Homework Exercises for Chapter 3

The Direct Stiffness Method II – Solutions

EXERCISE 3.1 The complete set of joint forces acting on the free-free structure, calculated in (3.18), is shown in Figure E3.3. Here is the verification of overall equilibrium.

$$\sum f_x \Rightarrow f_{x1} + f_{x2} + f_{x3} = -2 + 0 + 2 = 0,$$

$$\sum f_y \Rightarrow f_{y1} + f_{y2} + f_{y3} = -2 + 1 + 1 = 0,$$

$$\sum M_{wrt2} \Rightarrow f_{y1} L^{(1)} + f_{x3} L^{(2)} = -2 \times 10 + 2 \times 10 = 0. \tag{E3.2}$$

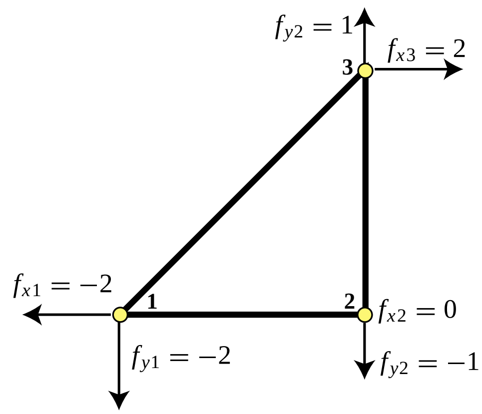

FIGURE E3.3. Joint forces on free-free example truss. Forces f_{x1}, f_{y1} and f_{y2} are reactions; the others are applied loads.

Note: A theorem of classical mechanics says that if a 2D force system is in translational equilibrium and its moment respect to a point is zero, the moment with respect to any other point vanishes. Therefore it is sufficient to verify moment equilibrium respect to just one point, here take to be node 2 for convenience.

EXERCISE 3.2 The computations are very simple, and yield $F^{(1)} = 0$, $F^{(2)} = -1$ (compression) and $F^{(3)} = 2\sqrt{2}$ (tension).

EXERCISE 3.3 For the example truss, joint forces may be also recovered from consideration of joint equilibrium, because the structure is statically determinate. Once the joint displacements (3.17) are known, the joint forces in the *local* system of member e and the internal (axial) force f^e may be recovered from the generic-member equilibrium relation

$$\bar{\mathbf{f}}^e = \begin{bmatrix} \bar{f}^e_{xi} \\ \bar{f}^e_{yi} \\ \bar{f}^e_{xj} \\ \bar{f}^e_{yj} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} f^e = \bar{\mathbf{K}}^e \bar{\mathbf{u}}^e = \bar{\mathbf{K}}^e \mathbf{T}^e \mathbf{u}^e. \tag{E3.3}$$

Carrying out the operations for the example truss we get

$$\bar{\mathbf{f}}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \bar{\mathbf{f}}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \qquad \bar{\mathbf{f}}^{(3)} = \begin{bmatrix} -2\sqrt{2} \\ 0 \\ 2\sqrt{2} \\ 0 \end{bmatrix}, \tag{E3.4}$$

from which it follows that the member axial forces $F^{(1)}$, $F^{(2)}$ and $F^{(3)}$ are 0, -1 (compression) and $+2\sqrt{2}$ (tension), respectively. See Figure E3.3 for physical interpretation. *This method is applicable only to statically determinate structures.*

EXERCISE 3.4 The reduced system is obtained by deleting the first three equations in (3.12):

$$\begin{bmatrix} 5 & 0 & -5 \\ 0 & 10 & 10 \\ -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

The coefficient matrix of this system is singular because the second row is the sum of the first and third rows. Because of this property, the system cannot be solved for the displacements. Physical interpretation: the support conditions $u_{x1} = u_{y1} = u_{x2} = 0$ are not sufficient to prevent an (infinitesimal) rigid body rotation about joint 1.

EXERCISE 3.5 Begin by clearing all entries of the 6×6 matrix $\mathbf{K}$ to zero, so that we effectively start with the null matrix

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{E3.5}$$

Merge member 1 ($e = 1$), for which $\text{EFT}^{(1)} = \{1, 2, 3, 4\}$. On completing the loops over i and j, we will have

$$\mathbf{K} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{E3.6}$$

Note that this is precisely the *augmented* member stiffness in (3.4). In fact (E3.6) may be viewed as the master stiffness matrix of a 3-node truss that consists of member 1 only. Next we merge member 2 ($e = 2$), for which $\text{EFT}^{(2)} = \{3, 4, 5, 6\}$. On completion of the i, j loops we will have

$$\mathbf{K} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \end{bmatrix}. \tag{E3.7}$$

This is the matrix that would result on adding the expanded matrices in (3.4) and (3.5), and may be interpreted as the master stiffness matrix of a structure that consists of members 1 and 2 only. Finally, upon merging member 3, for which $\text{EFT}^{(3)} = \{1, 2, 5, 6\}$, we get

$$\mathbf{K} = \begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix}. \tag{E3.8}$$

This is the complete free-free master stiffness (3.12).

