8

MultiFreedom Constraints I

Multifreedom Constraints

Single freedom constraint examples

$$u_{x4} = 0$$ ***linear, homogeneous***

$$u_{y9} = 0.6$$ ***linear, non-homogeneous***

Multifreedom constraint examples

$$u_{x2} = \frac{1}{2} u_{y2}$$ ***linear, homogeneous***

$$u_{x2} - 2u_{x4} + u_{x6} = 0.25$$ ***linear, non-homogeneous***

$$(x_5 + u_{x5} - x_3 - u_{x3})^2 + (y_5 + u_{y5} - y_3 - u_{y3})^2 = 0$$

nonlinear, homogeneous

Sources of Multifreedom Constraints

"Skew" displacement BCs

Coupling nonmatched FEM meshes

Global-local and multiscale analysis

Incompressibility

Model reduction

MFC Application Methods

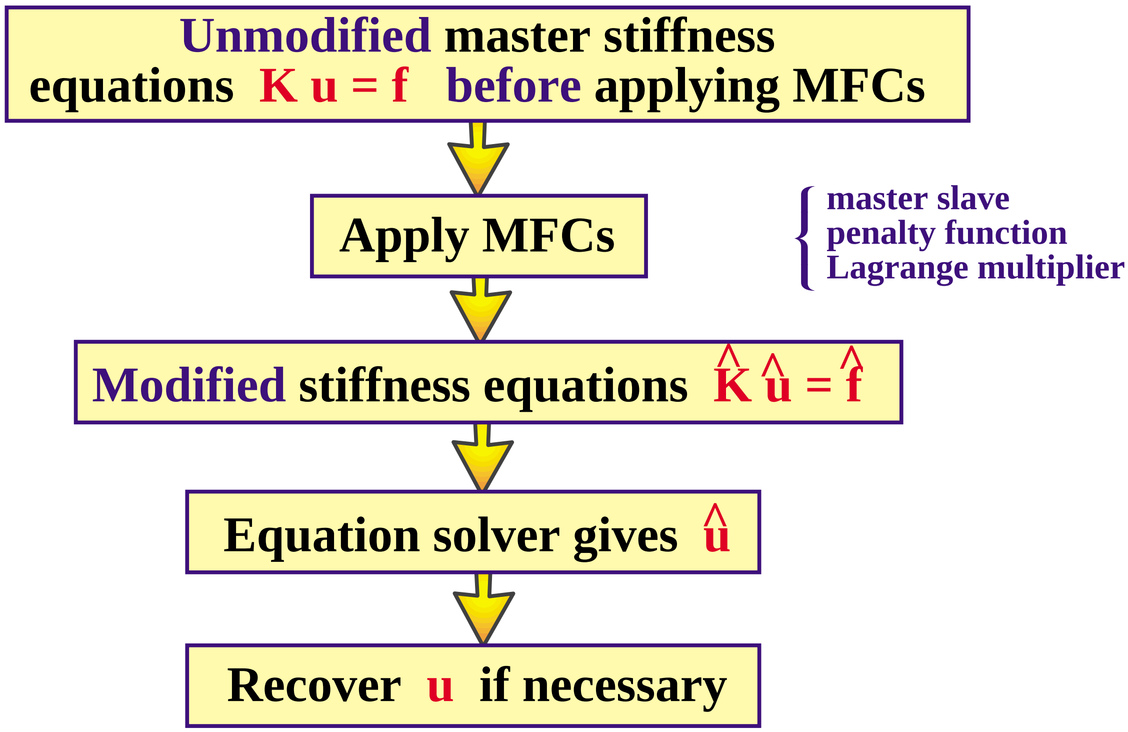
Master-Slave Elimination — **Chapter 8**

