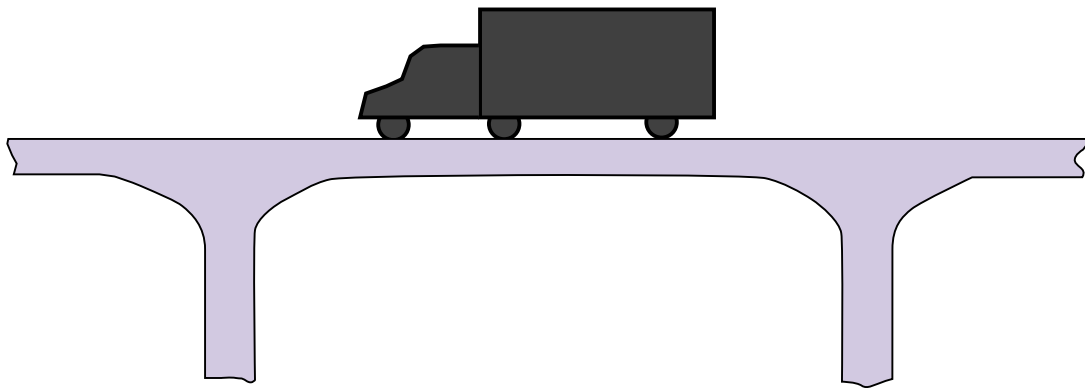


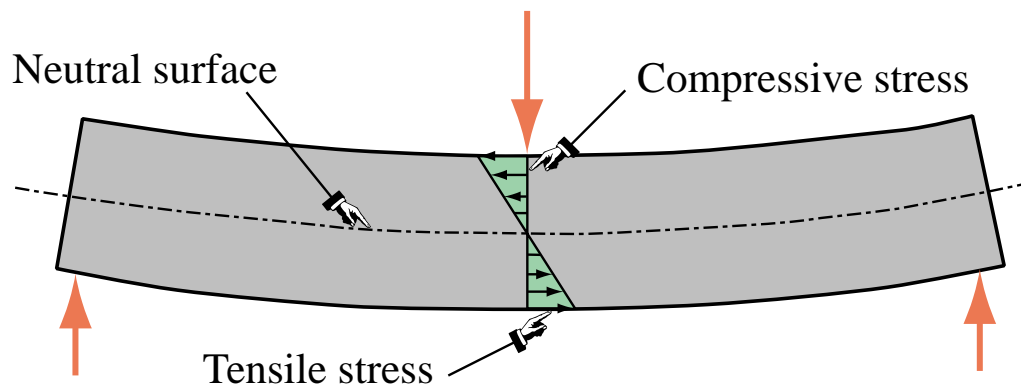
12

Variational Formulation of Plane Beam Element

**A Beam is a Structural Member Designed
to Resist Primarily Transverse Loads**



Transverse Loads are Transported to Supports by Flexural Action



Beam Configuration

Spatial (*General Beams*)

