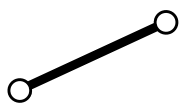
2
The Direct Stiffness Method Part I

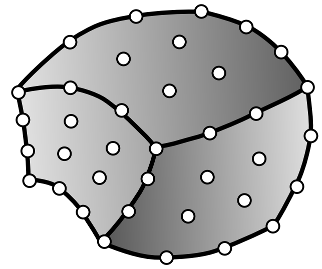
The Direct Stiffness Method (DSM)

Importance: DSM is used by all major commercial FEM codes

A democratic method, works the same no matter what the element:



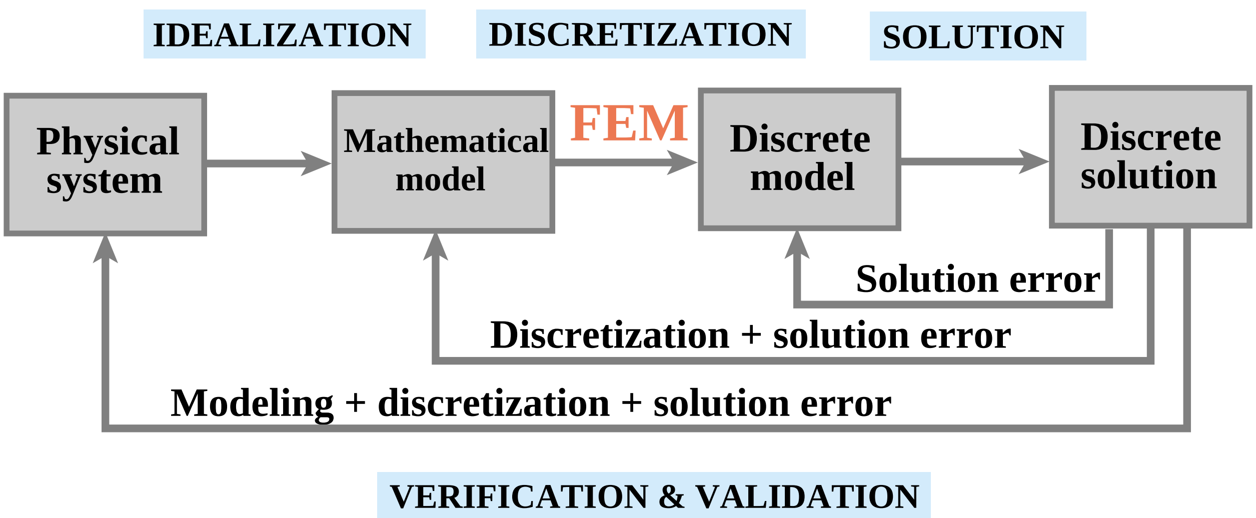
Bar (truss member) element,
2 nodes, 4 DOFs



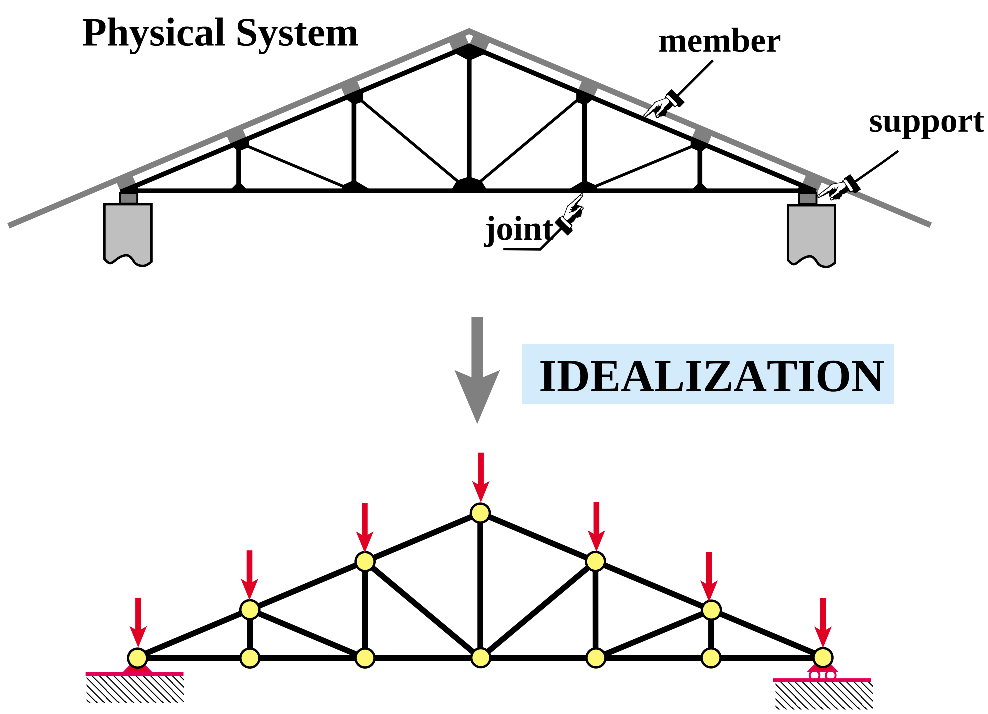
Tricubic brick element,
64 nodes, 192 DOFs