EXERCISE 3.6

(a) Recall that the generic member stiffness matrix in global coordinates is

$$\mathbf{K}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \tag{E3.9}$$

where $c = \cos\varphi^e$, $s = \sin\varphi^e$, E^e, A^e and L^e are the elastic modulus, cross-section area and length, respectively. For member (element) (1): $E^{(1)} = 1000$, $A^{(1)} = 2$, $L^{(1)} = \sqrt{4^2 + 3^2} = 5$, $\varphi^{(1)} =$

arctan (3/4), $c = 4/5 = 0.8$, $s = 3/5 = 0.6$. Therefore

$$\mathbf{K}^{(1)} = \frac{1000 \times 2}{5} \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} = \begin{bmatrix} 256 & 192 & -256 & -192 \\ 192 & 144 & -192 & -144 \\ -256 & -192 & 256 & 192 \\ -192 & -144 & 192 & 144 \end{bmatrix} \tag{E3.10}$$

For member (element) (2): $E^{(2)} = 1000$, $A^{(2)} = 4$, $L^{(2)} = \sqrt{4^2 + (-3)^2} = 5$, $\varphi^{(2)} = \arctan(-3/4)$, $c = 4/5 = 0.8$, $s = -3/5 = -0.6$. Therefore

$$\mathbf{K}^{(2)} = \frac{1000 \times 4}{5} \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} = \begin{bmatrix} 512 & -384 & -512 & 384 \\ -384 & 288 & 384 & -288 \\ -512 & 384 & 512 & -384 \\ 384 & -288 & -384 & 288 \end{bmatrix} \tag{E3.11}$$

Next the member stiffness equations are augmented by adding zero rows and columns as appropriate to complete the force and displacement vectors. Compatibility is used to drop the member index on the displacements. For member (1):

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \end{bmatrix} = \begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 256 & 192 & 0 & 0 \\ -192 & -144 & 192 & 144 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{E3.12}$$

For member (2):

$$\begin{bmatrix} f_{x1}^{(2)} \\ f_{y1}^{(2)} \\ f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 512 & -384 & -512 & 384 \\ 0 & 0 & -384 & 288 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{E3.13}$$

Adding the two equations and using the force equilibrium condition $\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} = (\mathbf{K}^{(1)} + \mathbf{K}^{(2)})\mathbf{u} = \mathbf{K}\mathbf{u}$, we arrive at the master stiffness equations

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \tag{E3.14}$$

whence

$$\mathbf{K} = \begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix} \tag{E3.15}$$

The master equations can also be derived using the *freedom pointer* technique described in §3.5.1, which is the way assembly is actually programmed. For this structure the Element Freedom Tables are $\text{EFT}^{(1)} = \{1, 2, 3, 4\}$ and $\text{EFT}^{(2)} = \{3, 4, 5, 6\}$. These tables may be used to obtain the same entries of the master stiffness matrix. For hand computations this technique is more prone to error.

(b) We apply the displacement boundary conditions:

$$u_{x1} = u_{y1} = u_{x3} = u_{y3} = 0,\ f_{x2} = P = 12, \text{ and } f_{y2} = 0, \tag{E3.16}$$

by removing equations 1, 2, 5, and 6 from the system. This is done by deleting rows and columns 1, 2, 5, and 6 from $\mathbf{K}$, and the corresponding components from $\mathbf{f}$ and $\mathbf{u}$. The reduced two-equation system is

$$\begin{bmatrix} 768 & -192 \\ -192 & 432 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \end{bmatrix} = \begin{bmatrix} f_{x2} \\ f_{y2} \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix} \tag{E3.17}$$

Solving this linear system by any method gives

$$u_{x2} = \frac{9}{512} \quad \text{and} \quad u_{y2} = \frac{1}{128}. \tag{E3.18}$$

(c) To recover all the joint forces note that the complete displacement vector is $\mathbf{u} = [\,0\ \ 0\ \ 9/512\ \ 1/128\ \ 0\ \ 0\,]^T$. Using the original master stiffness equations (E3.14):

$$\mathbf{f} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \mathbf{K}\mathbf{u} = \begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 9/512 \\ 1/128 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ -9/2 \\ 12 \\ 0 \\ -6 \\ 9/2 \end{bmatrix}. \tag{E3.19}$$

Overall equilibrium is verified by summing the forces in the $\{x, y\}$ directions and taking z-moments about any joint:

$$\begin{aligned} \sum f_x &\Rightarrow f_{x1} + f_{x2} + f_{x3} = -6 + 12 - 6 = 0, \\ \sum f_y &\Rightarrow f_{y1} + f_{y2} + f_{y3} = -\frac{9}{2} + 0 + \frac{9}{2} = 0, \\ \sum M_{wrt1} &\Rightarrow 4\,f_{y2} - 3\,f_{x2} + 8\,f_{y3} = 0 - 36 + 36 = 0, \\ \sum M_{wrt2} &\Rightarrow\ - 3\,f_{x1} + 4\,f_{y1} - 3\,f_{x3} + 4\,f_{y3} = -18 + 18 - 18 + 18 = 0, \\ \sum M_{wrt3} &\Rightarrow 8\,f_{y1} - 4\,f_{y2} - 3\,f_{x2} = 36 - 0 - 36 = 0. \end{aligned} \tag{E3.20}$$

