Homework Exercises for Chapter 9. MultiFreedom Constraints II

Solutions

EXERCISE 9.1 The result of running the *Mathematica* script on version 4.2 on a Mac G4 are shown in Figure E9.3. Some intermediate printout produced in the weight loop has been deleted to save space. The log-log plot has been massaged through Adobe Illustrator to boost line widths.

Weight w = 1. x 10^3
L2 solution error = 4.05286 x 10^{-2}
Weight w = 1. x 10^4
L2 solution error = 4.14382 x 10^{-3}
Weight w = 1. x 10^5
L2 solution error = 4.15314 x 10^{-4}
Weight w = 1. x 10^6
L2 solution error = 4.15407 x 10^{-5}
[Intermediate output deleted to save space]
Weight w = 1. x 10^{15}
L2 solution error = 3.23124 x 10^{-4}
Weight w = 1. x 10^{16}
L2 solution error = 1.05892 x 10^{-3}
Weight w = 1. x 10^{17}
L2 solution error = 1.09243 x 10^{-1}

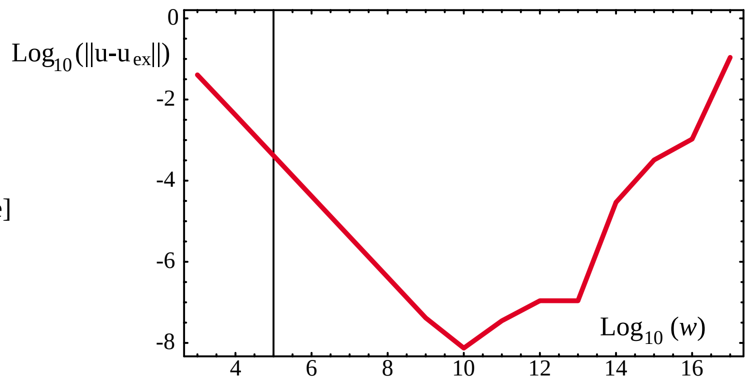

FIGURE E9.3. Results from Exercise 9.1.

The minimum solution error is obtained for $w \approx 10^{10}$, which gives roughly 8 digits of accuracy. The square root rule suggests taking $w \approx 10^2 \times 10^{16/2} = 10^{10}$ so for this simple problem it works well.

Note: The plot minimum and ascending branch shape may depend on the floating-point hardware used. On PCs in which *Mathematica* takes advantage of the 80-bit floating-point registers of the Pentium, the maximum accuracy is significantly better (about 14 places) and the optimal weight moves up to $w \approx 10^{16}$.

EXERCISE 9.2 The results of running the given *Mathematica* script for the Lagrangian Multiplier method are shown in Figure E9.4.

$$\text{Kmod} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 200 & -100 & 0 & 0 & 0 & 0 & 1 \\ 0 & -100 & 200 & -100 & 0 & 0 & 0 & 0 \\ 0 & 0 & -100 & 200 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & -100 & 200 & -100 & 0 & 0 \\ 0 & 0 & 0 & 0 & -100 & 200 & -100 & -1 \\ 0 & 0 & 0 & 0 & 0 & -100 & 100 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

fmod = $\{ 0,\ 2, 3, 4, 5, 6, 7, \frac{1}{5} \}$
Solution u = { 0., 0.27, 0.275, 0.25, 0.185, 0.07, 0.14 }, lambda = –24.5
Recovered node forces = { –27., 26.5, 3., 4., 5., –18.5, 7.}

FIGURE E9.4. Results from Exercise 10.2.

EXERCISE 9.3 The penalty elements for the stated MFCs are obtained with the rules explained in §9.1.4:

$$w \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{E9.9}$$

$$w \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} u_2 \\ u_6 \end{bmatrix} = \tfrac{1}{3} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \tag{E9.10}$$

EXERCISE 9.4

(a) Master-slave transformation:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{E9.11}$$

Transformed system:

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \tag{E9.12}$$

which yields $u_1 = u_2 = u_3 = 1.5$.