Obvious decision: use the truss to teach the DSM

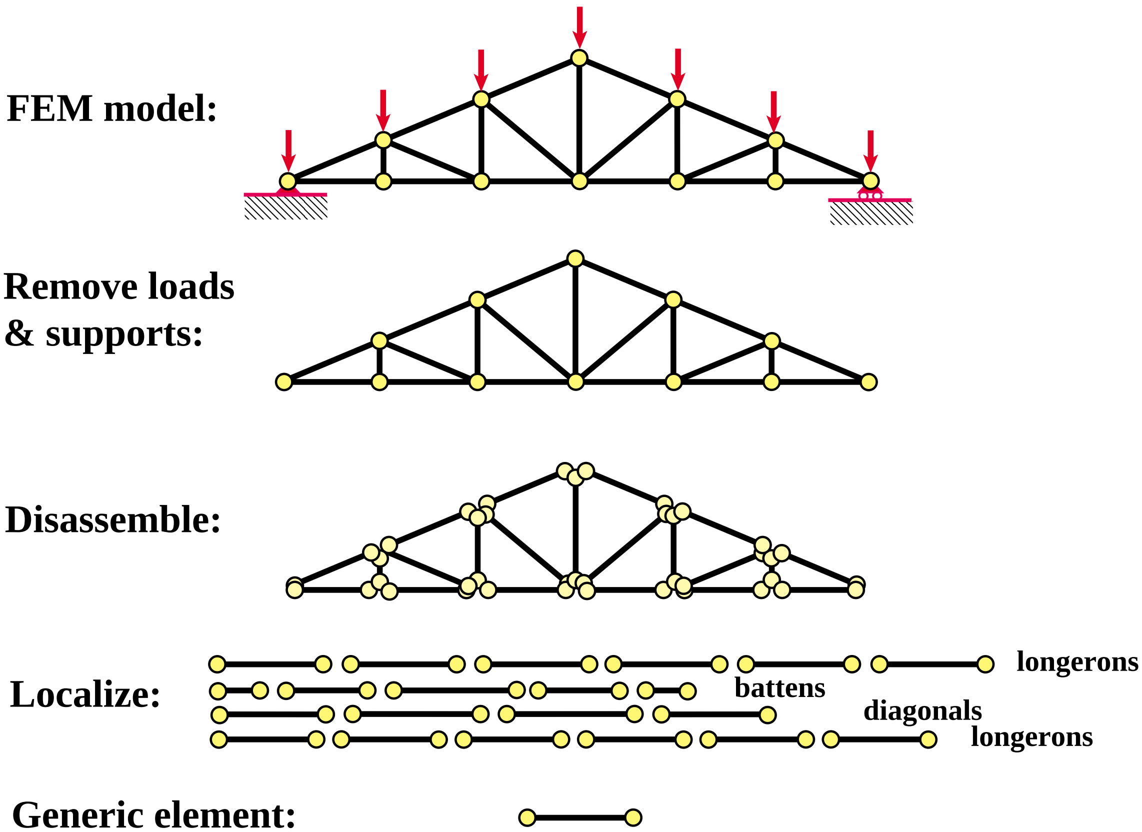
Model Based Simulation
(a simplification of diagrams of Chapter 1)



