Homework Exercises for Chapter 5

Constructing MoM Members – Solutions

EXERCISE 5.1 Take $d_y = \gamma L = \bar{u}_{yj} - \bar{u}_{yi}$ as deformation variable v paired to the transverse shear force V. See Figure E5.3(a). The constitutive equation is $V = (G A_s/L)\, d_y$. It is easily checked that $\mathbf{B} = [\,-1 \;\; 1\,] = \mathbf{A}$. Thus d_y and V are conjugate. The Tonti diagram is shown in Figure E5.3(b).

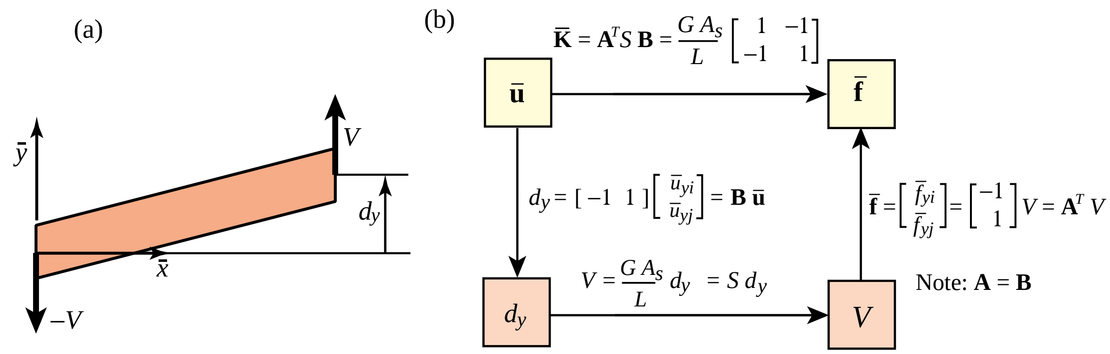

FIGURE E5.3. Solutions to Exercise 5.1: (a): conjugate variable d_y (displacements grossly exaggerated for visibility); (b) Tonti diagram.

EXERCISE 5.2 The displacement transformation matrix $\mathbf{T}^e$ is obtained by extracting rows 2 and 4 from that of (2.17). With s and c defined as there,

$$\begin{bmatrix} \bar{u}_{yi} \\ \bar{u}_{yj} \end{bmatrix} = \begin{bmatrix} -s & c & 0 & 0 \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix} = \mathbf{T}^e \mathbf{u}^e, \tag{E5.5}$$

Carrying out the multiplication $(\mathbf{T}^e)^T\, \bar{\mathbf{K}}^e \mathbf{T}^e$ gives the closed form of the spar element stiffness in global coordinates

$$\mathbf{K}^e = (\mathbf{T}^e)^T \bar{\mathbf{K}}^e \mathbf{T}^e = \frac{G A_s}{L} \begin{bmatrix} s^2 & -sc & -s^2 & sc \\ -sc & c^2 & sc & -c^2 \\ -s^2 & sc & s^2 & -sc \\ sc & -c^2 & -sc & c^2 \end{bmatrix}. \tag{E5.6}$$

EXERCISE 5.3 The displacement transformation matrix $\mathbf{T}^e$ is the same as that in (2.13) with the u_x and u_y replaced by θ_x and θ_y, respectively, because infinitesimal rotations transform exactly as vectors. As usual, the force transformation matrix is $(\mathbf{T}^e)^T$. The result for the globalized element stiffness matrix $\mathbf{K}^e = (\mathbf{T}_e)^T\, \bar{\mathbf{K}}^e\, \mathbf{T}^e$ is the same as (2.18), with factor EA/L replaced by GJ/L.

EXERCISE 5.4

(a) By elementary kinematics:

$$\bar{u}_x = u_x c_{xji} + u_y c_{yji} + u_z c_{zji} \tag{E5.7}$$

where $\{c_{xji}, c_{yji}, c_{zji}\}$ are the 3 direction cosines of longitudinal $\bar{x}$, defined by end nodes i and j ($i \to j$) with respect to $\{x, y, z\}$, respectively. These 3 numbers are easily computed as $\{x_{ji}/L, y_{ji}/L, z_{ji}/L\}$, respectively, in which $x_{ji} = x_j - x_i$, $y_{ji} = y_j - y_i$, $z_{ji} = z_j - z_i$ are the node coordinate differences and

$L = \sqrt{x_{ji}^2 + y_{ji}^2 + z_{ji}^2}$ is the element length. Evaluating this formula at end nodes i and j and puting in matrix form gives the stated form.