(b) The penalty augmented system is

$$\begin{bmatrix} 2+w & -1 & -w \\ -1 & 2 & -1 \\ -w & -1 & 2+w \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}. \tag{E9.13}$$

Solving this system by Cramer's rule yields

$$u_1 = \frac{6w+5}{4w+4}, \qquad u_2 = 1.5 \quad \text{for any } w, \qquad u_3 = \frac{6w+7}{4w+4}. \tag{E9.14}$$

As $w \to \infty$ the solution tends to $u_1 = u_2 = u_3 = 1.5$, which is the same found in (a). For sample finite values of w we get

$$\begin{array}{llll} w = 0 & u_1 = 1.250, & u_2 = 1.500, & u_3 = 1.750 \\ w = 1 & u_1 = 1.375, & u_2 = 1.500, & u_3 = 1.625 \\ w = 10 & u_1 = 1.477, & u_2 = 1.500, & u_3 = 1.523 \\ w = 100 & u_1 = 1.498, & u_2 = 1.500, & u_3 = 1.502 \end{array} \tag{E9.15}$$

(c) Lagrange-multiplier augmented system:

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}. \tag{E9.16}$$

EXERCISE 9.5 The finite element equations are

$$\begin{bmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \end{bmatrix} = \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix}, \tag{E9.17}$$

Introducing $P = \alpha EA$ and $I = \beta AL^2$ converts this to the scaled form

$$\frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 12\beta & 6\beta L \\ 0 & 6\beta L & 4\beta L^2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \end{bmatrix} = \begin{bmatrix} \alpha EA \\ 0 \\ 0 \end{bmatrix}. \tag{E9.18}$$

(a) Master-slave method. The MFC is $u_{x1} = u_{y1}$. Transformation equation with u_{x1} as master:

$$\begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ \theta_1 \end{bmatrix}, \qquad \text{or} \quad \mathbf{u} = \mathbf{T}\hat{\mathbf{u}}. \tag{E9.19}$$

Applying the congruent transformation to (E9.18) yields

$$\frac{EA}{L} \begin{bmatrix} 1+12\beta & 6\beta L \\ 6\beta L & 4\beta L^2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ \theta_1 \end{bmatrix} = \begin{bmatrix} \alpha EA \\ 0 \end{bmatrix}. \tag{E9.20}$$

Solving:

$$u_{x1} = \frac{\alpha L}{1+3\beta}, \qquad \theta_1 = -\frac{3\alpha}{2(1+3\beta)}, \tag{E9.21}$$

from which the physical solution in terms of P, E, A, I, L is easily recovered.

(b) Penalty function method. The penalty element stiffness equation is

$$wEA \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{E9.22}$$

where w is dimensionless. Augmenting (E9.18) with this yields

$$\frac{EA}{L} \begin{bmatrix} 1+wL & -wL & 0 \\ -wL & 12\beta + wL & 6\beta L \\ 0 & 6\beta L & 4\beta L^2 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \end{bmatrix} = \begin{bmatrix} \alpha EA \\ 0 \\ 0 \end{bmatrix}, \tag{E9.23}$$

Solving for u_{x1} by Cramer's rule we obtain

$$u_{x1} = \frac{\alpha L(3\beta + wL)}{3\beta + wL(1+3\beta)}. \tag{E9.24}$$

As $w \to \infty$ the displacement u_{x1} approaches that in (E9.21). Likewise for u_{y1} (which approaches u_{x1}in the limit) and θ_1.

Physical interpretation: the penalty element may be viewed as a fictious truss element of rigidity wEA attached to node 1 and oriented 135° with restect to x. In fact this is a way to implement skew rollers in FEM programs that lack other means of implementing MFC, as long as a truss element is available in the element library.

Note: if the penalty weight w is not divided by EA as in (E9.22), EA appears in the solution as reported in an email last week. That solution is also considered correct.

(c) The Lagrange Multiplier method gives

$$\begin{bmatrix} \frac{EA}{L} & 0 & 0 & 1 \\ 0 & 12\beta\frac{EA}{L} & 6\beta EA & -1 \\ 0 & 6\beta EA & 4\beta EAL & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ \lambda \end{bmatrix} = \begin{bmatrix} \alpha EA \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{E9.25}$$

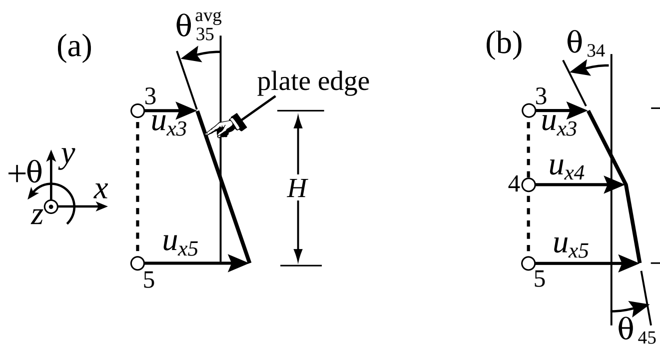

FIGURE E9.5. Geometric interpretation of plate edge rotations for Exercise 10.6.

