# 10

# Superelements and Global-Local Analysis

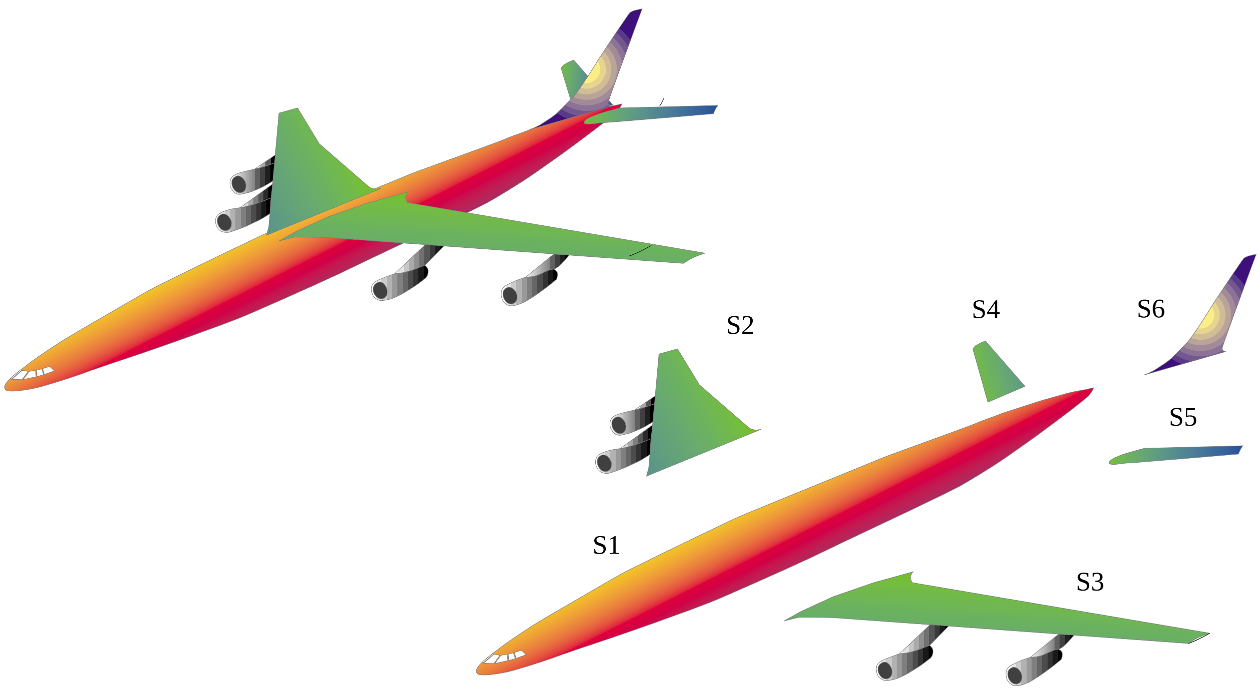
# Superelements

## Two extremes

**Macroelements** **"bottom up"**

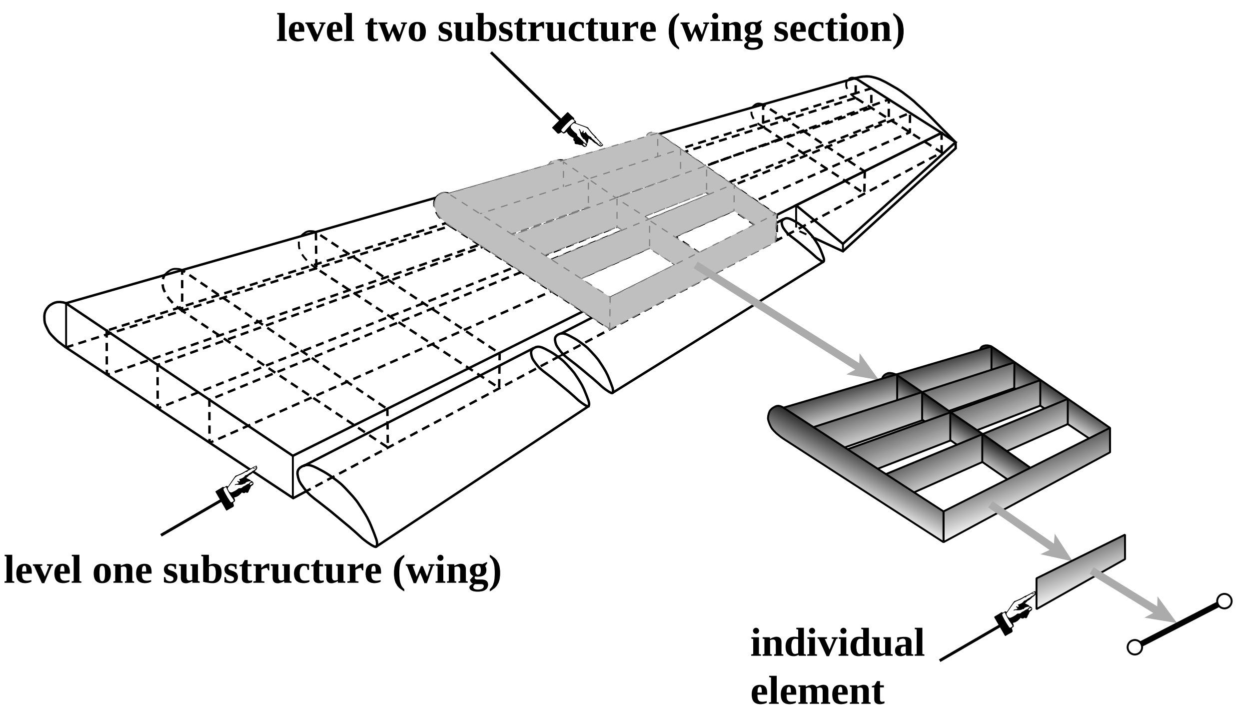
