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Constructing MoM Members
What Are MoM Members?

Skeletal structural members whose stiffness equations can be constructed by Mechanics of Materials (MoM) methods

Can be locally modeled as 1D elements
MoM Members Tend to Look Alike ...

One dimension (longitudinal) much larger than the other two (transverse)
But Receive Different Names
According to Structural Function

Bars: transmit axial forces
Beams: transmit bending
Shafts: transmit torque
Spars (aka Webs): transmit shear
Beam-columns: transmit bending + axial force
Common Features of MoM
Finite Element Models

End quantities are defined at the joints

Internal quantities are defined in the member
Governing Matrix Equations for Simplex MoM Element

From node displacements to internal deformations (strains)

\[ v = B \bar{u} \quad \text{Kinematic} \]

From deformations to internal forces

\[ p = S v \quad \text{Constitutive} \]

From internal forces to node forces

\[ \bar{f} = A p \quad \text{Equilibrium} \]

If \( \bar{f} \) and \( \bar{u} \) are PVW (Virtual Work) conjugate, \( B = A \)
Tonti Diagram of Governing Matrix Equations for Simplex MoM Element

Stiffness

\[ \bar{f} = A^T S B \bar{u} = \bar{K} \bar{u} \]

Kinematic

\[ \bar{v} = B \bar{u} \]

Equilibrium

\[ \bar{f} = A^T p \]

Constitutive

\[ p = S \bar{v} \]
Elimination of the Internal Quantities \( v \) and \( p \) gives the Element Stiffness Equations through Simple Matrix Multiplications

\[
\tilde{f} = A^T S B \tilde{u} = \tilde{K}\tilde{u}
\]

\[
\tilde{K} = A^T S B
\]

If \( B = A \)

\[
\tilde{K} = B^T S B \quad \text{symmetric if } S \text{ is}
\]
The Bar Element Revisited

(a) Axial rigidity $EA$, length $L$

(b) $f_{xi}, u_{xi}$

$EA$

$L$
The Bar Element Revisited (cont'd)

\[ d = [-1, 1] \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{xj} \end{bmatrix} = \mathbf{B}\bar{u} \]

\[ F = \frac{EA}{L} d = S d, \]

\[ \bar{f} = \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{xj} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} F = \mathbf{A}^T F \]

\[ \bar{K} = \mathbf{A}^T S \mathbf{B} = S \mathbf{B}^T \mathbf{B} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

Can be expanded to the 4 x 4 of Chapter 2 by adding two zero rows and columns to accommodate \( \bar{u}_{yi} \) and \( \bar{u}_{yj} \)
Discrete Tonti Diagram for Bar Element

\[ d = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_{xi} \\ \ddot{u}_{xj} \end{bmatrix} = B\ddot{u} \]

\[ \bar{f} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \ddot{u} \]

\[ F = \frac{EA}{L} d = Sd \]

\[ \bar{f} = \begin{bmatrix} \ddot{f}_{xi} \\ \ddot{f}_{xj} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} F = \Lambda^T F \]
The Spar (a.k.a. Shear-Web) Element

(a)

(b)

Shear rigidity $G A_s$, length $L$

$V$

$y$

$z$

$x$

$y$

$z$

$x$

$y$

$z$

$x$

$y$

$z$

$x$

$y$

$z$

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$x$

$y$

$z$

$x$

$y$

$z$

$x$

$y$

$z$

$x$

$y$

$z$
Spars used in Wing Structure
(Piper Cherokee)
The Spar Element (cont'd)

\[ \gamma = \frac{1}{L} \left[ -1 \quad 1 \right] \begin{bmatrix} \bar{u}_{yi} \\ \bar{u}_{yj} \end{bmatrix} = B\bar{u} \]

\[ V = GA_s \gamma = S \gamma \]

\[ \bar{f} = \begin{bmatrix} \bar{f}_{yi} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} V = A^T V \]

\[ \bar{f} = \begin{bmatrix} \bar{f}_{yi} \\ \bar{f}_{yj} \end{bmatrix} = A^T S B\bar{u} = \frac{GA_s}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_{yi} \\ \bar{u}_{yj} \end{bmatrix} = \tilde{K}\bar{u} \]

\[ \tilde{K} = \frac{GA_s}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]
The Shaft Element

(a) Torsional rigidity $GJ$, length $L$

(b) For stiffness derivation details see Notes
Matrix Equations for Non-Simplex MoM Element

From node displacements to internal deformations \textit{at each section}

\[ v = B \ddot{u} \quad \text{Kinematic} \]

From deformations to internal forces \textit{at each section}

\[ p = Rv \quad \text{Constitutive} \]

From internal forces to node forces

\[ d\ddot{f} = A^T dp \quad \text{Equilibrium} \]
Equilibrium

Constitutive (at each section)

Kinematic (at each section)

Stiffness

Tonti Diagram of Matrix Equations for Non-Simplex MoM Element (with A=B)

\[ \tilde{f} = \int_0^L B^T R B \, d\tilde{x} \, \tilde{u} \]

\[ v = B \, \tilde{u} \]

\[ d \tilde{f} = B^T \, dp \]

\[ p = R \, v \]
High-Aspect Wing, Constellation (1952)
Low-Aspect Delta Wing, F-117 (1975)
Low-Aspect Delta Wing, Blackhawk (1972)