Idealization Process



DSM: Breakdown Steps



DSM: Assembly & Solution Steps

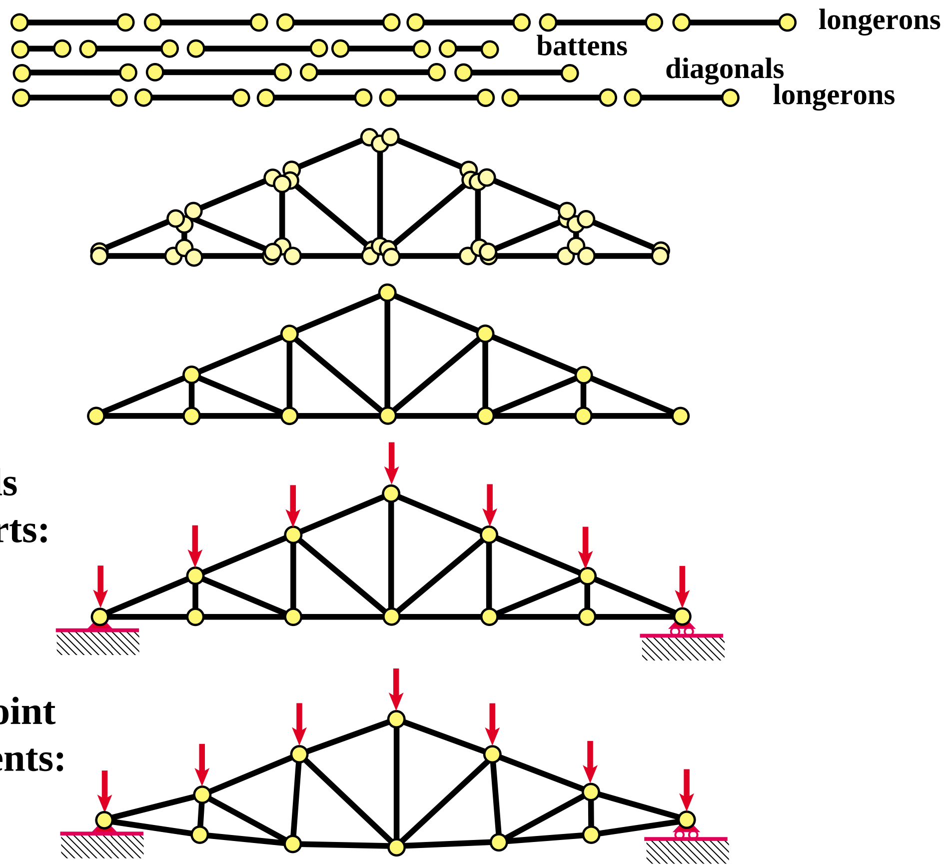
Form elements:

Globalize:

Merge:

Apply loads and supports:

Solve for joint displacements:

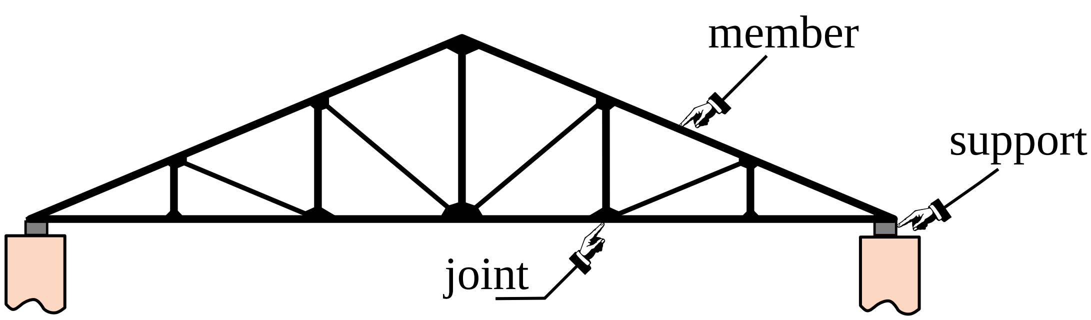


The Direct Stiffness Method (DSM) Steps
Starting with: Idealization

Breakdown (Chapter 2)
- Disconnection
- Localization
- Member (Element) Formation

