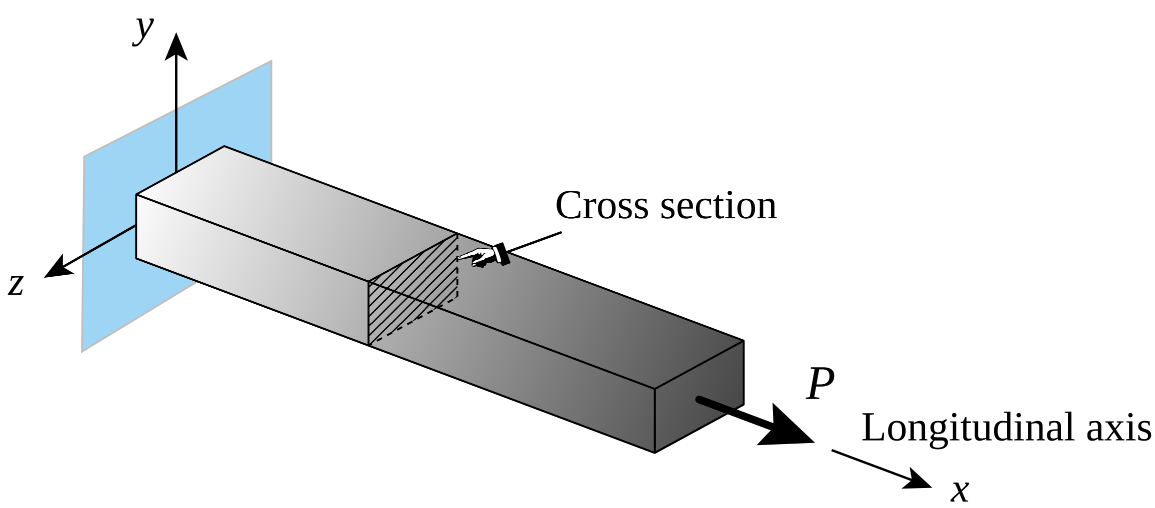
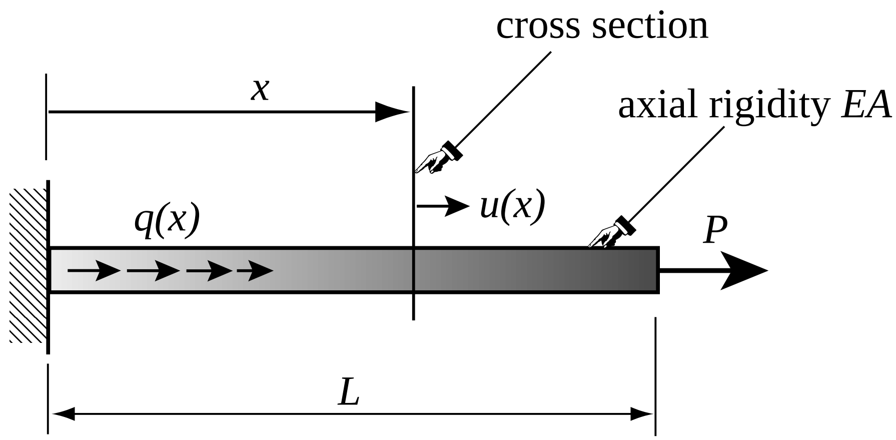
# 11
# Variational Formulation of Bar Element