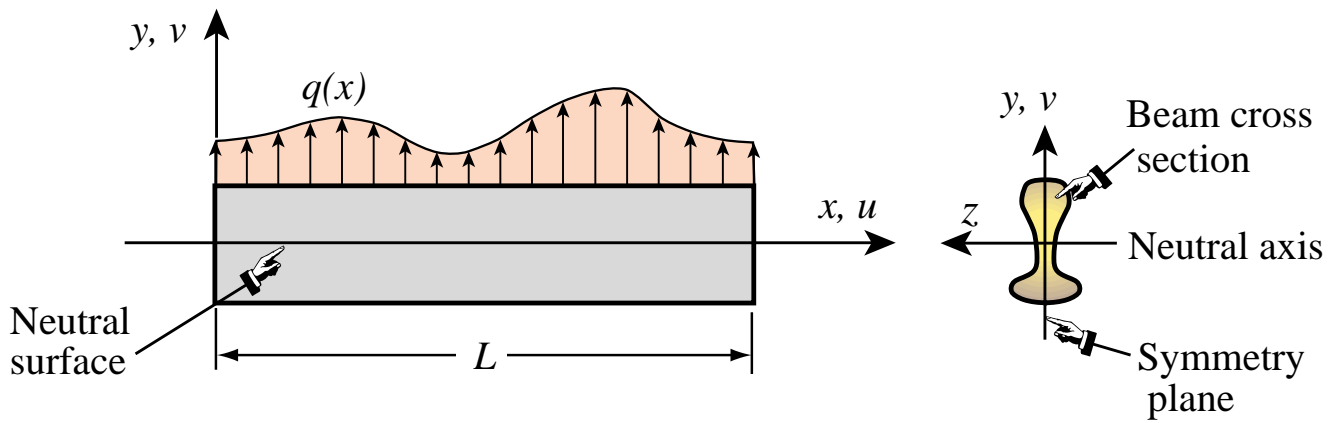
Plane (*This Chapter*)

Beam Models

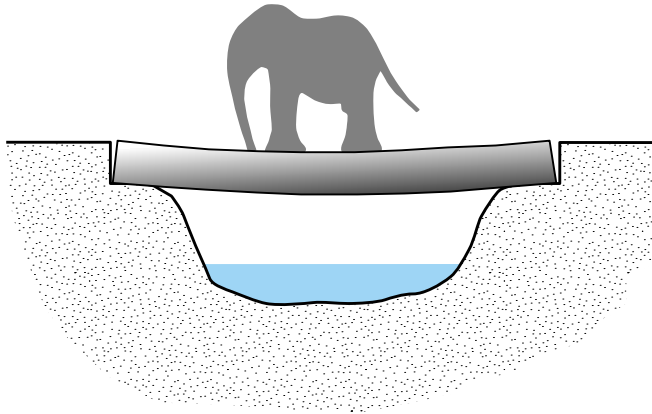
Bernoulli-Euler

Timoshenko (*more advanced topic: described in
Chapter 13 but not covered in course*)

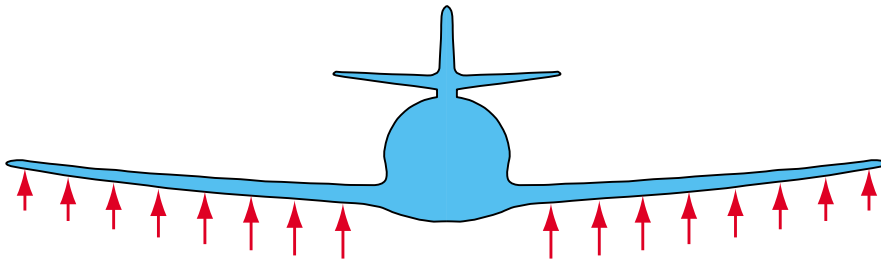
Plane Beam Terminology



Common Support Conditions



Simply Supported



Cantilever

Basic Relations for **Bernoulli-Euler** Model of Plane Beam

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} -y \frac{\partial v(x)}{\partial x} \\ v(x) \end{bmatrix} = \begin{bmatrix} -yv' \\ v(x) \end{bmatrix} = \begin{bmatrix} -y\theta \\ v(x) \end{bmatrix}$$