**Substructures** **"top down"**

# Substructuring was Invented in the Aerospace Industry (early 1960s)



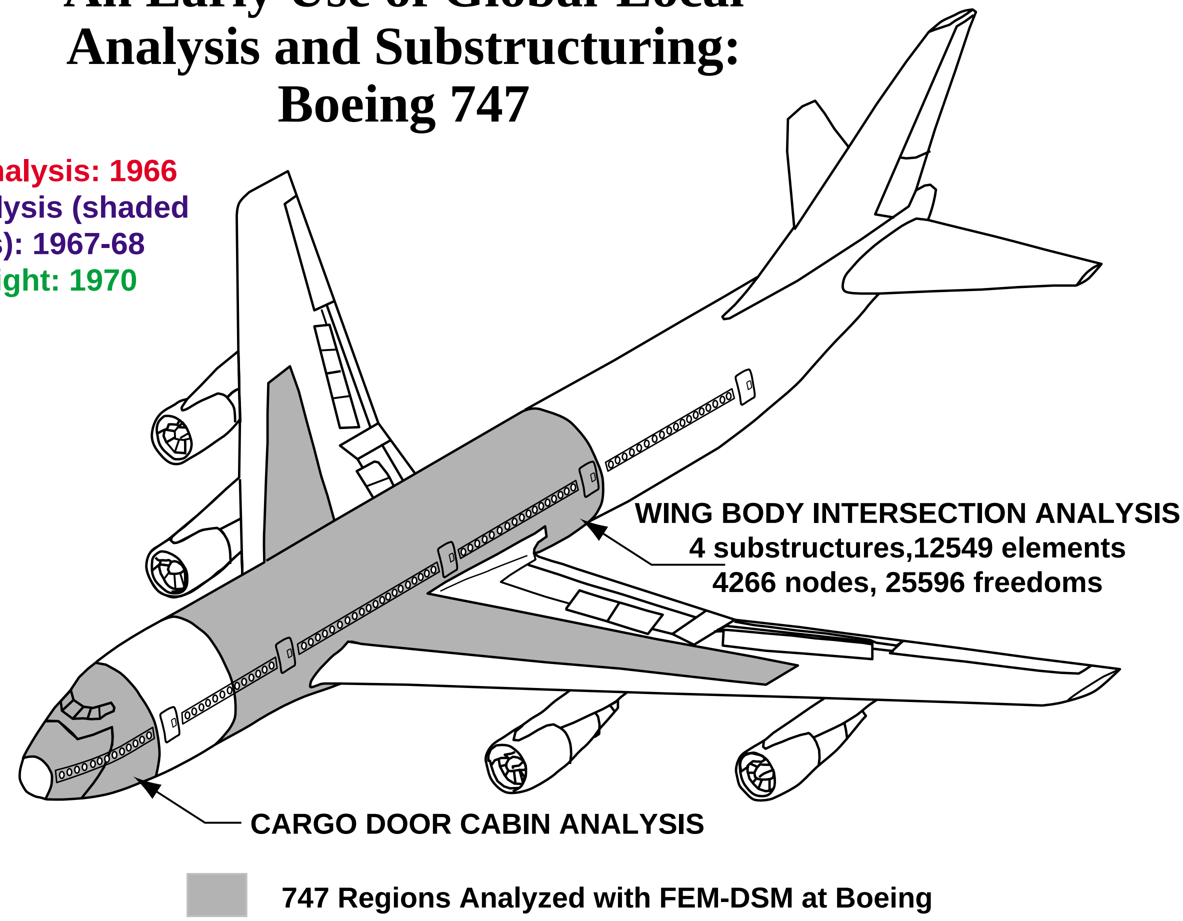

**First level substructuring**

# Substructures (cont'd)

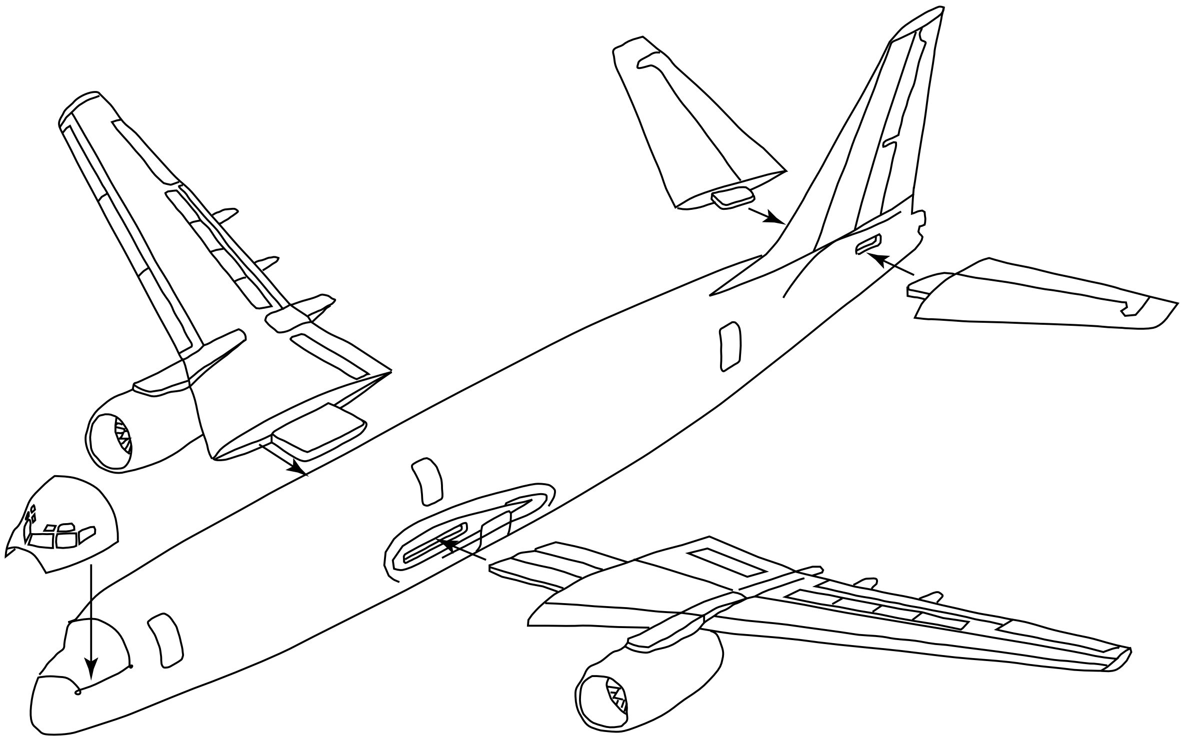


# An Early Use of Global-Local Analysis and Substructuring: Boeing 747

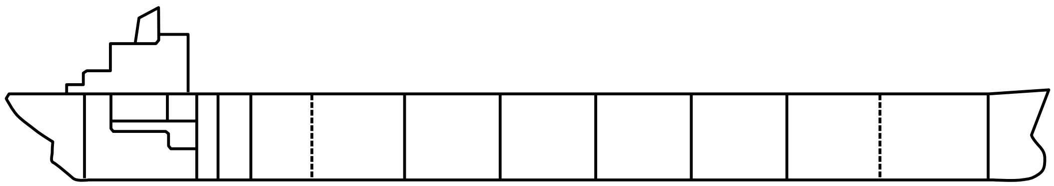
**Global Analysis: 1966**
**Local Analysis (shaded regions): 1967-68**
**First flight: 1970**

