F

Fitting Element Fields

TABLE OF CONTENTS

§F.1. Introduction

This Appendix studies the problem of Polynomial Field Fitting over a finite element domain. This is a subset of the more general curve- and surface-fitting problem, as covered for example in [456]. This special case is defined as follows.

(I) A polynomial interpolation of a scalar function f, called the *base function*, is defined over a finite element domain using n nodal values of f and n shape functions expressed in natural coordinates.

(II) A polynomial interpolation of a scalar function g, called the *fitted function*, is defined over the same element domain in terms of $m \leq n$ nodal values and m shape functions. This function approximates the base function f in a least square sense.

The interpolations involved in (I) and (II) will be called the *base* and *fitted* interpolants of f and g, respectively.

The case $m = n$ is not excluded, as it covers situations in which the same polynomial order is defined from different nodal values. A well know example is the linear triangle with $n = 3$ corner values (the Turner triangle) versus that with $m = 3$ midpoint values (the Veubeke triangle).

Passing from f to g involves a *least square fitting*. This is an operation that has been extensively studied in the applied mathematics literature. However, most of the work available there does not consider the use of natural coordinates as well as integration over the element domain. These two features are essential for finite element applications.

Passing from g to f is a *collocation* operation, which simply involves evaluating g at the f nodes.

§F.2. Least-Square Fitting

Let the node values for the base and reduced interpolant be arranged in column vectors denoted by $\mathbf{f}$ and $\mathbf{g}$ of lengths n and $m \leq n$, respectively. The associated shape functions are arranged in row vectors $\mathbf{N}_f$ and $\mathbf{N}_g$, respectively. The shape functions of both f and g will be assumed to be linearly independent. The interpolations may be expressed as the dot products

$$f = \mathbf{N}_f\,\mathbf{f}, \qquad g = \mathbf{N}_g\,\mathbf{g}. \tag{F.1}$$

The distance between g and f is $d = g - f = \mathbf{N}_g\,\mathbf{g} - \mathbf{N}_f\,\mathbf{f}$. Its square is

$$d^2 = (\mathbf{N}_g\,\mathbf{g} - \mathbf{N}_f\,\mathbf{f})^T\,(\mathbf{N}_g\,\mathbf{g} - \mathbf{N}_f\,\mathbf{f}) = \mathbf{g}^T\,\mathbf{N}_g^T\,\mathbf{N}_g\mathbf{g} - 2\,\mathbf{g}^T\,\mathbf{N}_g^T\,\mathbf{N}_f\,\mathbf{f} + \mathbf{f}^T\,\mathbf{N}_f^T\,\mathbf{N}_f\,\mathbf{f}. \tag{F.2}$$

As the distance norm we take

$$s = \tfrac{1}{2}\int_\Omega d^2\,d\Omega = \tfrac{1}{2}\left(\mathbf{f}^T\,\mathbf{A}\,\mathbf{f} - 2\,\mathbf{g}^T\,\mathbf{B}\,\mathbf{f} + \mathbf{g}^T\,\mathbf{C}\mathbf{g}\right) = \tfrac{1}{2}\,[\,\mathbf{f}^T \quad \mathbf{g}^T\,]\begin{bmatrix}\mathbf{A} & -\mathbf{B}^T\\ -\mathbf{B} & \mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{f}\\ \mathbf{g}\end{bmatrix}. \tag{F.3}$$

in which Ω denotes the element domain, and

$$\mathbf{A} = \int_\Omega \mathbf{N}_f^T\,\mathbf{N}_f\;d\Omega, \quad \mathbf{B} = \int_\Omega \mathbf{N}_g^T\,\mathbf{N}_f\;d\Omega, \quad \mathbf{C} = \int_\Omega \mathbf{N}_g^T\,\mathbf{N}_g\;d\Omega. \tag{F.4}$$

Matrices $\mathbf{A}$ and $\mathbf{C}$ are called the *self kernels* for f and g, respectively. They are symmetric and positive definite (because the shape functions are assumed to be linearly independent). Matrix $\mathbf{B}$ of order $m \times n$ is the *coupled kernel*, which is rectangular unless $n = m$.[1]

Minimizing s with respect to $\mathbf{g}$ yields

$$\frac{\partial s}{\partial \mathbf{g}} = \mathbf{C}\,\mathbf{g} - \mathbf{B}\,\mathbf{f} = \mathbf{0}, \quad \text{whence} \quad \mathbf{g} = \mathbf{C}^{-1}\,\mathbf{B}\,\mathbf{f} = \mathbf{G}\,\mathbf{f}. \tag{F.5}$$

$\mathbf{G} = \mathbf{C}^{-1}\,\mathbf{B}$ will be called the $f \rightarrow g$ *fitting matrix*. When identification of orders m and n are convenient, the matrix will be denoted $\mathbf{G}_m^n$.