Penalty Function Augmentation

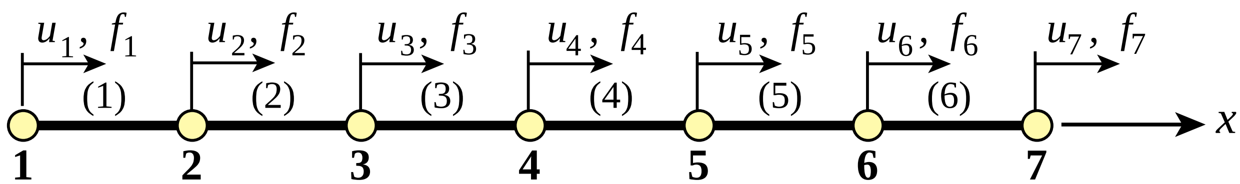
Lagrange Multiplier Adjunction

} **Chapter 9**

Procedure Summary in Static Analysis Valid for the Three Methods



Example 1D Structure to Illustrate MFCs

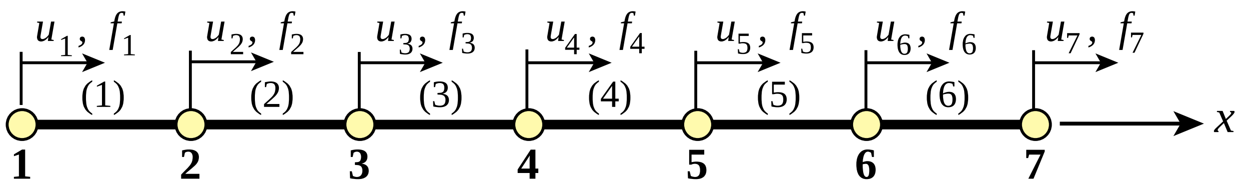


Multifreedom constraint:

$$u_2 = u_6 \qquad \text{or} \qquad u_2 - u_6 = 0$$

Linear homogeneous MFC

Example 1D Structure (Cont'd)



Unconstrained master stiffness equations

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

or $$\mathbf{K}\,\mathbf{u} = \mathbf{f}$$

Master-Slave Method for Example Structure

Recall: $u_2 = u_6$ or $u_2 - u_6 = 0$

Taking u_2 as master and u_6 as slave

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix}$$

or

$$\mathbf{u} = \mathbf{T}\,\hat{\mathbf{u}}$$

Forming the Modified Stiffness Equations

Unconstrained master stiffness equations:

$$\mathbf{K}\,\mathbf{u} = \mathbf{f}$$

Master-slave transformation:

$$\mathbf{u} = \mathbf{T}\,\hat{\mathbf{u}}$$

Replace u & premultiply both sides by $\mathbf{T}^T$:

$$\mathbf{K}\,\mathbf{T}\,\hat{\mathbf{u}} = \mathbf{f}$$

$$\mathbf{T}^T\mathbf{K}\,\mathbf{T}\,\hat{\mathbf{u}} = \mathbf{T}^T\mathbf{f}$$

On renaming $\hat{\mathbf{K}} = \mathbf{T}^T\mathbf{K}\,\mathbf{T}$ and $\hat{\mathbf{f}} = \mathbf{T}^T\mathbf{f}$ we get the modified stiffness equations:

$$\hat{\mathbf{K}}\,\hat{\mathbf{u}} = \hat{\mathbf{f}}$$

Modified Stiffness Equations for Example Structure

$$\hat{\mathbf{K}}\,\hat{\mathbf{u}} = \hat{\mathbf{f}}$$

In full

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+K_{66} & K_{23} & 0 & K_{56} & K_{67} \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 \\ 0 & K_{56} & 0 & K_{45} & K_{55} & 0 \\ 0 & K_{67} & 0 & 0 & 0 & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + f_6 \\ f_3 \\ f_4 \\ f_5 \\ f_7 \end{bmatrix}$$

Solve for $\hat{\mathbf{u}}$, then recover $\mathbf{u} = \mathbf{T}\,\hat{\mathbf{u}}$

Multiple MFCs

Suppose

$$u_2 - u_6 = 0, \qquad u_1 + 4\,u_4 = 0, \qquad 2u_3 + u_4 + u_5 = 0$$

Pick 3, 4 and 6 as slaves while 1, 2, 5 and 7 are masters

$$u_6 = u_2 \qquad u_4 = -\tfrac{1}{4}u_1 \qquad u_3 = -\tfrac{1}{2}(u_4 + u_5) = \tfrac{1}{8}u_1 - \tfrac{1}{2}u_5$$

Put in matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_5 \\ u_7 \end{bmatrix}$$

This is $\mathbf{u} = \mathbf{T}\,\hat{\mathbf{u}}$ - then proceed as before

Non-homogeneous MFCs

$$u_2 - u_6 = 0.2$$

Pick again u_6 as slave, put into matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.2 \\ 0 \end{bmatrix}$$