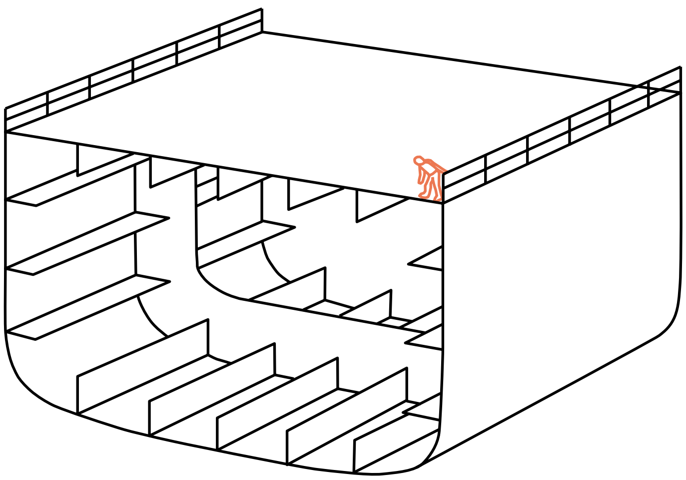


# Another Example (McDonnell-Douglas DC-10, ca 1968)

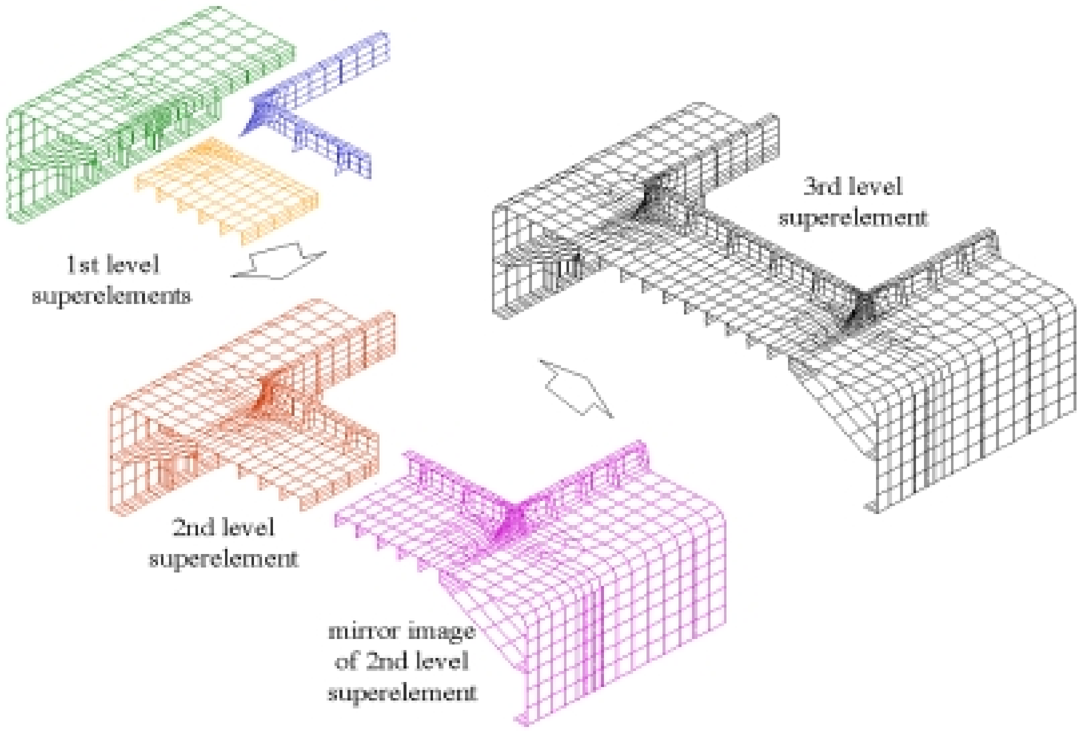


# Multilevel FEM Substructuring was Invented in the Norwegian Offshore Industry in the mid/late 60s





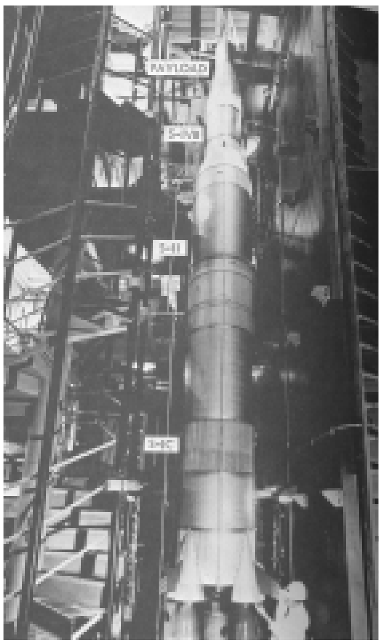