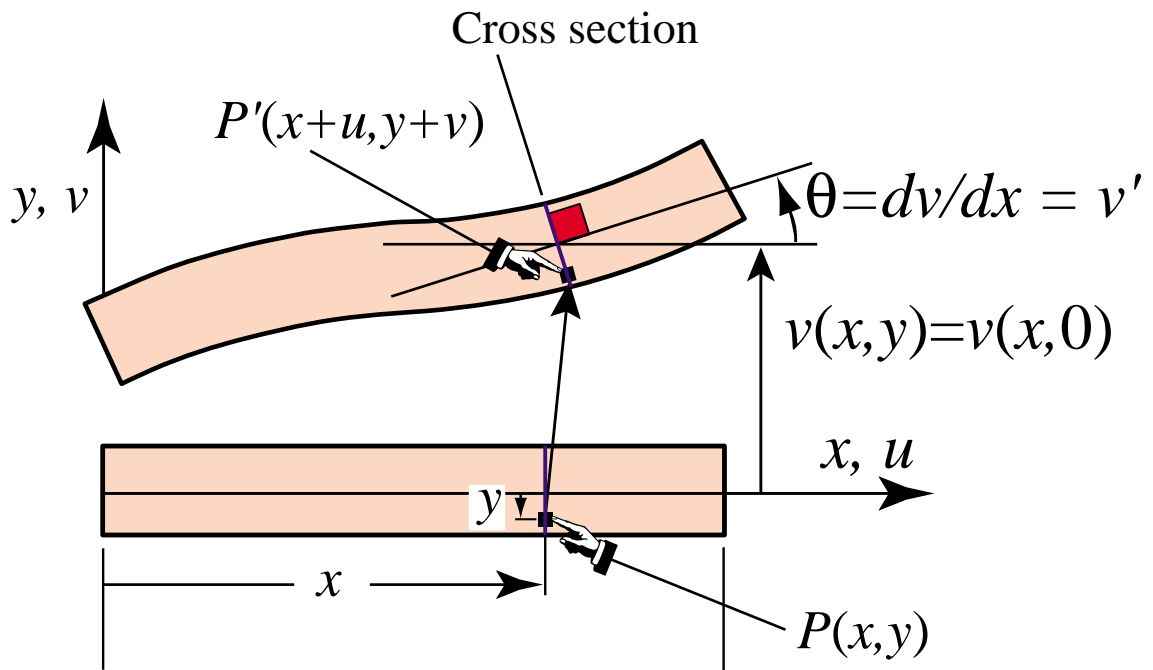
$$e = \frac{\partial u}{\partial x} = -y \frac{\partial^2 v}{\partial x^2} = -y \frac{d^2 v}{dx^2} = -y\kappa$$

$$\sigma = Ee = -Ey \frac{d^2 v}{dx^2} = -Ey\kappa$$

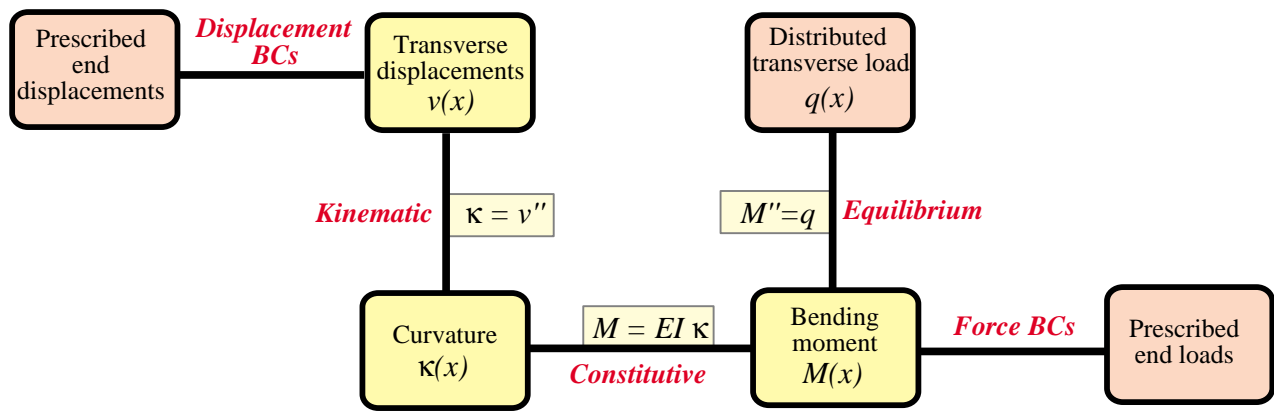
$$M = EI\kappa$$

Plus equilibrium equation $M'' = q$ (replaced by MPE principle in FEM)