# Bar Member - Variational Derivation



# Bar Member (cont'd)

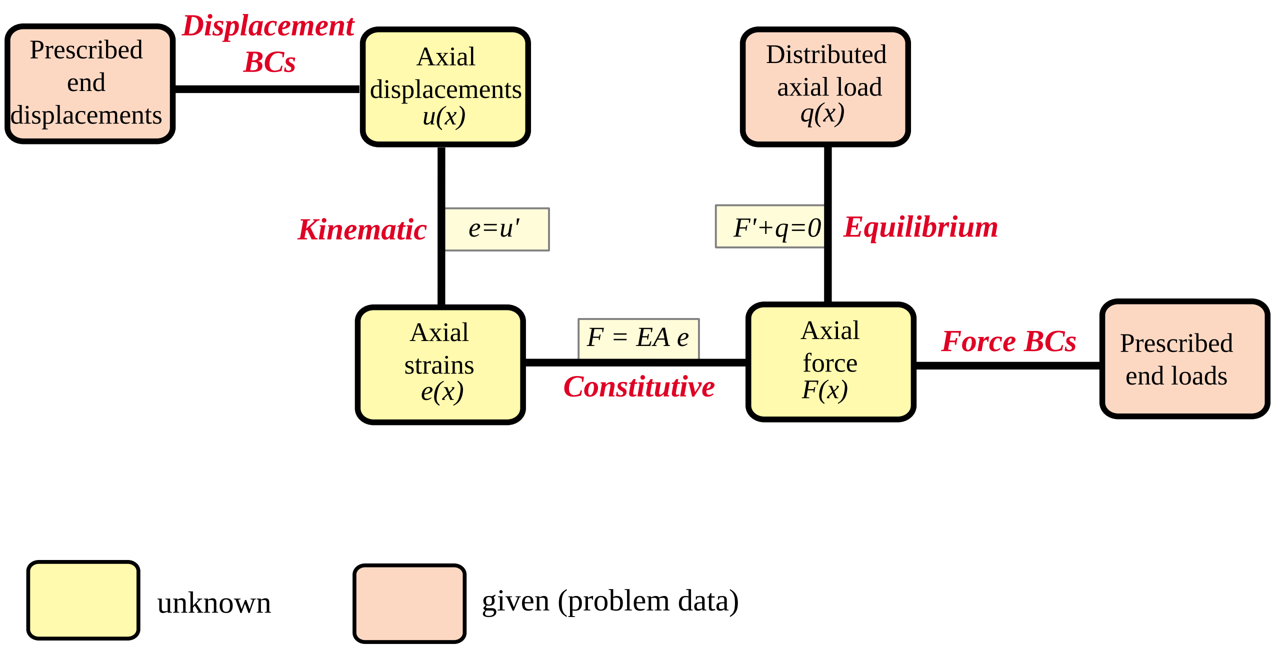


# The Bar Revisited - Notation

| *Quantity* | *Meaning* |
|---|---|
| $x$ | Longitudinal bar axis * |
| $(.)'$ | $d(.)/dx$ |
| $u(x)$ | Axial displacement |
| $q(x)$ | Distributed axial force, given per unit of bar length |
| $L$ | Total length of bar member |
| $E$ | Elastic modulus |
| $A$ | Cross section area, may vary with x |
| $EA$ | Axial rigidity |
| $e = du/dx = u'$ | Infinitesimal axial strain |
| $\sigma = E\,e = E\,u'$ | Axial stress |
| $F = A\,\sigma = EA\,e = EA\,u'$ | Internal axial force |
| $P$ | Prescribed end load |

* $x$ is used in this Chapter instead of $\bar{x}$ (as in Chapters 2-3) to simplify the notation

# Tonti Diagram of Governing Equations



# Potential Energy of the Bar Member
## (before discretization)

**Internal energy (= strain energy)**

$$U = \tfrac{1}{2}\int_0^L F e \, dx = \tfrac{1}{2}\int_0^L (E A u') u' \, dx = \tfrac{1}{2}\int_0^L u' E A u' \, dx$$

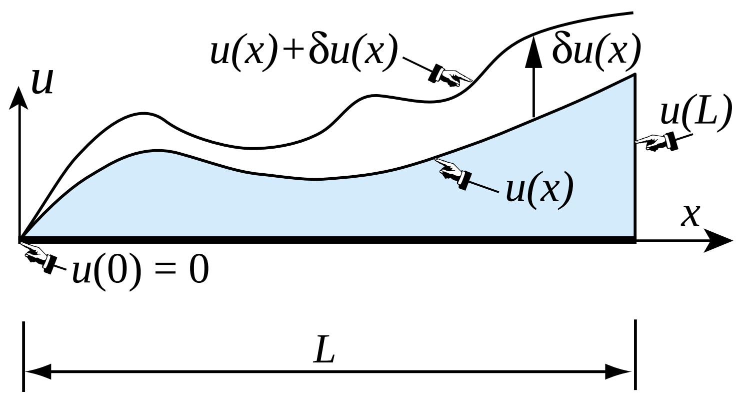
**External work**

$$W = \int_0^L q u \, dx$$

**Total potential energy**

$$\Pi = U - W$$

# Concept of Kinematically Admissible Variation




$\delta u(x)$ is ***kinematically admissible*** if $u(x)$ and $u(x) + \delta u(x)$

(i) are ***continuous*** over bar length, i.e. $u(x) \in \mathcal{C}_0$ in $x \in [0, L]$.