Nonhomogeneous MFCs (cont'd)

$$\mathbf{u} = \mathbf{T}\,\hat{\mathbf{u}} + \mathbf{g}$$

$\mathbf{g}$: "gap" vector

Premultiply both sides by $\mathbf{T}^T\mathbf{K}$, replace $\mathbf{K}\,\mathbf{u} = \mathbf{f}$ and pass data to RHS. This gives

$$\hat{\mathbf{K}}\,\hat{\mathbf{u}} = \hat{\mathbf{f}}$$

with $\hat{\mathbf{K}} = \mathbf{T}^T\mathbf{K}\,\mathbf{T}$ and $\hat{\mathbf{f}} = \mathbf{T}^T(\mathbf{f} - \mathbf{K}\,\mathbf{g})$

a modified force vector

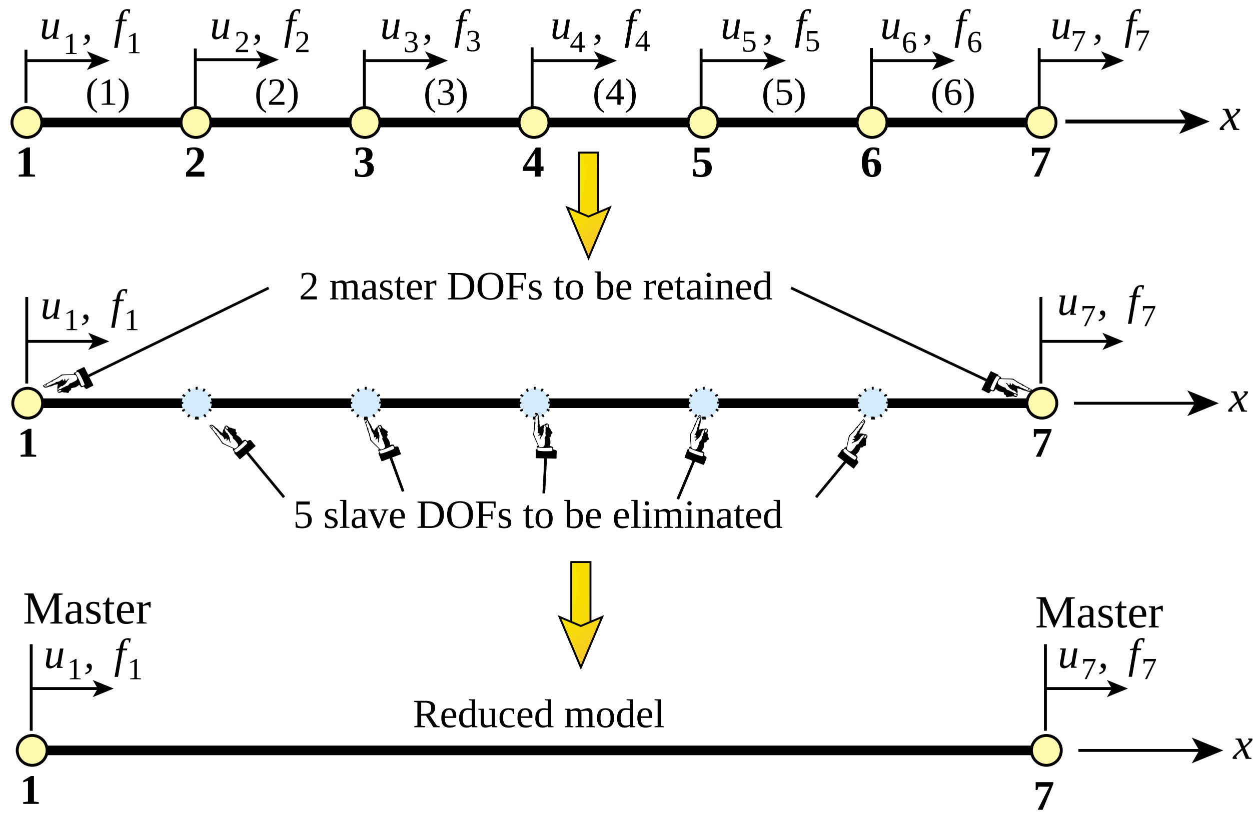
Nonhomogeneous MFCs (cont'd)

For the example structure

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+K_{66} & K_{23} & 0 & K_{56} & K_{67} \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 \\ 0 & K_{56} & 0 & K_{45} & K_{55} & 0 \\ 0 & K_{67} & 0 & 0 & 0 & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + f_6 - 0.2K_{66} \\ f_3 \\ f_4 \\ f_5 - 0.2K_{56} \\ f_7 - 0.2K_{67} \end{bmatrix}$$

Solve for $\hat{\mathbf{u}}$, then recover $\mathbf{u} = \mathbf{T}\,\hat{\mathbf{u}} + \mathbf{g}$

