Homework Exercises for Chapter 2
The Direct Stiffness Method I — Solutions

EXERCISE 2.1 Loads applied between truss joints will generally have components transverse to the member axis. These components cause bending, which by hypothesis cannot be resisted by the members of an idealized truss. (Members of actual trusses do have finite bending resistance but it is better not to call upon this property in the design process, except to resist own weight.)

EXERCISE 2.2 Take as node displacements $u_{xi} = u_{yi} = 1$ for $i = 1, 2, \ldots N$, where N is the number of truss joints (nodes). Each force component in $\mathbf{f} = \mathbf{K}\mathbf{u}$ is then a $\mathbf{K}$ row sum. This force must vanish because those displacements are associated with a translational rigid body motion: $\{u_x = 1, u_y = 1\}$ of the free-free truss structure.[10] Answering for the example truss or a single element is OK.

Since $\mathbf{K}$ is symmetric, the same conclusion applies to columns.

EXERCISE 2.3 Start from (2.9), which is reproduced here:

$$F = \bar{f}_{xj} = -\bar{f}_{xi}, \qquad d = \bar{u}_{xj} - \bar{u}_{xi}, \tag{E2.4}$$

in which F and d are connected by (2.8): $F = (EA/L)d$. Express these equations in matrix form:

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{xj} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} F = \frac{EA}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} d, \qquad d = \bar{u}_{xj} - \bar{u}_{xi} = [\,-1 \quad 1\,] \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{xj} \end{bmatrix}. \tag{E2.5}$$

and combine as matrix product:

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{xj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} [\,-1 \quad 1\,] \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{xj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{xj} \end{bmatrix}. \tag{E2.6}$$

The joint displacements and forces in the $\bar{y}$ direction have no effect on the stiffness of the truss member, and may be incorporated by adding null rows and columns to get (2.10). Exactly the same matrix-multiply technique is applied in Chapter 5 to other elements.

Alternatively one may incorporate x and y components from scratch by inserting zero entries in the foregoing matrices:

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} F = \frac{EA}{L} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} d, \qquad d = \bar{u}_{xj} - \bar{u}_{xi} = [\,-1 \quad 0 \quad 1 \quad 0\,] \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yj} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}. \tag{E2.7}$$

Combining by matrix multiplication we obtain the 4×4 element stiffness matrix directly:

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} [\,-1 \quad 0 \quad 1 \quad 0\,] \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yj} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yj} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}. \tag{E2.8}$$

[10] Recall that the master stiffness equations of a plane truss relates node forces and displacements of a structure with all supports removed. Such "floating structure" may experience rigid body motions in the $\{x, y\}$ plane.

EXERCISE 2.4

$$\bar{\mathbf{f}}^T \bar{\mathbf{u}} = \bar{f}_{xi}\bar{u}_{xi} + \bar{f}_{yi}\bar{u}_{yi} + \bar{f}_{xj}\bar{u}_{xj} + \bar{f}_{yj}\bar{u}_{yj} = F(\bar{u}_{xj} - \bar{u}_{xi}) = Fd. \tag{E2.9}$$

This equation expresses the invariance of external energy in two different bases.

EXERCISE 2.5 If $\bar{\mathbf{f}} = F\mathbf{T}_f = \mathbf{T}_f F$, transposing both sides gives $\bar{\mathbf{f}}^T = F\mathbf{T}_f^T$. Then

$$\bar{\mathbf{f}}^T \bar{\mathbf{u}} = F\mathbf{T}_f^T \bar{\mathbf{u}}, \qquad Fd = F\mathbf{T}_d \bar{\mathbf{u}}. \tag{E2.10}$$

Imposing energy invariance $\bar{\mathbf{f}}^T \bar{\mathbf{u}} = Fd$, checked in the last exercise, we equate the right hand sides of the two previous expressions:

$$F\mathbf{T}_f^T \bar{\mathbf{u}} = F\mathbf{T}_d \bar{\mathbf{u}}. \tag{E2.11}$$

Because F and $\bar{\mathbf{u}}$ are arbitrary we must have

$$\mathbf{T}_f^T = \mathbf{T}_d, \qquad \text{or} \qquad \mathbf{T}_f = \mathbf{T}_d^T. \tag{E2.12}$$

EXERCISE 2.6 The bar equilibrium equation is satisfied by a constant $A\sigma$. Because A is constant, so are the axial stress σ and axial strain $e = \sigma/E = d\bar{u}/d\bar{x}$. Therefore the displacement $\bar{u}$ must vary linearly in $\bar{x}$, and the interpolation (E2.2) is correct. Differentiating it gives $e = (u_{xj} - u_{xi})/L = d/L$, which combined with $\sigma = Ee = Ed/L$ and $F = A\sigma = EAd/L = k_s d$ yields $k_s = EA/L$. [This argument may be reversed: assume (E2.2) is correct; it gives constant strain e and constant stress $\sigma = Ee$; thus it satisfies the bar equations identically.]

EXERCISE 2.7 From (E2.2) one gets $e = d\bar{u}/d\bar{x} = (u_{xj} - u_{xi})/L = d/L$. Hooke's law gives $\sigma = Ee = Ed/L$. Both e and σ do not depend on $\bar{x}$. Substitution into (E2.3) gives

$$\Pi(d) = \tfrac{1}{2}\int_0^L \frac{EA}{L^2} d^2 \, d\bar{x} - Fd = \frac{EA}{2L^2} d^2 \int_0^L d\bar{x} - Fd = \frac{EA}{2L} d^2 - Fd. \tag{E2.13}$$

Applying the MPE yields

$$\frac{\partial \Pi}{\partial d} = \frac{EA}{L} d - F = 0, \tag{E2.14}$$

whence $F = (EA/L)\, d$ follows.

EXERCISE 2.8 Premultiply both sides of $\bar{\mathbf{K}}^e \, \bar{\mathbf{u}}^e = \bar{\mathbf{f}}^e$ by $(\mathbf{T}^e)^T$:

$$(\mathbf{T}^e)^T \, \bar{\mathbf{f}}^e = (\mathbf{T}^e)^T \bar{\mathbf{K}}^e \bar{\mathbf{u}}^e. \tag{E2.15}$$

Using the transformation equations in (2.16), namely $\bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e$ and $\mathbf{f}^e = (\mathbf{T}^e)^T \, \bar{\mathbf{f}}^e$, the preceding equation becomes

$$\mathbf{f}^e = [(\mathbf{T}^e)^T \, \bar{\mathbf{K}}^e \mathbf{T}^e] \, \mathbf{u}^e \stackrel{\text{def}}{=} \mathbf{K}^e \, \mathbf{u}^e. \tag{E2.16}$$

Identifying $\mathbf{K}^e$ with the triple matrix product in brackets yields (2.17).

For the truss member the transformation matrices are (2.13) and (2.14). The triple product involved in the congruential transformation may be conveniently carried out through Falk's scheme (Appendix B):

$$
\begin{array}{cc}
 & \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} = \mathbf{T}^e \\
\frac{E^e A^e}{L^e} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} c & s & -c & -s \\ 0 & 0 & 0 & 0 \\ -c & -s & c & s \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \frac{E^e A^e}{L^e} = \bar{\mathbf{K}}^e \mathbf{T}^e \\
\begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} & \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \times \frac{E^e A^e}{L^e} = \mathbf{K}^e,
\end{array}
\tag{E2.17}
$$

which agrees with (2.18).

EXERCISE 2.9 Those steps are not part in the computer implementation of FEM. They exist only in the analyst's mind.