(ii) ***satisfy exactly displacement BC***; in the figure, $u(0) = 0$

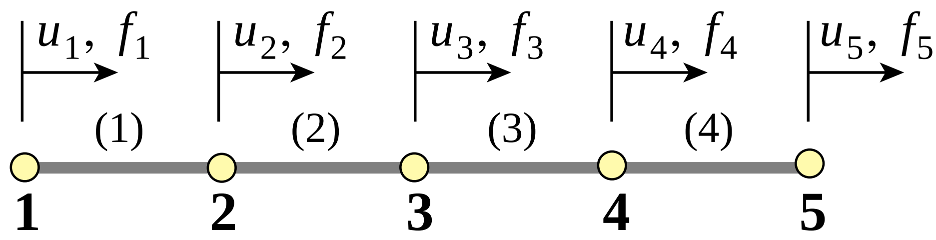
# The Minimum Potential Energy (MPE) Principle

The MPE principle states that the actual displacement solution $u^*(x)$ that satisfies the governing equations is that which renders the TPE functional $\Pi[u]$ stationary:

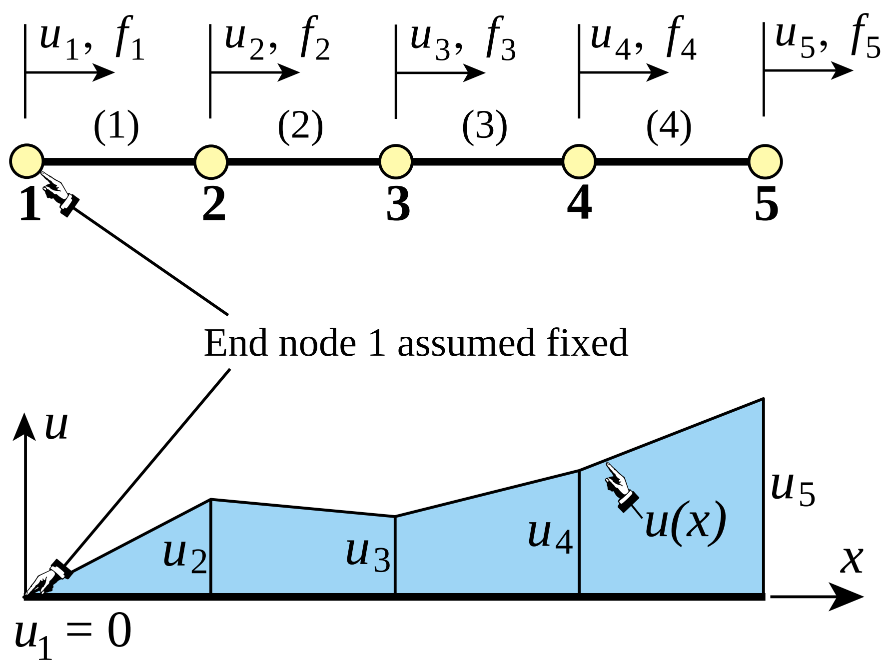
$$\delta\Pi = \delta U - \delta W = 0 \quad \text{iff} \quad u = u^*$$

with respect to *admissible* variations $u = u^* + \delta u$ of the exact displacement solution $u^*(x)$

# FEM Discretization of Bar Member



# FEM Displacement Trial Function




Axial displacement plotted normal to *x*
for visualization convenience

# Total Potential Energy Principle and Decomposition over Elements

$$\delta\Pi = \delta U - \delta W = 0 \quad \text{iff} \quad u = u^* \quad \text{(exact solution)}$$