Kinematics of Bernoulli-Euler Beam



Tonti Diagram for Bernoulli-Euler Model of Plane Beam (Strong Form)



Total Potential Energy of Beam Member

$$\Pi = U - W$$

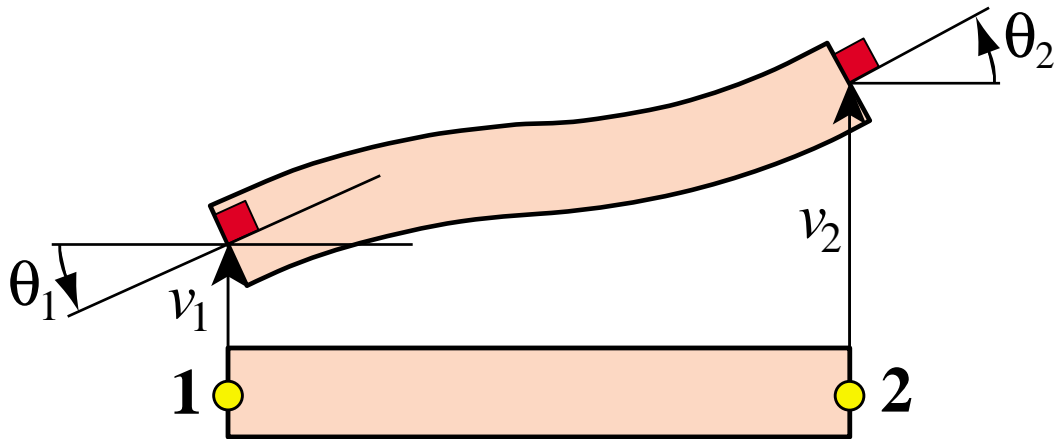
Internal External

$$U = \frac{1}{2} \int_V \sigma_{xx} e_{xx} dV = \frac{1}{2} \int_0^L M \kappa dx = \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx$$

$$= \frac{1}{2} \int_0^L EI \kappa^2 dx \quad \textit{Internal energy due to bending}$$

$$W = \int_0^L qv dx \quad \textit{External energy due to transverse load } q$$

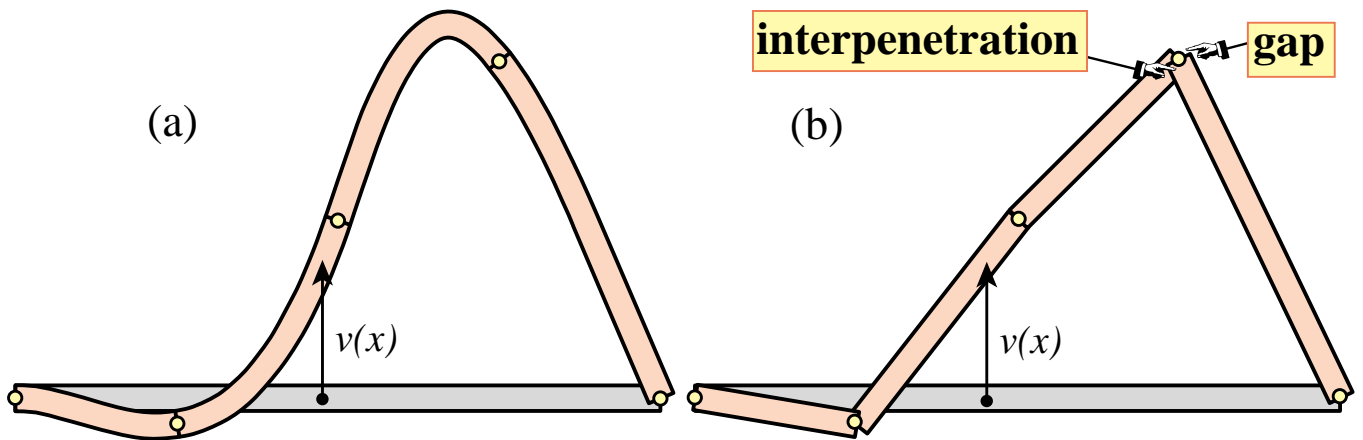
Degrees of Freedom of Plane Beam Element