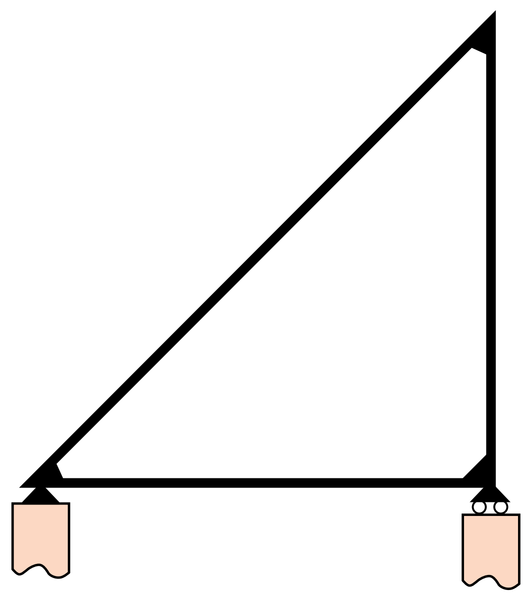
Assembly & Solution (Chapter 3)
- Globalization
- Merge
- Application of BCs
- Solution
- Recovery of Derived Quantities

A Physical Plane Truss

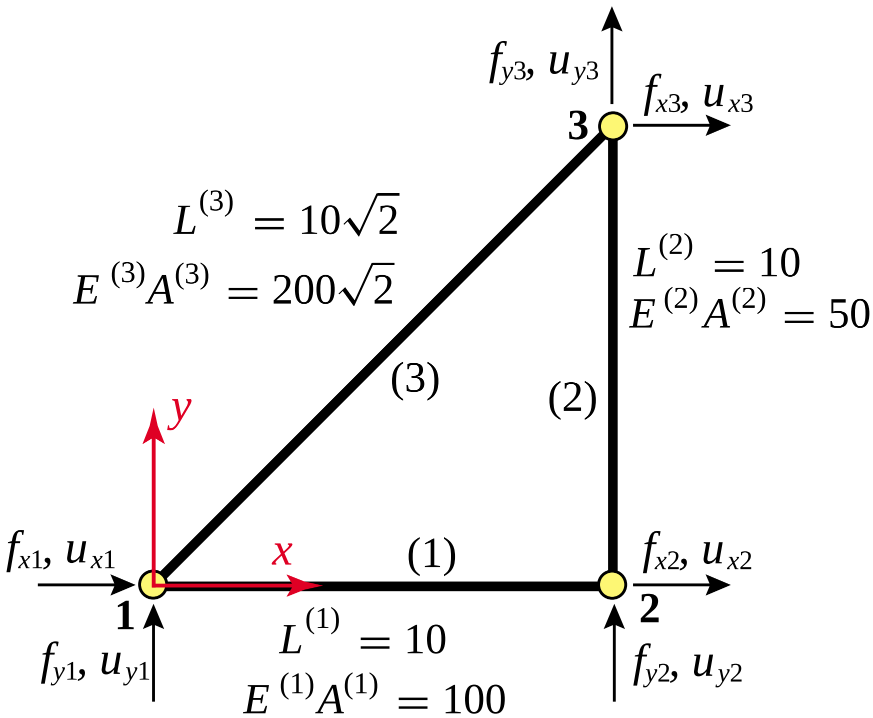


Too complicated to do by hand. We will use a simpler one to illustrate DSM steps

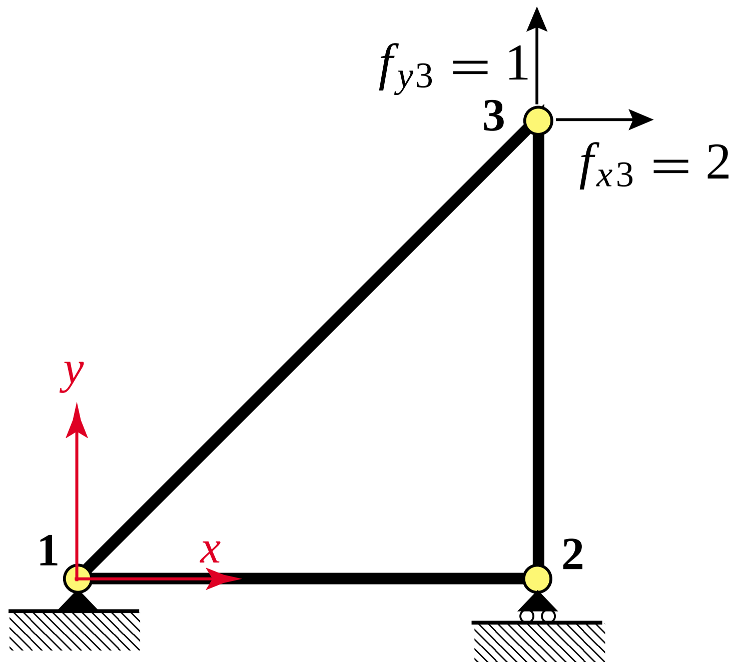
The Example Truss: Physical Model (Loads not shown)



