

Homework Exercises for Chapter 12 - Variational Formulation of Plane Beam Element Solutions

EXERCISE 12.1 A *Mathematica* script for \mathbf{f}^e by analytical integration is shown in Figure E12.1.

```

ClearAll[EI,EI1,EI2,Le,xi,l]; Le=l;
Be={{6*xi,(3*xi-1)*Le,-6*xi,(3*xi+1)*Le}};
EI=EI1*(1-xi)/2+EI2*(1+xi)/2;
Ke=1/(2*Le^3)*Integrate[EI*Transpose[Be].Be,{xi,-1,1}];
Ke=Simplify[Ke];
Print["Ke for variable xsec beam:\n", Ke//MatrixForm];
ClearAll[EI]; Ke=Simplify[Ke/.{EI2->EI,EI2->EI}];
Print["Ke for EI1=EI2=EI is ", Ke//MatrixForm];

```

Ke for variable xsec beam:

$$\begin{pmatrix} \frac{6(EI_1+EI_2)}{\ell^3} & \frac{2(2EI_1+EI_2)}{\ell^2} & -\frac{6(EI_1+EI_2)}{\ell^3} & \frac{2(EI_1+2EI_2)}{\ell^2} \\ \frac{2(2EI_1+EI_2)}{\ell^2} & \frac{3EI_1+EI_2}{\ell} & -\frac{2(2EI_1+EI_2)}{\ell^2} & \frac{EI_1+EI_2}{\ell} \\ -\frac{6(EI_1+EI_2)}{\ell^3} & -\frac{2(2EI_1+EI_2)}{\ell^2} & \frac{6(EI_1+EI_2)}{\ell^3} & -\frac{2(EI_1+2EI_2)}{\ell^2} \\ \frac{2(EI_1+2EI_2)}{\ell^2} & \frac{EI_1+EI_2}{\ell} & -\frac{2(EI_1+2EI_2)}{\ell^2} & \frac{EI_1+3EI_2}{\ell} \end{pmatrix}$$

FIGURE E12.1. Script to solve Exercise 12.1

Transcribing the result:

$$\mathbf{K}^e = \frac{1}{\ell^3} \begin{bmatrix} 6(EI_1 + EI_2) & 2(2EI_1 + EI_2)\ell & -6(EI_1 + EI_2) & 2(EI_1 + EI_2)\ell \\ & (3EI_1 + EI_2)\ell^2 & -2(2EI_1 + EI_2)\ell & (EI_1 + EI_2)\ell^2 \\ \text{symm} & & 6(EI_1 + EI_2) & -2(2EI_1 + 2EI_2)\ell \\ & & & (EI_1 + 2EI_2)\ell^2 \end{bmatrix}. \quad (\text{E12.10})$$

The print output of the check $EI_1 = EI_2 = EI$ is omitted to save space, but it reproduces the matrix (12.20).

EXERCISE 12.2 A *Mathematica* script for \mathbf{f}^e by analytical integration is shown in Figure E12.2.

```

ClearAll[q,q1,q2,xi,l]; Le=l;
Ne={2*(1-xi)^2*(2+xi),(1-xi)^2*(1+xi)*Le,
     2*(1+xi)^2*(2-xi),-(1+xi)^2*(1-xi)*Le}/8;
q=q1*(1-xi)/2+q2*(1+xi)/2;
fe=Simplify[(Le/2)*Integrate[q*Ne,{xi,-1,1}]];
Print["fe^T for lin varying load q:\n",fe];
ClearAll[q]; fe=Simplify[fe/.{q1->q,q2->q}];
Print["check for q1=q2=q: ",fe];

```

fe^T for lin varying load q:

$$\left\{ \frac{1}{20}(7q_1 + 3q_2)\ell \quad \frac{1}{60}(3q_1 + 2q_2)\ell^2 \quad \frac{1}{20}(3q_1 + 7q_2)\ell \quad -\frac{1}{60}(2q_1 + 3q_2)\ell^2 \right\}$$

FIGURE E12.2. Script to solve Exercise 12.2

Transcribing the result:

$$\mathbf{f}^e = \frac{\ell}{60} [3(7q_1 + 3q_2) \quad \ell(3q_1 + 2q_2) \quad 3(3q_1 + 7q_2) \quad \ell(2q_1 + 3q_2)]^T. \quad (\text{E12.11})$$

The output of the check $q_i = q_j = q$ is omitted to save space, but it does reproduce (12.21).