$$\mathbf{u}^e = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

Physical Interpretation of C^1 Continuity

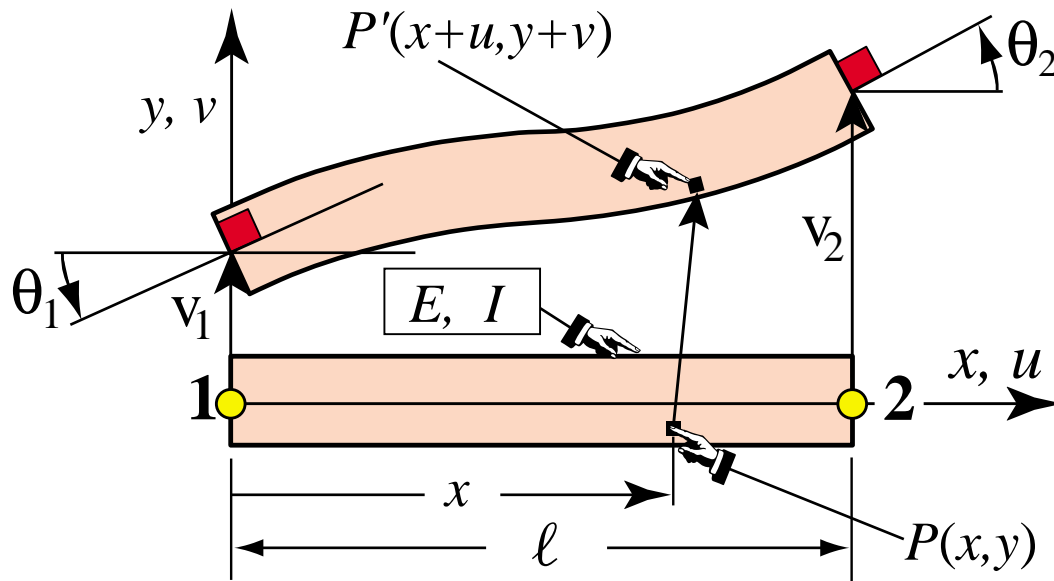
(deflections grossly exaggerated for visibility)



Piecewise cubic interpolation provides required C^1 continuity

Piecewise linear interpolation gives unacceptable C^0 continuity

Bernoulli-Euler Kinematics of Plane Beam Element



Shape Functions in Terms of Natural Coordinate ξ

$$v^e = [N_{v1}^e \quad N_{\theta1}^e \quad N_{v2}^e \quad N_{\theta2}^e] \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \mathbf{N} \mathbf{u}^e$$

