Homework Exercises for Chapter 10 – Superelements and Global-Local Analysis
Solutions

EXERCISE 10.1 To apply the matrix form of static condensation, prepare the partitions

$$\mathbf{K}_{ii} = \begin{bmatrix} 132 & -44 \\ -44 & 176 \end{bmatrix}, \quad \mathbf{K}_{bi} = \begin{bmatrix} -44 & -44 \\ -44 & -44 \end{bmatrix}, \quad \mathbf{K}_{bb} = \begin{bmatrix} 88 & 0 \\ 0 & 220 \end{bmatrix}, \quad \mathbf{f}_i = \begin{bmatrix} 10 \\ 15 \end{bmatrix}, \quad \mathbf{f}_b = \begin{bmatrix} 5 \\ 20 \end{bmatrix}. \tag{E10.8}$$

Then

$$\tilde{\mathbf{K}}_{bb} = \mathbf{K}_{bb} - \mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib} = \begin{bmatrix} 52 & -36 \\ -36 & 184 \end{bmatrix}, \quad \tilde{\mathbf{f}}_b = \mathbf{f}_b - \mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\mathbf{f}_i = \begin{bmatrix} 15 \\ 30 \end{bmatrix}. \tag{E10.9}$$

Performing symmetric Gauss elimination on u_2 and u_3 gives the same answer. This was actually done with *Mathematica* by testing the module of the next Exercise.

EXERCISE 10.2

Module `CondenseFreedom` listed in Figure E10.1 statically condenses an arbitrary DOF. The test statements given in Figure E10.2 process the system of Exercise 10.1. The results reproduce (E10.9).

```
CondenseFreedom[K_,f_,k_]:=Module[{c,pivot,Kc,fc,
   ii,jj,n=Length[K]}, If [n<=0,Return[{K,f}]];
   If [k<=0 || k>n, Return[{K,f}]];
   Kc=Table[0,{n-1},{n-1}]; fc=Table[0,{n-1}];
   pivot=K[[k,k]]; If [pivot==0,Print["CondenseFreedom:",
         "Singular Matrix"]; Return[{K,f}]];  ii=0;
   For [i=1,i<=n,i++,  If [i==k, Continue[]]; ii++;
        c=K[[i,k]]/pivot;
        fc[[ii]]=f[[i]]-c*f[[k]]; jj=0;
        For [j=1,j<=n,j++, If [j==k,Continue[]]; jj++;
             Kc[[ii,jj]]=K[[i,j]]-c*K[[k,j]]
            ];
       ];
       Return[{Kc,fc}]
   ];
```

FIGURE E10.1. *Mathematica* module for Exercise 10.2.

```
K=44*{{2,-1,-1,0},{-1,3,-1,-1},{-1,-1,4,-1},{0,-1,-1,5}};
f={1,2,3,4}*5; Print["K=",K//MatrixForm,"  f=",f//MatrixForm];
{K,f}=CondenseFreedom[K,f,3]; Print["Upon condensing DOF 3:",
 "  K=",K//MatrixForm,"  f=",f//MatrixForm];
{K,f}=CondenseFreedom[K,f,2]; Print["Upon condensing DOF 2:",
 "  K=",K//MatrixForm,"  f=",f//MatrixForm];
```

$$\text{K}=\begin{pmatrix} 88 & -44 & -44 & 0 \\ -44 & 132 & -44 & -44 \\ -44 & -44 & 176 & -44 \\ 0 & -44 & -44 & 220 \end{pmatrix} \quad \text{f}=\begin{pmatrix} 5 \\ 10 \\ 15 \\ 20 \end{pmatrix}$$

Upon condensing DOF 3: $\text{K}=\begin{pmatrix} 77 & -55 & -11 \\ -55 & 121 & -55 \\ -11 & -55 & 209 \end{pmatrix} \quad \text{f}=\begin{pmatrix} \frac{35}{4} \\ \frac{55}{4} \\ \frac{95}{4} \end{pmatrix}$

Upon condensing DOF 2: $\text{K}=\begin{pmatrix} 52 & -36 \\ -36 & 184 \end{pmatrix} \quad \text{f}=\begin{pmatrix} 15 \\ 30 \end{pmatrix}$

FIGURE E10.2. Test of *Mathematica* module `CondenseFreedom` on system (E10.1).

EXERCISE 10.3 Similarity: both are staged processes. Difference: in superelements staging is controlled by the way a single model is decomposed, whereas in global-local analysis staging is dictated by the use of two models.