EXERCISE 12.3 Following the hint, the shape function matrix is evaluated at $x = a$, or $\xi = 2a/\ell - 1$:

$$\mathbf{f}^e = P \mathbf{N}^T \Big|_{\xi \rightarrow 2a/\ell - 1} = \frac{P}{\ell^2} \left[\frac{(a - \ell)^2(2a + \ell)}{\ell} \quad a(a - \ell)^2 \quad -\frac{a^2(2a - 3\ell)}{\ell} \quad a^2(a - \ell) \right]^T. \quad (\text{E12.12})$$

If $a = 0$ and $a = \ell$ the force vector reduces to $[P \ 0 \ 0 \ 0]^T$ and $[0 \ 0 \ P \ 0]^T$, respectively, as expected.

EXERCISE 12.4 Figure E12.3 shows a *Mathematica* script that implements (E12.5) for a linearly varying distributed moment m .

```
ClearAll[m, Le, xi, l]; Le=l;
Ne={2*(1-xi)^2*(2+xi), (1-xi)^2*(1+xi)*Le,
     2*(1+xi)^2*(2-xi), -(1+xi)^2*(1-xi)*Le}/8;
m=m1*(1-xi)/2+m2*(1+xi)/2;
fe=Integrate[m*D[Ne, xi], {xi, -1, 1}];
fe=Simplify[fe]; Print["fe: ", fe];
```

fe: $\{-\frac{1}{2}(m_1 + m_2) \quad \frac{1}{12}(m_1 - m_2)\ell \quad \frac{1}{2}(m_1 + m_2) \quad -\frac{1}{12}(m_1 - m_2)\ell\}$

FIGURE E12.3. Script to solve Exercise 12.4

This gives

$$\mathbf{f}^e = \frac{1}{12} [-6(m_1 + m_2) \quad (m_1 - m_2)\ell \quad 6(m_1 + m_2) \quad (m_1 - m_2)\ell]^T. \quad (\text{E12.13})$$

There is a shortcut for uniform m ; that is, $m_1 = m_2 = m$. Moving m out of the integral (E12.5) gives

$$\mathbf{f}^e = m \int_{-1}^1 \mathbf{N}_\xi^T d\xi = m \mathbf{N}^T \Big|_{\xi=-1}^{\xi=1} = m [-1 \quad 0 \quad 1 \quad 0]^T. \quad (\text{E12.14})$$

EXERCISE 12.5 Replacing $m(x) = C \delta(a)$ in the integral (E12.5) reduces it to

$$\mathbf{f}^e = C \frac{d\mathbf{N}^T}{dx} \Big|_{x \rightarrow a} = \frac{2C}{\ell} \frac{d\mathbf{N}^T}{d\xi} \Big|_{\xi \rightarrow 2a/\ell - 1} = \frac{C}{\ell^2} \left[\frac{6a(a - \ell)}{\ell} \quad (\ell - 3a)(\ell - a) \quad \frac{6a(\ell - a)}{\ell} \quad a(3a - 2\ell) \right]^T. \quad (\text{E12.15})$$

If $a = 0$ and $a = \ell$, \mathbf{f}^e reduces to $[0 \ C \ 0 \ 0]^T$ and $[0 \ 0 \ 0 \ C]^T$, respectively, as expected. If $a = \frac{1}{2}\ell$ we get $C [-3/(2\ell) \quad -1/4 \quad 3/(2\ell) \quad -1/4]^T$, which is not that obvious.

EXERCISE 12.6 The results of this exercise are collected on Table 12.1. The rules with 1, 2 and 3 points fail for monomial degrees 2, 4 and 6, respectively, and are exact for degrees up to one less. The property is extendible to any rule order: a Gauss rule with p points integrates exactly polynomials of degree $2p - 1$ or less, as proven in texts on numerical analysis. Note that since the rules are symmetric about $x = 0$ they are exact for any *odd* function about $x = 0$ because the integral of any such function from -1 to 1 is zero. Consequently odd-degree monomials, such as ξ or ξ^3 , are integrated exactly by *all* rules.