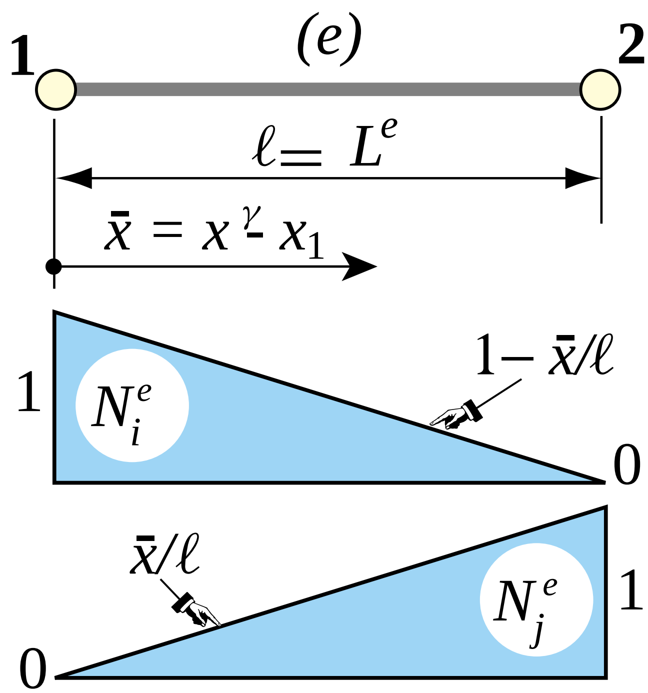
But $\quad \Pi = \Pi^{(1)} + \Pi^{(2)} + \ldots + \Pi^{(N^e)}$

and $\quad \delta\Pi = \delta\Pi^{(1)} + \delta\Pi^{(2)} + \ldots + \delta\Pi^{(N^e)} = 0$

From fundamental lemma of variational calculus, each ***element variation must vanish***, giving

$$\delta\Pi^e = \delta U^e - \delta W^e = 0$$

# Element Shape Functions



# Element Shape Functions (cont'd)

Linear displacement interpolation:

$$u^e(x) = N_1^e u_1^e + N_2^e u_2^e = [\; N_1^e \;\; N_2^e \;] \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \mathbf{N}\,\mathbf{u}^e$$

in which

$$N_1^e = 1 - \frac{x - x_1}{\ell} = 1 - \zeta\,, \qquad N_2^e = \frac{x - x_1}{\ell} = \zeta$$

$$\zeta = \frac{x - x_1}{\ell}$$ dimensionless (natural) coordinate

# Displacement Variation Process Yields the Element Stiffness Equations

$$\Pi^e = U^e - W^e \qquad \begin{cases} U^e = \frac{1}{2}(\mathbf{u}^e)^T\ \mathbf{K}^e\ \mathbf{u}^e \\ W^e = (\mathbf{u}^e)^T\ \mathbf{f}^e \end{cases}$$

$$\delta\Pi^e = 0 \quad \rightarrow \quad (\delta\mathbf{u}^e)^T\ [\mathbf{K}^e\mathbf{u}^e - \mathbf{f}^e] = 0$$

**(Appendix D)**

since $\delta\mathbf{u}^e$ is arbitrary [...] = 0 ↓

$$\mathbf{K}^e\,\mathbf{u}^e = \mathbf{f}^e$$

***the element stiffness equations***

# The Bar Element Stiffness

$$U^e = \tfrac{1}{2}\int_0^{\ell} e\, E A\, e \; dx \qquad e = u'$$

$$U^e = \tfrac{1}{2}\int_0^{\ell} [\, u_1 \quad u_2 \,] \; \frac{1}{\ell}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{1}{\ell}[\, -1 \quad 1 \,] \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} dx$$

$$U^e = \tfrac{1}{2}[\, u_1^e \quad u_2^e \,] \int_0^{\ell} \frac{EA}{\ell^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \tfrac{1}{2}\left(\mathbf{u}^e\right)^T \mathbf{K}^e \;\mathbf{u}^e$$

$$\mathbf{K}^e = \int_0^{\ell} EA\, \mathbf{B}^T \mathbf{B}\, dx = \int_0^{\ell} \frac{EA}{\ell^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$$

If *EA* is constant over element

$$\mathbf{K}^e = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

# The Consistent Nodal Force Vector

$$W^e = \int_0^\ell q\,u\;dx = \int_0^\ell (\mathbf{u}^e)^T\,\mathbf{N}^T q\;dx = \left(\mathbf{u}^e\right)^T \int_0^\ell q \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = \left(\mathbf{u}^e\right)^T \mathbf{f}^e$$

Since $\mathbf{u}^e$ is arbitrary

$$\mathbf{f}^e = \int_0^\ell q \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx$$

in which $\zeta = \dfrac{x-x_1}{\ell}$

# Bar Consistent Force Vector (cont'd)

If $q$ is ***constant*** along element

$$\mathbf{f}^e = q \int_0^{\ell} \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = q\ell \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

the ***same result as with EbE load lumping*** (i.e., assigning one half of the total load to each node)