# Among Other Things, to Take Advantage of Repetition

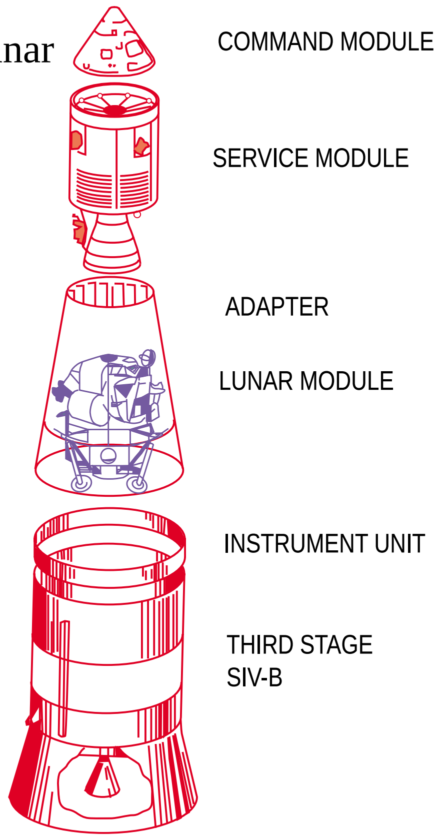


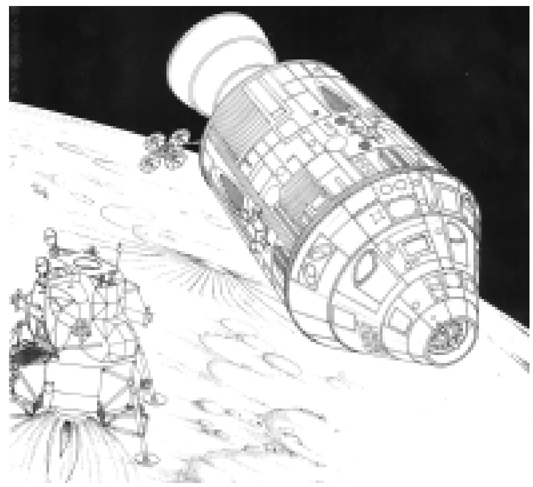

From DNV (Det Norske Veritas) web-posted brochure.
Permission requested for inclusion in book proper.