The Example Truss - FEM Model: Nodes, Elements and DOFs



The Example Truss - FEM Model BCs:
Applied Loads and Supports
Saved for Last



Master (Global) Stiffness Equations

$$\mathbf{f} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Linear structure:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} K_{x1x1} & K_{x1y1} & K_{x1x2} & K_{x1y2} & K_{x1x3} & K_{x1y3} \\ K_{y1x1} & K_{y1y1} & K_{y1x2} & K_{y1y2} & K_{y1x3} & K_{y1y3} \\ K_{x2x1} & K_{x2y1} & K_{x2x2} & K_{x2y2} & K_{x2x3} & K_{x2y3} \\ K_{y2x1} & K_{y2y1} & K_{y2x2} & K_{y2y2} & K_{y2x3} & K_{y2y3} \\ K_{x3x1} & K_{x3y1} & K_{x3x2} & K_{x3y2} & K_{x3x3} & K_{x3y3} \\ K_{y3x1} & K_{y3y1} & K_{y3x2} & K_{y3y2} & K_{y3x3} & K_{y3y3} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Nodal forces **Master stiffness matrix** **Nodal displacements**

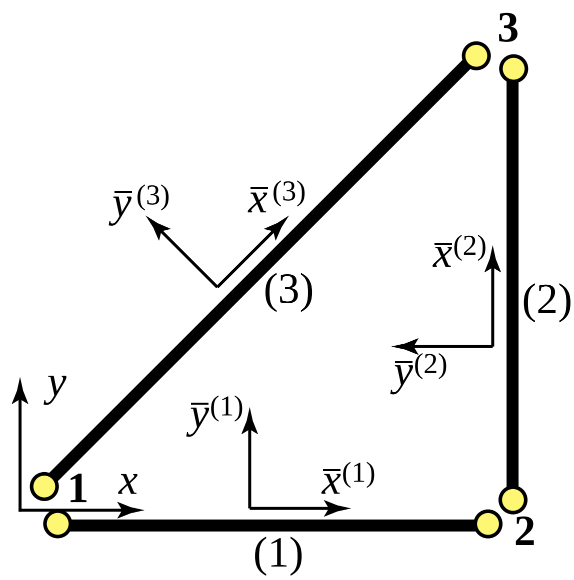
or $$\mathbf{f} = \mathbf{K}\,\mathbf{u}$$