(b)

$$\mathbf{T} = \tfrac{1}{7} \begin{bmatrix} 2 & 6 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 6 & 3 \end{bmatrix} \tag{E5.8}$$

EXERCISE 5.5

(a) From kinematics, at any point on the line i-j (the longitudinal axis of the spar) we can write the displacement in the $\bar{y}$ direction as the linear combination of global displacements:

$$\bar{u}_y = c_{x\bar{y}}\, u_x + c_{y\bar{y}}\, u_y + c_{z\bar{y}}\, u_z \tag{E5.9}$$

where $c_{x\bar{y}}$, $c_{y\bar{y}}$ and $c_{z\bar{y}}$ are the direction cosines formed by $\bar{y}$ with the x, y and z global axes, respectively. But the direction $\bar{y}$ can be defined by that of the vector joining m to k, where m is the projection of the orientation node on i-j; see Figure E5.4. Consequently $c_{x\bar{y}} = c_{xkm}$, $c_{y\bar{y}} = c_{ykm}$, and $c_{z\bar{y}} = c_{zkm}$ and

$$\bar{u}_y = c_{xkm}\, u_x + c_{ykm}\, u_y + c_{zkm}\, u_z \tag{E5.10}$$

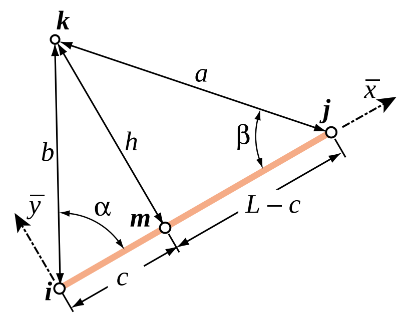

FIGURE E5.4. Reproduction of Figure E5.2(b) for the reader's convenience.

Evaluate this relation at node i: $\bar{u}_{yi} = c_{xkm}\, u_{xi} + c_{ykm}\, u_{yi} + c_{zkm}\, u_{zi}$ and at node j: $\bar{u}_{yj} = c_{xkm}\, u_{xj} + c_{ykm}\, u_{yj} + c_{zkm}\, u_{zj}$. Collecting and passing to matrix form gives

$$\bar{\mathbf{u}}^e = \begin{bmatrix} \bar{u}_{yi} \\ \bar{u}_{yj} \end{bmatrix} = \begin{bmatrix} c_{xmk} & c_{ymk} & c_{zmk} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{xmk} & c_{ymk} & c_{zmk} \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ \vdots \\ u_{zj} \end{bmatrix} = \mathbf{T}\,\mathbf{u}^e. \tag{E5.11}$$

The global components of vector $m{\rightarrow}k$ are $x_{km} = x_k - x_m$, $y_{km} = y_k - y_m$ and $z_{km} = z_k - z_m$ and its length is $h = \sqrt{x_{km}^2 + y_{km}^2 + z_{km}^2}$. Consequently $c_{xkm} = x_{km}/h$, $c_{ykm} = x_{ym}/h$ and $c_{zkm} = z_{km}/h$. This proves the form of $\mathbf{T}$ given in (E5.2). This assumes that the location of m is known, which is the topic of the next item.

(b) Before proceeding to locate m the lengths a, b and L are computed from the coordinate data for $\{i, j, k\}$. These are the triangle side lengths pictured in Figure E5.4, are always positive. Then compute $2A$ from Heron's area formula given in the Exercise assignment, taking the + sign of the square root so $A \geq 0$. If $A = 0$ orientation node k is colinear with $\{i, j\}$ and the computation aborts. Since $2A = Lh$, obviously $h = 2A/L$, which must be positive. The location of point m is determined if the signed distance c from node i is computed. This can be done in several ways. Two methods are described below.

Method I. The method outlined in the Exercise proceeds as follows. Compute $\cos\alpha$ from the cosine law at vertex i: $\cos\alpha = (L^2 + b^2 - a^2)/(2bL)$, Then $c = b\cos\alpha = b(L^2 + b^2 - a^2)/(2bL) = (L^2 + b^2 - a^2)/(2L)$. The global coordinates of m are obtained by linear interpolation of those of i and j: $x_m = x_i(L - c)/L + x_j c/L$, $y_m = x_i(L - c)/L + y_j c/L$ and $z_m = z_i(L - c)/L + z_j c/L$. Then form x_{km}, y_{km}, z_{km}, divide by h to get the direction cosines and store in $\mathbf{T}$ as per (E5.11).

Method II. (Suggested in footnote to Exercise assignment). Apply Pithagoras' theorem twice: $h^2 = (b^2 - c^2)$ and $h^2 = a^2 - (L - c)^2$. On subtracting, h^2 and c^2 cancel out giving $b^2 - a^2 + L^2 - 2Lc = 0$, whence $c = (L^2 + b^2 - a^2)/(2L)$, which is the same result as above. The coordinates of m are defined by interpolation, and $\mathbf{T}$ constructed as described for Method I.