(d) Using the method described in §3.4.2 we proceed as follows. For member (1), $c = 4/5$, $s = 3/5$, $\mathbf{u}^{(1)} = [\,0\quad 0\quad 9/512\quad 1/128\,]^T$. The local joint displacements are recovered by

$$\bar{\mathbf{u}}^{(1)} = \begin{bmatrix} \bar{u}_{xi}^{(1)} \\ \bar{u}_{yi}^{(1)} \\ \bar{u}_{xj}^{(1)} \\ \bar{u}_{yj}^{(1)} \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 & 0 & 0 \\ -3/5 & 4/5 & 0 & 0 \\ 0 & 0 & 4/5 & 3/5 \\ 0 & 0 & -3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 9/512 \\ 1/128 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3/160 \\ -11/2560 \end{bmatrix} \tag{E3.21}$$

The elongation is $d^{(1)} = 3/160 - 0 = 3/160$, from which $F^{(1)} = 1000 \times 2 \times (3/160)/5 = 15/2 = 7.5$. The positive sign indicates that member (1) is in tension.

For member (2), $c = 4/5$, $s = -3/5$, $\mathbf{u}^{(2)} = [\,9/512 \quad 1/128 \quad 0 \quad 0\,]^T$. The local joint displacements are recovered by

$$\bar{\mathbf{u}}^{(2)} = \begin{bmatrix} \bar{u}_{xi}^{(2)} \\ \bar{u}_{yi}^{(2)} \\ \bar{u}_{xj}^{(2)} \\ \bar{u}_{yj}^{(2)} \end{bmatrix} = \begin{bmatrix} 4/5 & -3/5 & 0 & 0 \\ 3/5 & 4/5 & 0 & 0 \\ 0 & 0 & 4/5 & -3/5 \\ 0 & 0 & 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 9/512 \\ 1/128 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/320 \\ 43/2560 \\ 0 \\ 0 \end{bmatrix} \tag{E3.22}$$

The elongation is $d^{(2)} = 0 - 3/320 = -3/320$, from which $F^{(2)} = 1000 \times 4 \times (-3/320)/5 = -15/2 = -7.5$. The negative sign indicates that member (2) is in compression.

EXERCISE 3.7

(a) The master stiffness matrix is the same as in the previous Exercise:

$$\mathbf{K} = \begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix} \tag{E3.23}$$

However, now we have a prescribed displacement, $u_{y3} = -\frac{1}{2}$:

$$\begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_{x2} \\ u_{y2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ 12 \\ 0 \\ f_{x3} \\ f_{y3} \end{bmatrix} \tag{E3.24}$$

(b) For hand computation, reduce the system by removing rows 1, 2, 5 and 6 that pertain to the prescribed displacements:

$$\begin{bmatrix} -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_{x2} \\ u_{y2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix} \tag{E3.25}$$

Next, columns 1, 2, 5 and 6 are removed by tranferring all known terms from the left to the right hand side:

$$\begin{bmatrix} 768 & -192 \\ -192 & 432 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix} - \begin{bmatrix} (-256)(0) + (-192)(0) + (-512)(0) + (384)(-\frac{1}{2}) \\ (-192)(0) + (-144)(0) + (384)(0) + (-288)(-\frac{1}{2}) \end{bmatrix} \tag{E3.26}$$

which gives

$$\begin{bmatrix} 768 & -192 \\ -192 & 432 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \end{bmatrix} = \begin{bmatrix} 204 \\ -144 \end{bmatrix} \tag{E3.27}$$

Solution by Gausss elimination yields

$$\boxed{u_{x2} = \frac{105}{512} \quad \text{and} \quad u_{y2} = -\frac{31}{128}} \tag{E3.28}$$

(c) To recover all the joint forces we complete the node displacement vector with the known values:

$$\boxed{\mathbf{u}^T = [\,0 \quad 0 \quad \tfrac{105}{512} \quad -\tfrac{31}{128} \quad 0 \quad -\tfrac{1}{2}\,]} \tag{E3.29}$$

and use $\mathbf{f} = \mathbf{K}\mathbf{u}$ to get

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 256 & 192 & -256 & -192 & 0 & 0 \\ 192 & 144 & -192 & -144 & 0 & 0 \\ -256 & -192 & 768 & -192 & -512 & 384 \\ -192 & -144 & -192 & 432 & 384 & -288 \\ 0 & 0 & -512 & 384 & 512 & -384 \\ 0 & 0 & 384 & -288 & -384 & 288 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{105}{512} \\ -\frac{31}{128} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -6 \\ -\frac{9}{2} \\ 12 \\ 0 \\ -6 \\ \frac{9}{2} \end{bmatrix} \tag{E3.30}$$

Horizontal force equilibrium is verified by

$$\boxed{f_{x1} + f_{x2} + f_{x3} = -6 + 12 - 6 = 0} \tag{E3.31}$$

and likewise for y-force and moment equilibrium.