Introduce the natural (isoparametric) coordinate

$$\xi = \frac{2x}{\ell} - 1$$

$$N_{v1}^e(\xi) = \frac{1}{4} (1 - \xi)^2 (2 + \xi)$$

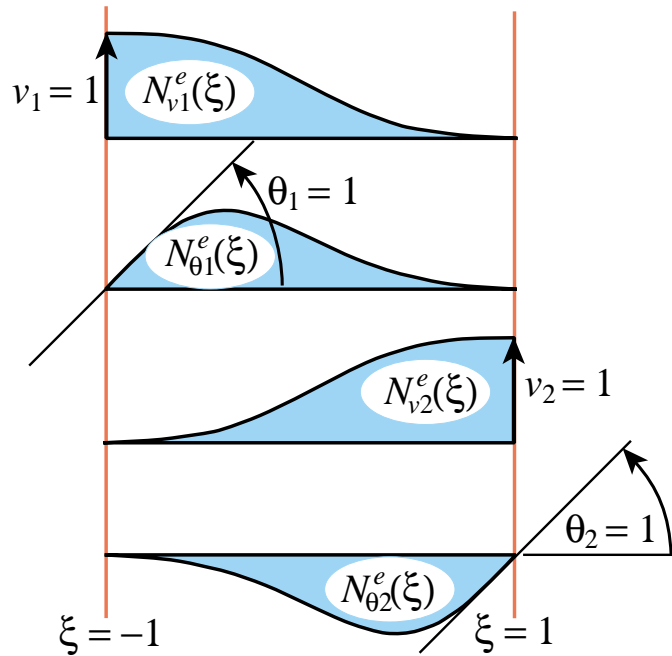
$$N_{v2}^e(\xi) = \frac{1}{4} (1 + \xi)^2 (2 - \xi)$$

$$N_{\theta1}^e(\xi) = \frac{1}{8} \ell (1 - \xi)^2 (1 + \xi)$$

$$N_{\theta2}^e(\xi) = -\frac{1}{8} \ell (1 + \xi)^2 (1 - \xi)$$

Plots on next slide

Beam Element Shape Function Plots



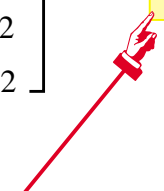
$$N_{v1}^e(\xi) = \frac{1}{4} (1 - \xi)^2 (2 + \xi)$$

$$N_{\theta1}^e(\xi) = \frac{1}{8} l (1 - \xi)^2 (1 + \xi)$$

$$N_{v2}^e(\xi) = \frac{1}{4} (1 + \xi)^2 (2 - \xi)$$

$$N_{\theta2}^e(\xi) = -\frac{1}{8} l (1 + \xi)^2 (1 - \xi)$$

Getting Curvatures from the Displacement Interpolation

$$\begin{aligned}
 \kappa &= \frac{d^2 v(x)}{dx^2} = \frac{d^2 \mathbf{N}}{dx^2} \mathbf{u}^e + \mathbf{N} \frac{d^2 \mathbf{u}^e}{dx^2} = \frac{d^2 \mathbf{N}}{dx^2} \mathbf{u}^e \\
 &= \begin{bmatrix} \frac{d^2 N_{v1}^e}{dx^2} & \frac{d^2 N_{\theta1}^e}{dx^2} & \frac{d^2 N_{v2}^e}{dx^2} & \frac{d^2 N_{\theta2}^e}{dx^2} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \stackrel{\text{def}}{=} \mathbf{B} \mathbf{u}^e
 \end{aligned}$$


B is the 1 x 4 curvature-displacement matrix constructed in next slide

Curvatures from Displacements (Cont'd)

To construct **B**, apply the chain rule

$$\frac{df(x)}{dx} = \frac{df(\xi)}{d\xi} \frac{d\xi}{dx} = \frac{2}{l} \frac{df(\xi)}{d\xi}$$

$$\frac{d^2f(x)}{dx^2} = \frac{d}{dx} \left(\frac{2}{l} \frac{df(\xi)}{d\xi} \right) = \frac{2}{l} \frac{d}{dx} \left(\frac{df(\xi)}{d\xi} \right) = \frac{4}{l^2} \frac{d^2f(\xi)}{d\xi^2}$$

to differentiate the shape functions. Result is

$$\mathbf{B} = \frac{1}{l} \left[6 \frac{\xi}{l} \quad 3\xi - 1 \quad -6 \frac{\xi}{l} \quad 3\xi + 1 \right]$$

Element Stiffness and Consistent Node Forces

Varying the element TPE

$$\Pi^e = \frac{1}{2} \mathbf{u}^{eT} \mathbf{K}^e \mathbf{u}^e - \mathbf{u}^{eT} \mathbf{f}^e$$

we get

$$\mathbf{K}^e = \int_0^\ell EI \mathbf{B}^T \mathbf{B} dx = \int_{-1}^1 EI \mathbf{B}^T \mathbf{B} \frac{1}{2} \ell d\xi$$

$$\mathbf{f}^e = \int_0^\ell \mathbf{N}^T q dx = \int_{-1}^1 \mathbf{N}^T q \frac{1}{2} \ell d\xi$$

