Homework Exercises for Chapter 11. Variational Formulation of Bar Element

Solutions

EXERCISE 11.1 If the function $u(x)$ is discontinuous at $x = a$, material gap or interpenetration at that cross section occurs. The first requirement rules that out at an arbitrary cross section. The second requirement does the same for supports.

EXERCISE 11.2 Only A in (11.28) is a function of x. Consequently

$$\mathbf{K}^e = \int_0^\ell \frac{EA}{\ell^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx = \frac{E}{\ell^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^\ell A(x)\, dx. \tag{E11.2}$$

But

$$\int_0^\ell A(x)\, dx = \ell \int_0^1 A(\zeta)\, d\zeta = \ell \int_0^1 \big[A_i(1-\zeta) + A_j\zeta\big]\, d\zeta = \ell\, \tfrac{1}{2}(A_i + A_j) = \ell \bar{A}, \tag{E11.3}$$

where $\bar{A} = \frac{1}{2}(A_i + A_j)$ is the average area. Therefore

$$\mathbf{K}^e = \frac{E\bar{A}}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{E11.4}$$

EXERCISE 11.3

$$\mathbf{f}^e = \int_0^\ell \rho g A \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = \rho g A \ell \int_0^1 \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} d\zeta = \rho g A \ell \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \tfrac{1}{2} \rho g A \ell \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{E11.5}$$

This is the same answer of element-by-element load lumping.

EXERCISE 11.4

$$\mathbf{f}^e = \int_0^\ell \rho g A \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = \rho g \ell \int_0^1 \begin{bmatrix} \big[A_i(1-\zeta) + A_j\zeta\big](1-\zeta) \\ \big[A_i(1-\zeta) + A_j\zeta\big]\zeta \end{bmatrix} d\zeta = \rho g \ell \begin{bmatrix} \frac{1}{3}A_i + \frac{1}{6}A_j \\ \frac{1}{6}A_i + \frac{1}{3}A_j \end{bmatrix}. \tag{E11.6}$$

Obviously if $A_i = A_j$ we recover (E11.5).

EXERCISE 11.5 Here is a *Mathematica* solution:

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ClearAll[Le,q,A,Ai,Aj,ρ,g];
ζ=x/Le; Ne={1-ζ,ζ}; A=Ai*(1-ζ)+Aj*ζ; q=ρ*A*ω^2*x;
fe=Integrate[q*Ne,{x,0,Le}];
fe=Simplify[fe]; Print["fe for centrifugal load q: ",fe];
ClearAll[A];
Print["fe check:",Simplify[fe/.{Ai->A,Aj->A}]];
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fe for centrifugal load q: $\{\frac{1}{12}(\mathrm{Ai}+\mathrm{Aj})\,\mathrm{Le}^2\rho\,\omega^2,\ \frac{1}{12}(\mathrm{Ai}+3\mathrm{Aj})\,\mathrm{Le}^2\rho\,\omega^2\}$

fe check: $\{\frac{1}{6}\mathrm{A}\,\mathrm{Le}^2\rho\,\omega^2,\ \frac{1}{3}\mathrm{A}\,\mathrm{Le}^2\rho\,\omega^2\}$

So the answer is $f^e_{xi} = \frac{1}{12}(A_i + A_j)\ell^2\rho\omega^2$, $f^e_{xj} = \frac{1}{12}(A_i + 3A_j)\ell^2\rho\omega^2$.

EXERCISE 11.6

$$\mathbf{f}^e = \int_0^\ell \begin{bmatrix} 1 - x/\ell \\ x/\ell \end{bmatrix} Q\,\delta(a)\,dx = Q \begin{bmatrix} 1-a \\ a \end{bmatrix}. \tag{E11.7}$$

in which the δ-function property $\int_{-\infty}^{\infty} g(x)\,\delta(a)\,dx = g(a)$, $g(x)$ being any continuous function, has been used.

EXERCISE 11.7 Three zero energy mechanisms occur because the bars can hinge without restraints about the midpoint nodes 4, 5 and 6. Removal would require messy MFC constraints. But the addition of these nodes is unnecessary since one element per member gives the exact answer if the loads are applied at the nodes, as discussed in §11.5.

EXERCISE 11.8 The consistent node force computed from (11.26) is $f_j = 2\ell(q/2! + \ell^2 q''/4! + \ell^4 q''''/6! + \ell^6 q''''''/8! + \ldots)$, where q and its derivatives are evaluated at $x = x_2$. Insert this into the RHS of (11.30). Differentiate both sides repeatedly with respect to x keeping only even derivatives up to a certain order. The configuration is illustrated here when keeping up to the eighth derivative of $u(x)$ and sixth of $q(x)$:

$$-2EA\ell \begin{bmatrix} \frac{1}{2!} & \frac{\ell^2}{4!} & \frac{\ell^4}{6!} & \frac{\ell^6}{8!} \\ 0 & \frac{1}{2!} & \frac{\ell^2}{4!} & \frac{\ell^4}{6!} \\ 0 & 0 & \frac{1}{2!} & \frac{\ell^2}{12} \\ 0 & 0 & 0 & \frac{1}{2!} \end{bmatrix} \begin{bmatrix} u''_j \\ u''''_j \\ u''''''_j \\ u''''''''_j \end{bmatrix} = \begin{bmatrix} f_j \\ f''_j \\ f''''_j \\ f''''''_j \end{bmatrix} = 2\ell \begin{bmatrix} \frac{1}{2!} & \frac{\ell^2}{4!} & \frac{\ell^4}{6!} & \frac{\ell^6}{8!} \\ 0 & \frac{1}{2!} & \frac{\ell^2}{4!} & \frac{\ell^4}{6!} \\ 0 & 0 & \frac{1}{2!} & \frac{\ell^2}{12} \\ 0 & 0 & 0 & \frac{1}{2!} \end{bmatrix} \begin{bmatrix} q_j \\ q''_j \\ q''''_j \\ q''''''_j \end{bmatrix} \tag{E11.8}$$

Cancelling the common Toeplitz coefficient matrix, which is possible since it is obviously nonsingular,[14] and the ℓ factor, one gets (from the first row) $EAu''_j + q_j = 0$ identically for any differentiation order. Consequently the FEM solution is nodally exact for any load, as load as the consistent force computation is used.

As usual the fastest route, if and when applicable, is to use the Laplace transform $\mathcal{L}$. Let s be the transform variable, with $\mathcal{L}(d^k F(x)/dx^k) \Rightarrow s^k \tilde{F}(s)$. Transforming (11.30) with the nodal force f_j found above yields $-(2EA/\ell)\big(\cosh(\ell s) - 1\big)\,\tilde{u}(s) = (2/\ell)\big(\cosh(\ell s) - 1\big)s^{-2}\,\tilde{q}(s)$, or $EA\,s^2\tilde{u}(s) + \tilde{q}(s) = 0$, with subscript j suppressed for brevity. Backtransforming gives $EAu''_j + q_j = 0$.

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[14] The determinant of a triangular matrix is the product of the diagonal entries, all of which are nonzero.