[1] Readers familiar with computational dynamics may recognize $\mathbf{A}$ and $\mathbf{C}$ as the numerical part of the consistent mass matrices pertaining to the base and fitted elements, respectively.

Minimizing s with respect to $\mathbf{f}$ yields

$$\frac{\partial s}{\partial \mathbf{f}} = \mathbf{A}\,\mathbf{f} - \mathbf{B}^T\,\mathbf{g} = \mathbf{0}, \quad \text{whence} \quad \mathbf{f} = \mathbf{A}^{-1}\,\mathbf{B}^T\,\mathbf{g} = \mathbf{F}\,\mathbf{g}. \tag{F.6}$$

$\mathbf{F} = \mathbf{A}^{-1}\,\mathbf{B}$ will be called the $g \to f$ *collocation matrix*. It is a generalized inverse of $\mathbf{G}$ since

$$\mathbf{G}\,\mathbf{F} = \mathbf{I}, \qquad \mathbf{F}\,\mathbf{G} = \mathbf{H}, \tag{F.7}$$

in which $\mathbf{I}$ is the $m \times m$ identity matrix whereas $\mathbf{H}$ is an $n \times n$ projector (idempotent) matrix satisfying $\mathbf{H} = \mathbf{H}^2$. Matrix $\mathbf{H}$, which has rank m, is not generally symmetric.

Replacing the expressions of $\mathbf{G}$ and $\mathbf{F}$ into $\mathbf{G}\,\mathbf{F} = \mathbf{I}$ yields the property

$$\mathbf{C}^{-1}\mathbf{B}\,\mathbf{A}^{-1}\,\mathbf{B}^T = \mathbf{I}, \quad \text{or} \quad \mathbf{B}^T\,\mathbf{A}^{-1}\,\mathbf{B} = \mathbf{C}, \tag{F.8}$$

whereas replacing the expressions of $\mathbf{G}$ and $\mathbf{F}$ into $\mathbf{F}\,\mathbf{G} = \mathbf{H}$ yields the property

$$\mathbf{A}^{-1}\mathbf{B}^T\,\mathbf{C}^{-1}\,\mathbf{B} = \mathbf{H}, \quad \text{or} \quad \mathbf{B}^T\,\mathbf{C}^{-1}\,\mathbf{B} = \mathbf{A}\,\mathbf{H}. \tag{F.9}$$

Notice that $\mathbf{B}^T\,\mathbf{A}^{-1}\,\mathbf{B}$ and $\mathbf{B}^T\,\mathbf{C}^{-1}\,\mathbf{B}$ are the Schur complements of $\mathbf{C}$ and $\mathbf{A}$, respectively, in the supermatrix shown in (F.3). Since $\mathbf{B}^T\,\mathbf{C}^{-1}\,\mathbf{B}$ is symmetric, so is $\mathbf{A}\,\mathbf{H}$. It can be readily shown that the reverse product $\mathbf{H}\,\mathbf{A}$ is antisymmetric.

Inserting (F.5) into (F.3) gives the minimum distance as

$$s_{min} = \tfrac{1}{2}\,\mathbf{f}^T\,(\mathbf{A} - \mathbf{B}^T\,\mathbf{C}^{-1}\,\mathbf{B})\,\mathbf{f} = \tfrac{1}{2}\,\mathbf{f}^T\,(\mathbf{A} - \mathbf{A}\,\mathbf{H})\,\mathbf{f} = \tfrac{1}{2}\,\mathbf{f}^T\,\mathbf{A}\,\hat{\mathbf{H}}\,\mathbf{f}, \tag{F.10}$$

in which $\hat{\mathbf{H}} = \mathbf{I} - \mathbf{H}$. Since $\mathbf{H}$ is a projection matrix, so is $\hat{\mathbf{H}}$. As a quick check, take $\mathbf{N}_f \equiv \mathbf{N}_g$, whence $m = n$ and $\mathbf{C} = \mathbf{B} = \mathbf{A}$; if so $\mathbf{H} = \mathbf{I}$ and $s_{min} = 0$, as can be expected.

§F.3. Line Elements

Line elements are used for one-dimensional FEM models. In structural mechanics: bars, beams, shafts, etc. This Section tabulates results for the seven line elements shown in Figure F.1. The line segment is assumed straight with constant metric and length L^e along x. As a consequence, the integrals over the line can be evaluated exactly, without resource to numerical integration.