EXERCISE 10.4 Here are the necessary matrix manipulations:

$$\begin{aligned}\hat{\mathbf{K}} = \mathbf{T}^T\mathbf{K}\mathbf{T} &= [\,\mathbf{I} \quad -\mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\,]\begin{bmatrix}\mathbf{K}_{bb} & \mathbf{K}_{bi}\\ \mathbf{K}_{ib} & \mathbf{K}_{ii}\end{bmatrix}\begin{bmatrix}\mathbf{I}\\ -\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib}\end{bmatrix}\\ &= [\,\mathbf{I} \quad -\mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\,]\begin{bmatrix}\mathbf{K}_{bb} - \mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib}\\ \mathbf{0}\end{bmatrix} = \mathbf{K}_{bb} - \mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib} = \tilde{\mathbf{K}}_{bb}.\end{aligned} \tag{E10.10}$$

$$\begin{aligned}\hat{\mathbf{f}} = \mathbf{T}^T(\mathbf{f} - \mathbf{K}\mathbf{g}) &= [\,\mathbf{I} \quad -\mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\,]\left(\begin{bmatrix}\mathbf{f}_b\\ \mathbf{f}_i\end{bmatrix} - \begin{bmatrix}\mathbf{K}_{bb} & \mathbf{K}_{bi}\\ \mathbf{K}_{ib} & \mathbf{K}_{ii}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ -\mathbf{K}_{ii}^{-1}\mathbf{f}_i\end{bmatrix}\right)\\ &= [\,\mathbf{I} \quad -\mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\,]\begin{bmatrix}\mathbf{f}_b - \mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\mathbf{f}_i\\ \mathbf{0}\end{bmatrix} = \mathbf{f}_b - \mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\mathbf{f}_i = \tilde{\mathbf{f}}_b.\end{aligned} \tag{E10.11}$$

EXERCISE 10.5 The answers are not the same unless the process is iterated.

EXERCISE 10.6 For the original full-model vibration eigenproblem

$$\omega^2\begin{bmatrix}2&1&0&0\\1&4&1&0\\0&1&4&1\\0&0&1&2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}, \tag{E10.12}$$

the 4 squared frequencies are:

$$\omega_1^2 = 0, \quad \omega_2^2 = 1/5 = 0.2, \quad \omega_3^2 = 1, \quad \omega_4^2 = 2. \tag{E10.13}$$

To perform the Guyan reduction with matrix operations it is convenient to rearrange the system so that the DOF to be eliminated: v_2 and v_3, appear at the bottom of the eigenvector:

$$\omega^2\begin{bmatrix}2&0&1&0\\0&2&0&1\\1&0&4&1\\0&0&1&4\end{bmatrix}\begin{bmatrix}v_1\\v_4\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}1&0&-1&0\\0&1&0&-1\\-1&0&2&-1\\0&-1&-1&2\end{bmatrix}\begin{bmatrix}v_1\\v_4\\v_2\\v_3\end{bmatrix}, \tag{E10.14}$$

which in partitioned matrix form is

$$\omega^2\begin{bmatrix}\mathbf{M}_{bb} & \mathbf{M}_{bi}\\ \mathbf{M}_{ib} & \mathbf{M}_{ii}\end{bmatrix}\begin{bmatrix}\mathbf{v}_b\\ \mathbf{v}_i\end{bmatrix} = \begin{bmatrix}\mathbf{K}_{bb} & \mathbf{K}_{bi}\\ \mathbf{K}_{ib} & \mathbf{K}_{ii}\end{bmatrix}\begin{bmatrix}\mathbf{v}_b\\ \mathbf{v}_i\end{bmatrix}. \tag{E10.15}$$

The transformation matrix obtained from the stiffness equation is

$$\mathbf{T} = \begin{bmatrix}\mathbf{I}\\ -\mathbf{K}_{ii}^{-1}\,\mathbf{K}_{ib}\end{bmatrix} = \begin{bmatrix}1&0\\0&1\\2/3&1/3\\1/3&2/3\end{bmatrix} \quad \text{relating} \quad \begin{bmatrix}v_1\\v_4\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}1&0\\0&1\\2/3&1/3\\1/3&2/3\end{bmatrix}\begin{bmatrix}v_1\\v_4\end{bmatrix}. \tag{E10.16}$$

The reduced vibration eigenproblem is $\hat{\omega}^2\,\hat{\mathbf{M}}\,\hat{\mathbf{v}} = \hat{\mathbf{K}}\,\hat{\mathbf{v}}$ in which $\hat{\mathbf{K}} = \mathbf{T}^T\,\mathbf{K}\,\mathbf{T}$, $\hat{\mathbf{M}} = \mathbf{T}^T\,\mathbf{M}\,\mathbf{T}$ and $\hat{\mathbf{v}} = [\,v_1 \;\; v_4\,]^T$. Carrying out the congruential transformations we obtain the reduced eigensystem

$$\hat{\omega}^2\begin{bmatrix}6&3\\3&6\end{bmatrix}\begin{bmatrix}v_1\\v_4\end{bmatrix} = \begin{bmatrix}\frac{1}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{1}{3}\end{bmatrix}\begin{bmatrix}v_1\\v_4\end{bmatrix}. \tag{E10.17}$$

The 2 squared frequencies given by this reduced model are

$$\omega_1^2 = 0, \quad \omega_2^2 = 2/9 = 0.222222. \tag{E10.18}$$

The zero frequency (which is associated to a rigid body mode) is reproduced exactly. The first nonzero frequency ω_2 is found within an error of about 5%. The third and fourth frequencies are lost in the reduction process.

EXERCISE 10.7 Not assigned.