Table 12.1. Results From Exercise 12.6

	Monomial degree						
	0	1	2	3	4	5	6
Exact	2	0	$\frac{2}{3}$	0	$\frac{2}{5}$	0	$\frac{2}{7}$
One-point rule (E12.6)	2	0	0				
Two-point rule (E12.7)	2	0	$\frac{2}{3}$	0	$\frac{2}{9}$		
Three-point rule (E12.8)	2	0	$\frac{2}{3}$	0	$\frac{2}{5}$	0	$\frac{6}{25}$

EXERCISE 12.7 A *Mathematica* script to compute \mathbf{K}^e by two-point Gauss rule is shown in Figure E12.4. This result reproduces exactly (E12.10) because the polynomials being integrated for linearly varying EI are cubic in ξ (\mathbf{B} , \mathbf{B}^T and EI are linear in ξ). From Exercise 12.6, the 2-point Gauss rule integrates exactly polynomials of order up to and including 3.

```

ClearAll[EI,EI1,EI2,ξ,ℓ]; Le=ℓ;
Be={{6*ξ,(3*ξ-1)*Le,-6*ξ,(3*ξ+1)*Le}};
EI=EI1*(1-ξ)/2+EI2*(1+ξ)/2;
Ke1=(1/(2*Le^3))*(EI*Transpose[Be].Be)/.ξ->Sqrt[3]/3;
Ke2=(1/(2*Le^3))*(EI*Transpose[Be].Be)/.ξ->-Sqrt[3]/3;
Ke=Simplify[Ke1+Ke2];
Print["2-Gauss-pt Ke for var xsec beam:\n ",Ke//MatrixForm];

```

2-Gauss-pt Ke for var xsec beam:

$$\begin{pmatrix} \frac{6(EI1+EI2)}{\ell^3} & \frac{2(2EI1+EI2)}{\ell^2} & -\frac{6(EI1+EI2)}{\ell^3} & \frac{2(EI1+2EI2)}{\ell^2} \\ \frac{2(2EI1+EI2)}{\ell^2} & \frac{3EI1+EI2}{\ell} & -\frac{2(2EI1+EI2)}{\ell^2} & \frac{EI1+EI2}{\ell} \\ -\frac{6(EI1+EI2)}{\ell^3} & -\frac{2(2EI1+EI2)}{\ell^2} & \frac{6(EI1+EI2)}{\ell^3} & -\frac{2(EI1+2EI2)}{\ell^2} \\ \frac{2(EI1+2EI2)}{\ell^2} & \frac{EI1+EI2}{\ell} & -\frac{2(EI1+2EI2)}{\ell^2} & \frac{EI1+3EI2}{\ell} \end{pmatrix}$$

FIGURE E12.4. Script to solve Exercise 12.7

EXERCISE 12.8 A *Mathematica* script to compute \mathbf{K}^e by two-point Gauss rule is shown in Figure E12.5.

```

ClearAll[q,q1,q2,ξ,ℓ]; Le=ℓ;
Ne={2*(1-ξ)^2*(2+ξ),(1-ξ)^2*(1+ξ)*Le,
     2*(1+ξ)^2*(2-ξ),-(1+ξ)^2*(1-ξ)*Le}/8;
q=q1*(1-ξ)/2+q2*(1+ξ)/2;
fe1=(Le/2)*(Ne*q)/.ξ->Sqrt[3]/3;
fe2=(Le/2)*(Ne*q)/.ξ->-Sqrt[3]/3; fe=Simplify[fe1+fe2];
Print["2-Gauss-pt fe^T for var load q:\n ",fe];

```

2-pt Gauss for var load q:

$$\left\{ \frac{1}{36}(13q1+5q2)\ell, \frac{1}{36}(2q1+q2)\ell^2, \frac{1}{36}(5q1+13q2)\ell, -\frac{1}{36}(q1+2q2)\ell^2 \right\}$$

FIGURE E12.5. Script to solve Exercise 12.8

This result is different from (E12.11) because the polynomials being integrated for linearly varying q are quartic in ξ , and the 2-point Gauss rule is not exact for that order. But for uniform q the results would be the same.

EXERCISE 12.9 The derivation is presented in Przemieniecki [140], §5.6.

EXERCISE 12.10 The FEM result is $v_2 = qL^4(1 - 4\alpha^2)^2/(96EI)$. The ratio to the exact solution is $r = 4(1 - 4\alpha^2)/(5 - 4\alpha^2)$. For $\alpha = 0$ the ratio is $4/5$ and the FEM result is 20% in error. As α grows the error increases, for example if $\alpha = 1/4$, $r = 12/19 = 0.6317$ and the error is over 36%.