EXERCISE 9.6

(a) The three multifreedom constraints (MFCs) are

$$u_{x2} = u_{x4}, \quad u_{y2} = u_{y4}, \quad \theta_2 = \theta_{35}^{avg} = (u_{x5} - u_{x3})/2. \tag{E9.26}$$

These three MFCs are linear and homogeneous.

(b) The rotation θ_{35}^{avg} of the plate edge 3–5 about z is defined in terms of the displacements of the nodes 3–5. From geometry, see Figure E9.5(a):

$$\theta_{35}^{avg} \approx \tan\theta_{35}^{avg} \approx \frac{u_{x5} - u_{x3}}{H} \tag{E9.27}$$

where the replacements are justified by the small-displacements and small-angles assumptions of this course. Trying to refine this estimate by accounting for the displacement of node 4, we have two angles identified in Figure E9.5(b): $\theta_{34} \approx \tan\theta_{34} \approx (u_{x4} - u_{x3})/(H/2) = 2(u_{x4} - u_{x3})/H$ and $\theta_{45} \approx \tan\theta_{45} \approx (u_{x5} - u_{x4})/(H/2) = 2(u_{x5} - u_{x4})/H$. Assuming that the rotation at node 4 is the average of these (which is reasonable since the lengths 34 and 45 are the same) we get

$$\theta_4^{avg} = \tfrac{1}{2}(\theta_{34} + \theta_{45}) = \tfrac{1}{2}\left(\frac{2(u_{x4} - u_{x3})}{H} + \frac{2(u_{x5} - u_{x4})}{H}\right) = \frac{u_{x5} - u_{x3}}{H}. \tag{E9.28}$$

It is seen that u_{x4} cancels out and we get back (E9.27). Conclusion: the inclusion of node 4 makes no difference in this particular configuration.

Note. Another way to refine the rotation estimate would be to try a least-square linear fit through u_{x3}, u_{x4} and u_{x5}. If correctly explained and worked out, this alternative method is also acceptable as answer for the second part of the question.

(c) With u_{x2}, u_{y2} and θ_2 as slave freedoms the master-slave transformation is

$$\begin{bmatrix} u_{x2} \\ u_{y2} \\ \theta_2 \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \\ u_{x5} \\ u_{y5} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1/H & 0 & 0 & 0 & 1/H & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \\ u_{x5} \\ u_{y5} \end{bmatrix}. \tag{E9.29}$$

(d) The penalty elements for the MFCs (E9.26) are

$$\begin{aligned}
\text{For } u_{x2} = u_{x4}: &\quad w \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\text{For } u_{y2} = u_{y4}: &\quad w \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{y2} \\ u_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\text{For } \theta_2 = (u_{x5} - u_{x3})/H: &\quad w \begin{bmatrix} 1 & 1/H & -1/H \\ 1/H & 1/H^2 & -1/H^2 \\ -1/H & -1/H^2 & 1/H^2 \end{bmatrix} \begin{bmatrix} \theta_2 \\ u_{x3} \\ u_{x5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{E9.30}$$

Scaled versions of the latter (for example, using $H\theta_2 + u_{x3} - u_{x5} = 0$ as MFC) are also OK as answers.

(e) Three multipliers, one for each MFC.

EXERCISE 9.7 The stationarity conditions for functional Π_{MS} are

$$\begin{bmatrix} \mathbf{K} & \mathbf{I} & -\mathbf{T} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\mathbf{T}^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ \hat{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{E9.31}$$

Eliminating $\mathbf{u}$ and $\boldsymbol{\lambda}$, in that order, yields $\mathbf{T}^T\mathbf{K}\mathbf{T}\hat{\mathbf{u}} = \mathbf{T}\mathbf{u}$, which is the equation of the master-slave method.

EXERCISE 9.8 Never assigned.

EXERCISE 9.9 Solution given in paper [71].