Elements labeled `"LLn"` are Lagrangian, with function values at equally spaced n nodes, which include the end points if $n \geq 2$. Elements labeled `"LLnG"` have all-internal node values at `n`≥`2` Gauss points.[2] For all `"LLn"`-to-`"LLm"` mappings, it is convenient to set $L_e = 1$ *ab initio* to reduce clutter. If L_e is left arbitrary, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are scaled by it, but it cancels out in $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$.

That labeled `"HL4"` is the standard Hermite cubic-interpolated element with 4 DOF: two end-node values and two length-scaled x-derivatives at the end nodes: $[\,f_1 = (df/dx)_1\,L^e = \theta_1\,L^e,\ f_2 = (df/dx)_2\,L^e = \theta_2\,L^e]$. This scaling allows presetting $L_e = 1$ again in the fitting process. Should L_e be retained as symbolic value, all fitting matrices would contain entry-dependent powers of L_e. Such dependence may be introduced later using diagonal scaling matrices, as preparation for assembly, but for individual element fitting it works fine.

§F.3.1. Line Element Processing Modules

The shape functions for the line elements shown in Figure F.1 are returned by the *Mathematica* module `LineShapeFunctions` listed in Figure F.2. The module is invoked as

```
{Nsf,m,ζnc}=LineShapeFunctions[typ,ξ,Le]                (F.11)
```

in which

`typ` One of the identifiers shown in Figure F.1 supplied as a character string (e.g., `"LL3"`)

[2] The one-point line element is denoted simply by `"LL1"` rather than `"LL1G"` since no confusion can arise. These “Gauss elements” have specialized uses.

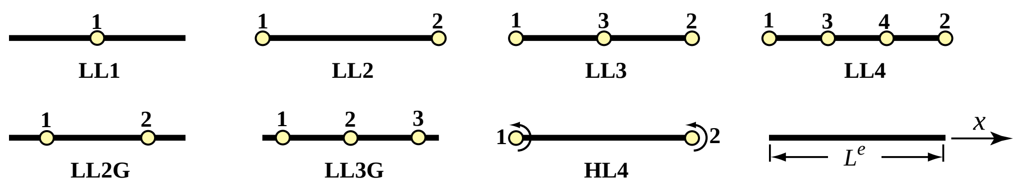

FIGURE F.1. Line element notation. LL and HL denote Lagrangian and Hermitian interpolation, respectively. Nodes shown as circles have only one DOF: the function value there. Nodes pictured with a rotation ("harpoon") symbol have two DOF there: the function value and its x-derivative scaled by the line length L^e.

```
LineShapeFunctions[typ_,ξ_]:=Module[{Nsf=Null,m=Null,ξnc=Null},
   If [typ=="LL1", Nsf={1}; m=0; ξnc={{0}}];
   If [typ=="LL2", Nsf={(1-ξ)/2,(1+ξ)/2}; m=1; ξnc={{-1},{1}}];
   If [typ=="LL3", Nsf={-(1-ξ)*ξ/2,(1+ξ)*ξ/2,1-ξ^2}; m=2;
          ξnc={{-1},{1},{0}}];
   If [typ=="LL4", Nsf={(1-ξ)*(9*ξ^2-1),(1+ξ)*(9*ξ^2-1),
          9*(3*ξ-1)*(ξ^2-1),9*(3*ξ+1)*(1-ξ^2)}/16; m=3;
          ξnc={{-1},{1},{-1/3},{1/3}}];
   If [typ=="LL2G", Nsf={1-Sqrt[3]*ξ,1+Sqrt[3]*ξ}/2;
          m=1; ξnc={{-Sqrt[1/3]},{Sqrt[1/3]}}];
   If [typ=="LL3G", Nsf={ξ*(-Sqrt[15]+5*ξ),6-10*ξ^2,ξ*(Sqrt[15]+5*ξ)}/6;
          m=2; ξnc={{-Sqrt[3/5]},{0},{Sqrt[3/5]}}];
   If [typ=="HL4", Nsf={2*(1-ξ)^2*(2+ξ),(1-ξ)^2*(1+ξ),2*(1+ξ)^2*(2-ξ),
        -(1+ξ)^2*(1-ξ)}/8; m=3; ξnc={{-1},{-1},{1},{1}}];
   Return[{Nsf,m,ξnc}]];
```