# Multistage Rockets Naturally Decompose into Substructures

Short stack
Apollo/Saturn lunar
rocket

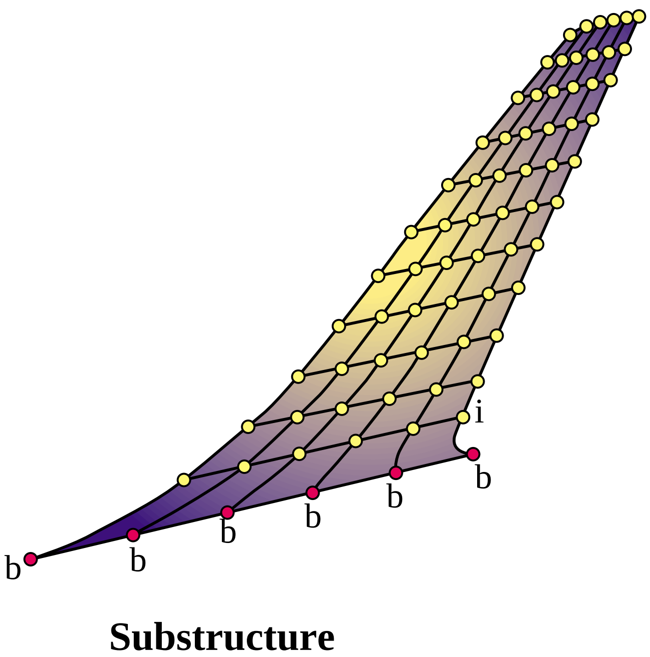


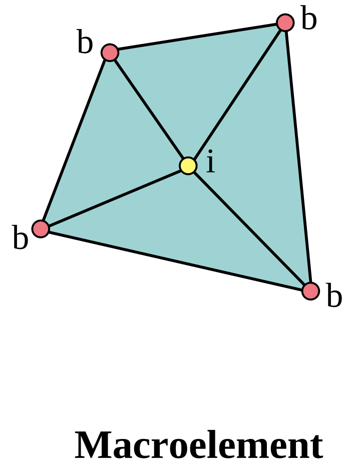


# Static Condensation

## A universal way to eliminate internal DOFs



# Static Condensation by Matrix Algebra

**Partition**

$$\begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{u}_b \\ \mathbf{u}_i \end{bmatrix} = \begin{bmatrix} \mathbf{f}_b \\ \mathbf{f}_i \end{bmatrix}$$

**Solve for interior displacements from 2nd matrix equation**

$$\mathbf{u}_i = \mathbf{K}_{ii}^{-1} (\mathbf{f}_i - \mathbf{K}_{ib} \mathbf{u}_b)$$

**replace into first matrix equation**

$$\tilde{\mathbf{K}}_{bb} \mathbf{u}_b = \tilde{\mathbf{f}}_b$$

*Condensed stiffness equations*

**where**

$$\tilde{\mathbf{K}}_{bb} = \mathbf{K}_{bb} - \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib}$$

$$\tilde{\mathbf{f}}_b = \mathbf{f}_b - \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{f}_i$$

# Static Condensation by Symmetric Gauss Elimination

$$\begin{bmatrix} 6 & -2 & -1 & -3 \\ -2 & 5 & -2 & -1 \\ -1 & -2 & 7 & -4 \\ -3 & -1 & -4 & 8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 4 \\ 0 \end{bmatrix}$$

8 is called the ***pivot***

**Task: eliminate $u_4$**

$$\begin{bmatrix} 6-\frac{(-3)\times(-3)}{8} & -2-\frac{(-1)\times(-3)}{8} & -1-\frac{(-4)\times(-3)}{8} \\ -2-\frac{(-3)\times(-1)}{8} & 5-\frac{(-1)\times(-1)}{8} & -2-\frac{(-4)\times(-1)}{8} \\ -1-\frac{(-3)\times(-4)}{8} & -2-\frac{(-1)\times(-4)}{8} & 7-\frac{(-4)\times(-4)}{8} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 3-\frac{0\times(-3)}{8} \\ 6-\frac{0\times(-1)}{8} \\ 4-\frac{0\times(-4)}{8} \end{bmatrix}$$

$$\begin{bmatrix} \frac{39}{8} & -\frac{19}{8} & -\frac{5}{2} \\ -\frac{19}{8} & \frac{39}{8} & -\frac{5}{2} \\ -\frac{5}{2} & -\frac{5}{2} & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