Model Reduction Example



Model Reduction Example (cont'd)

Lots of slaves, few masters. Only masters are left. Example of previous slide:

5 slaves → $$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5/6 & 1/6 \\ 4/6 & 2/6 \\ 3/6 & 3/6 \\ 2/6 & 4/6 \\ 1/6 & 5/6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_7 \end{bmatrix}$$ ← 2 masters

Applying the congruential transformation we get the reduced stiffness equations

$$\begin{bmatrix} \hat{K}_{11} & \hat{K}_{17} \\ \hat{K}_{17} & \hat{K}_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_7 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_7 \end{bmatrix}$$

where

$$\hat{K}_{11} = \tfrac{1}{36}(36K_{11}+60K_{12}+25K_{22}+40K_{23}+16K_{33}+24K_{34}+9K_{44}+12K_{45}+4K_{55}+4K_{56}+K_{66})$$
$$\hat{K}_{17} = \tfrac{1}{36}(6K_{12}+5K_{22}+14K_{23}+8K_{33}+18K_{34}+9K_{44}+18K_{45}+8K_{55}+14K_{56}+5K_{66}+6K_{67})$$
$$\hat{K}_{77} = \tfrac{1}{36}(K_{22}+4K_{23}+4K_{33}+12K_{34}+9K_{44}+24K_{45}+16K_{55}+40K_{56}+25K_{66}+60K_{67}+36K_{77})$$
$$\hat{f}_1 = \tfrac{1}{6}(6f_1+5f_2+4f_3+3f_4+2f_5+f_6), \qquad \hat{f}_7 = \tfrac{1}{6}(f_2+2f_3+3f_4+4f_5+5f_6+6f_7).$$

Model Reduction Example: Mathematica Script

```
(* Model Reduction Example *)
ClearAll[K11,K12,K22,K23,K33,K34,K44,K45,K55,K56,K66,
         f1,f2,f3,f4,f5,f6];
K={{K11,K12,0,0,0,0,0},{K12,K22,K23,0,0,0,0},
   {0,K23,K33,K34,0,0,0},{0,0,K34,K44,K45,0,0},
   {0,0,0,K45,K55,K56,0},{0,0,0,0,K56,K66,K67},
   {0,0,0,0,0,K67,K77}};  Print["K=",K//MatrixForm];
f={f1,f2,f3,f4,f5,f6,f7}; Print["f=",f];
T={{6,0},{5,1},{4,2},{3,3},{2,4},{1,5},{0,6}}/6;
Print["Transformation matrix T=",T//MatrixForm];
Khat=Simplify[Transpose[T].K.T];
fhat=Simplify[Transpose[T].f];
Print["Modified Stiffness:"];
Print["Khat(1,1)=",Khat[[1,1]],"\nKhat(1,2)=",Khat[[1,2]],
    "\nKhat(2,2)=",Khat[[2,2]] ];
Print["Modified Force:"];
Print["fhat(1)=",fhat[[1]],"  fhat(2)=",fhat[[2]] ];
```

Modified Stiffness:

(Some print output removed so slide fits)

$$\text{Khat}(1,1)=\frac{1}{36}(36\,K11+60\,K12+25\,K22+40\,K23+16\,K33+24\,K34+9\,K44+12\,K45+4\,K55+4\,K56+K66)$$

$$\text{Khat}(1,2)=\frac{1}{36}(6\,K12+5\,K22+14\,K23+8\,K33+18\,K34+9\,K44+18\,K45+8\,K55+14\,K56+5\,K66+6\,K67)$$

$$\text{Khat}(2,2)=\frac{1}{36}(K22+4\,K23+4\,K33+12\,K34+9\,K44+24\,K45+16\,K55+40\,K56+25\,K66+60\,K67+36\,K77)$$

Modified Force:

$$\text{fhat}(1)=\frac{1}{6}(6\,f1+5\,f2+4\,f3+3\,f4+2\,f5+f6)\qquad \text{fhat}(2)=\frac{1}{6}(f2+2\,f3+3\,f4+4\,f5+5\,f6+6\,f7)$$

Assessment of Master-Slave Method

☺ **ADVANTAGES**

exact if precautions taken
easy to understand
retains positive definiteness
important applications to model reduction

☹ **DISADVANTAGES**

requires user decisions
messy implementation for general MFCs
hinders sparsity of master stiffness equations
sensitive to constraint dependence
restricted to linear constraints