Member (Element) Stiffness Equations

$$\bar{\mathbf{f}} = \bar{\mathbf{K}}\,\bar{\mathbf{u}}$$

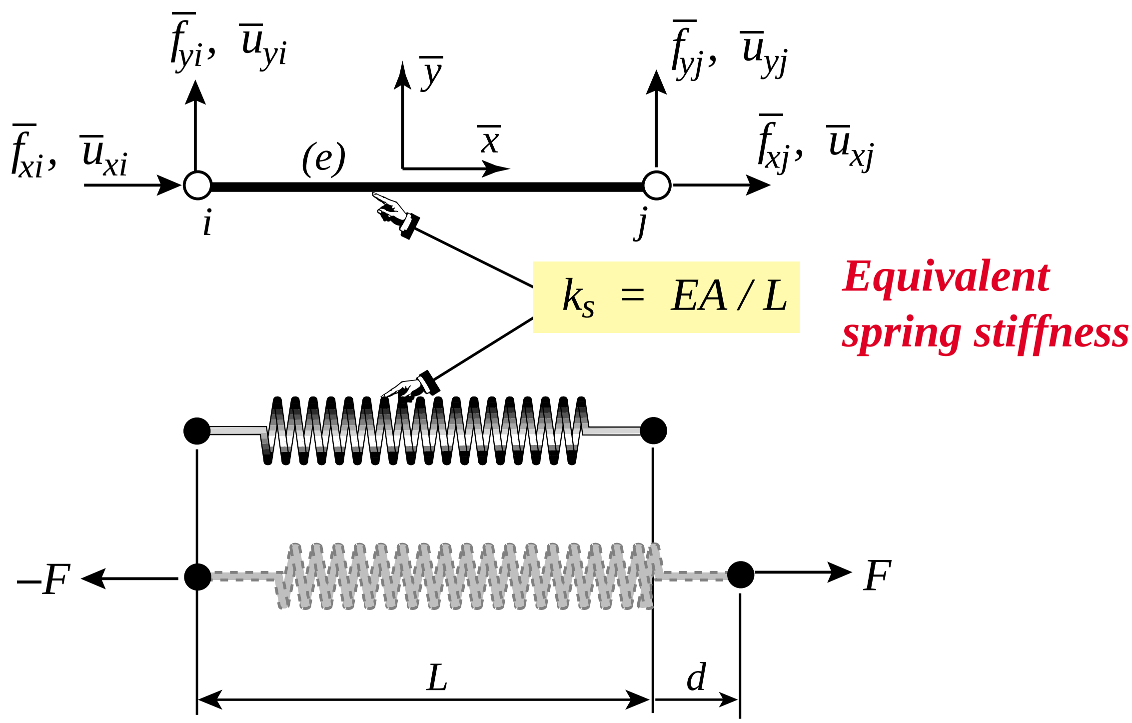
$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} \bar{K}_{xixi} & \bar{K}_{xiyi} & \bar{K}_{xixj} & \bar{K}_{xiyj} \\ \bar{K}_{yixi} & \bar{K}_{yiyi} & \bar{K}_{yixj} & \bar{K}_{yiyj} \\ \bar{K}_{xjxi} & \bar{K}_{xjyi} & \bar{K}_{xjxj} & \bar{K}_{xjyj} \\ \bar{K}_{yjxi} & \bar{K}_{yjyi} & \bar{K}_{yjxj} & \bar{K}_{yjyj} \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

First Two Breakdown Steps: Disconnection and Localization



These steps are conceptual (not actually programmed)

The 2-Node Truss (Bar) Element



Truss (Bar) Element Formulation by Mechanics of Materials (MoM)

$$F = k_s d = \frac{EA}{L} d, \quad F = \bar{f}_{xj} = -\bar{f}_{xi}, \quad d = \bar{u}_{xj} - \bar{u}_{xi}$$

Exercise 2.3

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

Element stiffness equations in local coordinates

from which

$$\overline{\mathbf{K}} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element stiffness matrix in local coordinates

Where We Are So Far in the DSM

we are done with this ...

Breakdown
(Chapter 2)

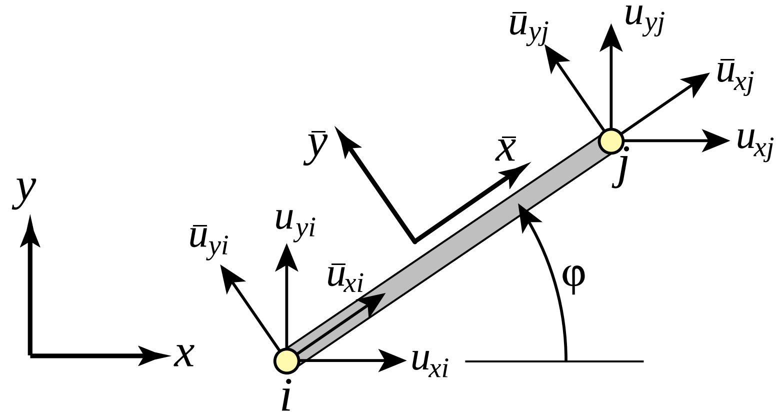
- **Disconnection**
- **Localization**
- **Member (Element) Formation**

we finish Chapter 2 with

Assembly &
Solution
(Chapter 3)

- **Globalization**
- **Merge**
- **Application of BCs**
- **Solution**
- **Recovery of Derived Quantities**

Globalization: Displacement Transformation



Node displacements transform as

$$\bar{u}_{xi} = u_{xi} c + u_{yi} s, \qquad \bar{u}_{yi} = -u_{xi} s + u_{yi} c\gamma$$

$$\bar{u}_{xj} = u_{xj} c + u_{yj} s, \qquad \bar{u}_{yj} = -u_{xj} s + u_{yj} c\gamma$$

in which $c = \cos\varphi \quad s = \sin\varphi$

Displacement Transformation (cont'd)