***Condensed equations***

# Static Condensation by Symmetric Gauss Elimination (cont'd)

$$\begin{bmatrix} \frac{39}{8} & -\frac{19}{8} & -\frac{5}{2} \\ -\frac{19}{8} & \frac{39}{8} & -\frac{5}{2} \\ -\frac{5}{2} & -\frac{5}{2} & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

**Now eliminate** $u_3$

$$\begin{bmatrix} \frac{39}{8} - \frac{(-5/2)\times(-5/2)}{5} & -\frac{19}{8} - \frac{(-5/2)\times(-5/2)}{5} \\ -\frac{19}{8} - \frac{(-5/2)\times(-5/2)}{5} & \frac{39}{8} - \frac{(-5/2)\times(-5/2)}{5} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 3 - \frac{4\times(-5/2)}{5} \\ 6 - \frac{4\times(-5/2)}{5} \end{bmatrix}$$

$$\begin{bmatrix} \frac{29}{8} & -\frac{29}{8} \\ -\frac{29}{8} & \frac{29}{8} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

***Condensed equations***

# Static Condensation Module
## (posted on web site)

```
CondenseLastFreedom[K_,f_]:=Module[{pivot,c,Kc,fc,
    n=Length[K]}, If [n<=1,Return[{K,f}]];
    Kc=Table[0,{n-1},{n-1}]; fc=Table[0,{n-1}];
    pivot=K[[n,n]]; If [pivot==0, Print["CondenseLastFreedom:",
          " Singular Matrix"]; Return[{K,f}]];
    For [i=1,i<=n-1,i++,  c=K[[i,n]]/pivot;
         fc[[i]]=f[[i]]-c*f[[n]];
         For [j=1,j<=i,j++,
             Kc[[j,i]]=Kc[[i,j]]=K[[i,j]]-c*K[[n,j]]
             ];
        ];
    Return[Simplify[{Kc,fc}]]
    ];

ClearAll[K,f];
K={{6,-2,-1,-3},{ -2,5,-2,-1},{ -1,-2,7,-4},{-3,-1,-4,8}};
f={3,6,4,0};
Print["Before condensation:"," K=",K//MatrixForm," f=",f//MatrixForm];
{K,f}=CondenseLastFreedom[K,f];Print["Upon condensing DOF 4:",
  " K=",K//MatrixForm," f=",f//MatrixForm];
{K,f}=CondenseLastFreedom[K,f];Print["Upon condensing DOF 3:",
  " K=",K//MatrixForm," f=",f//MatrixForm];
```

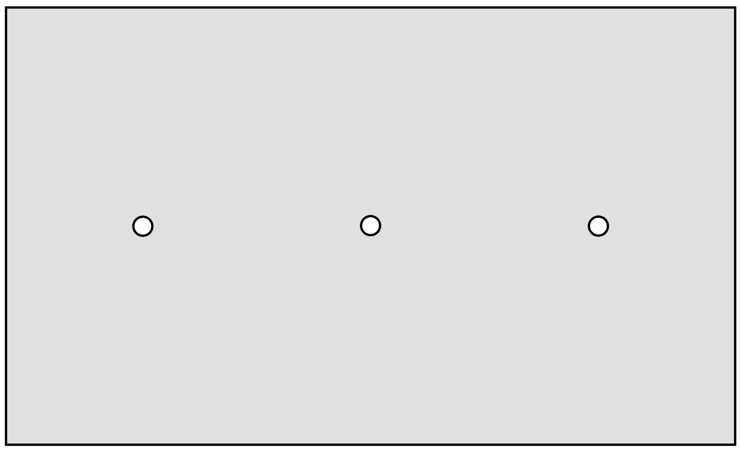
# Static Condensation Module Results on Notes Example

Before condensation: $K=\begin{pmatrix} 6 & -2 & -1 & -3 \\ -2 & 5 & -2 & -1 \\ -1 & -2 & 7 & -4 \\ -3 & -1 & -4 & 8 \end{pmatrix}$ $f=\begin{pmatrix} 3 \\ 6 \\ 4 \\ 0 \end{pmatrix}$