Analytical Computation of Prismatic Beam Element Stiffness

("prismatic" means constant EI)

$$\mathbf{K}^e = \frac{EI}{2l^3} \int_{-1}^1 \begin{bmatrix} 36\xi^2 & 6\xi(3\xi - 1) & -36\xi^2 & 6\xi(3\xi + 1)l \\ & (3\xi - 1)^2 l^2 & -6\xi(3\xi - 1)l & (9\xi^2 - 1)l^2 \\ \text{symm} & & 36\xi^2 & -6\xi(3\xi + 1)l \\ & & & (3\xi + 1)^2 l^2 \end{bmatrix} d\xi$$

$$= \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ \text{symm} & & 12 & -6l \\ & & & 4l^2 \end{bmatrix}$$

Mathematica Script for Symbolic Computation of Prismatic Plane Beam Element Stiffness

```

ClearAll[EI,l,ξ];
B={{6*ξ,(3*ξ-1)*l,-6*ξ,(3*ξ+1)*l}}/l^2;
Ke=(EI*l/2)*Integrate[Transpose[B].B,{ξ,-1,1}];
Ke=Simplify[Ke];
Print["Ke for prismatic beam:"];
Print[Ke//MatrixForm];

```

Ke for prismatic beam:

$$\begin{pmatrix} \frac{12 EI}{l^3} & \frac{6 EI}{l^2} & -\frac{12 EI}{l^3} & \frac{6 EI}{l^2} \\ \frac{6 EI}{l^2} & \frac{4 EI}{l} & -\frac{6 EI}{l^2} & \frac{2 EI}{l} \\ -\frac{12 EI}{l^3} & -\frac{6 EI}{l^2} & \frac{12 EI}{l^3} & -\frac{6 EI}{l^2} \\ \frac{6 EI}{l^2} & \frac{2 EI}{l} & -\frac{6 EI}{l^2} & \frac{4 EI}{l} \end{pmatrix}$$

Corroborates the result from hand integration.

Analytical Computation of Consistent Node Force Vector for **Uniform Load** q

$$\mathbf{f}^e = \frac{1}{2}q\ell \int_{-1}^1 \mathbf{N} d\xi = \frac{1}{2}q\ell \int_{-1}^1 \begin{bmatrix} \frac{1}{4}(1-\xi)^2(2+\xi) \\ \frac{1}{8}\ell(1-\xi)^2(1+\xi) \\ \frac{1}{4}(1+\xi)^2(2-\xi) \\ -\frac{1}{8}\ell(1+\xi)^2(1-\xi) \end{bmatrix} d\xi$$

$$= q\ell \begin{bmatrix} \frac{1}{2} \\ \frac{1}{12}\ell \\ \frac{1}{2} \\ -\frac{1}{12}\ell \end{bmatrix}$$

"fixed end moments"

Mathematica Script for Computation of Consistent Node Force Vector for Uniform q

```
ClearAll[q,l,ξ]
Ne={{2*(1-ξ)^2*(2+ξ), (1-ξ)^2*(1+ξ)*l,
      2*(1+ξ)^2*(2-ξ), -(1+ξ)^2*(1-ξ)*l}}/8;
fe=(q*l/2)*Integrate[Ne,{ξ,-1,1}]; fe=Simplify[fe];
Print["fe^T for uniform load q:"];
Print[fe//MatrixForm];
```

fe^T for uniform load q:

$$\left(\frac{lq}{2} \quad \frac{l^2q}{12} \quad \frac{lq}{2} \quad -\frac{l^2q}{12} \right)$$

Force vector printed as row vector to save space.