In matrix form

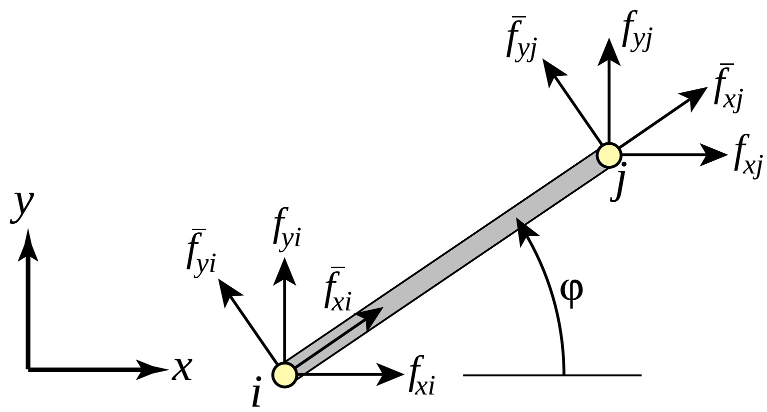
$$\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}$$

or

$$\bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e$$

Note:
global on RHS,
local on LHS

Globalization: Force Transformation



Node forces transform as

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}$$

Note:
global on LHS,
local on RHS

or

$$\mathbf{f}^e = (\mathbf{T}^e)^T \, \bar{\mathbf{f}}^e$$

Globalization: Congruential Transformation of Element Stiffness Matrices

$$\bar{\mathbf{K}}^e \ \mathbf{u}^e = \bar{\mathbf{f}}^e$$

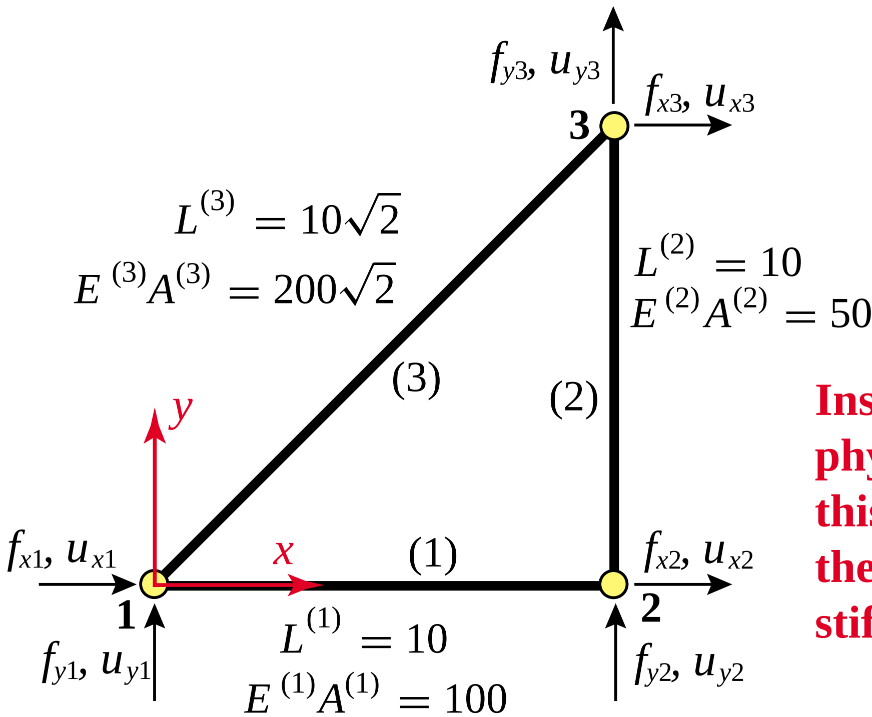
$$\bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e \qquad \mathbf{f}^e = (\mathbf{T}^e)^T \ \bar{\mathbf{f}}^e$$

Exercise 2.8

$$\mathbf{K}^e = (\mathbf{T}^e)^T \ \bar{\mathbf{K}}^e \ \mathbf{T}^e$$

$$\mathbf{K}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

The Example Truss - FEM Model (Recalled for Convenience)



Insert the geometric & physical properties of this model into the globalized member stiffness equations

We Obtain the Globalized Element Stiffness Equations of the Example Truss

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix}$$

In the next class we will put these to good use

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{bmatrix} = 20 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{bmatrix}$$