Upon condensing DOF 4: $K=\begin{pmatrix} \frac{39}{8} & -\frac{19}{8} & -\frac{5}{2} \\ -\frac{19}{8} & \frac{39}{8} & -\frac{5}{2} \\ -\frac{5}{2} & -\frac{5}{2} & 5 \end{pmatrix}$ $f=\begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix}$

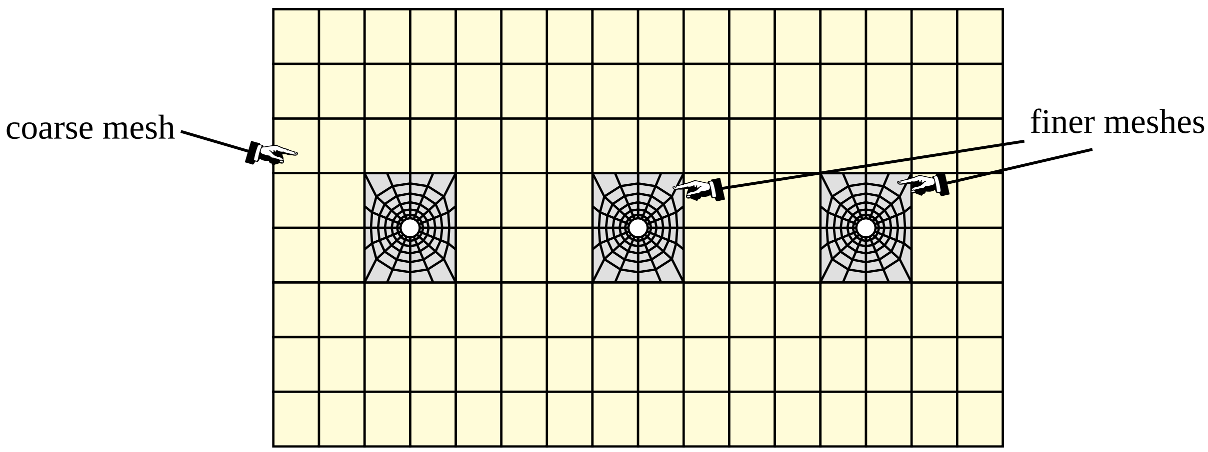
Upon condensing DOF 3: $K=\begin{pmatrix} \frac{29}{8} & -\frac{29}{8} \\ -\frac{29}{8} & \frac{29}{8} \end{pmatrix}$ $f=\begin{pmatrix} 5 \\ 8 \end{pmatrix}$

# Global-Local Analysis (an instance of Multiscale Analysis)

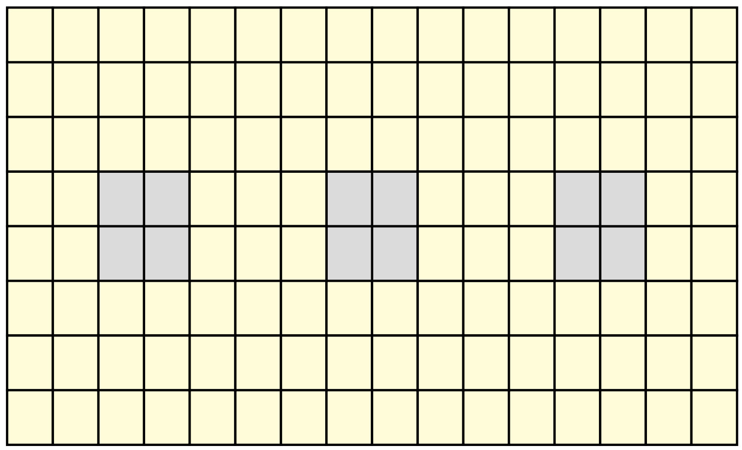


**Example structure: panel with small holes**

# Standard (one-stage) FEM Analysis



# Global-Local (two-stage) FEM Analysis



Global analysis with a coarse mesh, ignoring holes, followed by local analysis of the vicinity of the holes with finer meshes (next slide)

# Local Analysis

**BCs of displacement or (better) of force type using results extracted from the global analysis**