf{e}\,d\Omega$$

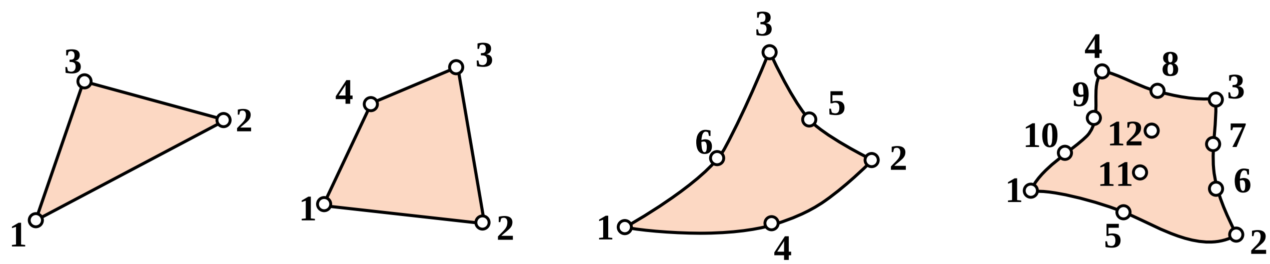
$$W = \int_\Omega h\,\mathbf{u}^T \mathbf{b}\,d\Omega + \int_{\Gamma_t} h\,\mathbf{u}^T \hat{\mathbf{t}}\,d\Gamma$$

body forces ***boundary tractions***

Discretization into Plane Stress Finite Elements

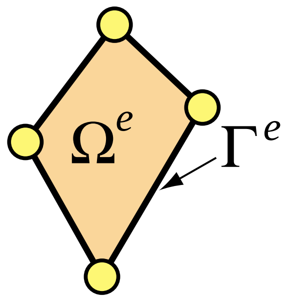


Plane Stress Element Geometries and Node Configurations



$n = 3$ $n = 4$ $n = 6$ $n = 12$

Total Potential Energy of Plane Stress Element

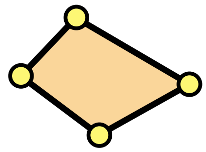


$$\Pi^{e\,\gamma} = U^e - W^e$$

$$U^e = \tfrac{1}{2}\int_{\Omega^e} h\,\boldsymbol{\sigma}^T \mathbf{e}\, d\Omega^e = \tfrac{1}{2}\int_{\Omega^e} h\,\mathbf{e}^T \mathbf{E}\,\mathbf{e}\, d\Omega^e$$

$$W^e = \int_{\Omega^e} h\,\mathbf{u}^T \mathbf{b}\, d\Omega^e + \int_{\Gamma^e} h\,\mathbf{u}^T \mathbf{t}\, d\Gamma^e$$

Constructing a Displacement Assumed Element



n nodes, n=4 in figure

Node displacement vector

$$\mathbf{u}^e = [\, u_{x1} \quad u_{y1} \quad u_{x2} \quad \ldots \quad u_{xn} \quad u_{yn} \,]^T$$

Displacement interpolation over element

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \ldots & N_n^e & 0 \\ 0 & N_1^e & 0 & N_2^e & \ldots & 0 & N_n^e \end{bmatrix} \mathbf{u}^e$$

$$= \mathbf{N} \;\; \mathbf{u}^e$$

N is called the shape function matrix
It has order 2 x 2n

Element Construction (cont'd)

Differentiate the displacement interpolation wrt *x,y*
to get the strain-displacement relation

$$\mathbf{e}(x,y) = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \dots & \frac{\partial N_n^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \dots & 0 & \frac{\partial N_n^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \dots & \frac{\partial N_n^e}{\partial y} & \frac{\partial N_n^e}{\partial x} \end{bmatrix} \mathbf{u}^e = \mathbf{B}\,\mathbf{u}^e$$

B is called the strain-displacement matrix
It has order 3 x **2***n*

Element Construction (cont'd)

Element total potential energy

$$\Pi^e = \tfrac{1}{2}\,\mathbf{u}^{e\,T}\mathbf{K}^e\,\mathbf{u}^e - \mathbf{u}^{e\,T}\mathbf{f}^e$$

Element stiffness matrix

$$\mathbf{K}^e = \int_{\Omega^e} h\,\mathbf{B}^T\mathbf{E}\,\mathbf{B}\,d\Omega^e$$

Consistent node force vector

$$\mathbf{f}^e = \int_{\Omega^e} h\,\mathbf{N}^T\mathbf{b}\,d\Omega^e + \int_{\Gamma^e} h\,\mathbf{N}^T\hat{\mathbf{t}}\,d\Gamma^e$$

due to: body force **due to: surface tractions**

Requirements on Finite Element Shape Functions

Interpolation Condition

N_i **takes on value 1 at node** i**, 0 at all other nodes**

Continuity (intra- and inter-element) and
Completeness Conditions

are covered later in the course (Chs. 18-19)