FIGURE F.2. Line element shape function module.

```
IntegrateOverLine[f_,ξ_,Le_]:=Module[{fint},
     fint=Integrate[(Le/2)*f,{ξ,-1,1}]; Return[fint]];

LineFit[ftyp_,gtyp_,ξ_,Le_]:=Module[{Nf={},Ng={},mf,mg,ζf,ζg,
  Am,Bm,Cm,Ainv,Cinv,F,G,GF,FG},
  {Nf,mf,ζf}=LineShapeFunctions[ftyp,ξ];
  {Ng,mg,ζg}=LineShapeFunctions[gtyp,ξ];
  If [Nf==Null, Print["Unknown ftyp=",ftyp]; Return[Null]];
  If [Ng==Null, Print["Unknown gtyp=",gtyp]; Return[Null]];
  Am=IntegrateOverLine[Transpose[{Nf}].{Nf},ξ,Le]; Ainv=Inverse[Am];
  Bm=IntegrateOverLine[Transpose[{Ng}].{Nf},ξ,Le]; BmT=Transpose[Bm];
  Cm=IntegrateOverLine[Transpose[{Ng}].{Ng},ξ,Le]; Cinv=Inverse[Cm];
  G=Cinv.Bm;  F=Ainv.BmT; FG=F.G; GF=G.F;
  Return[{Am,Bm,Cm,G,F,GF,FG}]];
```

FIGURE F.3. Line element fitting module.

ξ The isoparametric natural coordinate.

`Le` The element length. Set to 1 for fittings, but kept just in case.

The module returns

`Nfs` A list of shape functions.

`m` The polynomial order of the shape functions.

`ξnc` A list of isoparametric coordinates of the nodes.

If argument `typ` is not implemented, `Nsf` returns `Null`.

The polynomial fit operation is carried out by the *Mathematica* module `LTFit` listed in Figure F.3. The module is invoked as

```
{A,B,C,G,F,FG,GF}=LineFit[ftyp,gtyp,ξ,Le]
```
(F.12)

in which

`ftyp,gtyp` Identifiers of the base and fitted line element, respectively.

ξ The isoparametric natural coordinate.

`Le` The line segment length. Only relevant if `ftyp` or `gtyp` is `"HL4"`, but typically is set to 1 to reduce clutter in the fitting matrices.

The module returns a list of seven matrices: $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{G}$, $\mathbf{F}$, $\mathbf{F\,G}$ (which should be the identity), and $\mathbf{GF} = \mathbf{H}$ (which should be a projector). These matrices were introduced in §F.2.

§F.3.2. Line Self-Kernel Matrices

Self kernels are the matrices $\mathbf{A}$ and $\mathbf{C}$ defined in (F.4). Since $\mathbf{C} = \mathbf{A}$ if $\mathbf{N}_f \equiv \mathbf{N}_g$, only $\mathbf{A}$ is listed. Its subscript identifies the line element.

$$\mathbf{A}_{\rm LL1} = [\,1\,],\quad \mathbf{A}_{\rm LL2} = \frac{1}{6}\begin{bmatrix} 2 & 1\\ 1 & 2\end{bmatrix},\quad \mathbf{A}_{\rm LL3} = \frac{1}{30}\begin{bmatrix} 4 & -1 & 2\\ -1 & 4 & 2\\ 2 & 2 & 16\end{bmatrix},$$
$$\mathbf{A}_{\rm LL4} = \frac{1}{1680}\begin{bmatrix} 128 & 19 & 99 & -36\\ 19 & 128 & -36 & 99\\ 99 & -36 & 648 & -81\\ -36 & 99 & -81 & 648\end{bmatrix},\quad \mathbf{A}_{\rm LL2G} = \frac{1}{2}\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}, \tag{F.13}$$
$$\mathbf{A}_{\rm LL3G} = \frac{1}{18}\begin{bmatrix} 5 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 5\end{bmatrix},\quad \mathbf{A}_{\rm HL4} = \frac{1}{420}\begin{bmatrix} 156 & 22 & 54 & -13\\ 22 & 4 & 13 & -3\\ 54 & 13 & 156 & -22\\ -13 & -3 & -22 & 4\end{bmatrix},$$

Note that the $\mathbf{A}$ matrices for LL2G and LL3G are diagonal, since shape functions for Gauss collocated nodes are mutually orthogonal.

§F.3.3. Line Coupled Matrices

Some combinations are not listed.

Source: LL2, Fitted: LL2G

$$\mathbf{B} = \frac{1}{12}\begin{bmatrix} a_1 & a_2\\ a_2 & a_1\end{bmatrix},\quad \mathbf{G} = \frac{1}{6}\begin{bmatrix} a_1 & a_2\\ a_2 & a_1\end{bmatrix},\quad \mathbf{F} = \frac{1}{2}\begin{bmatrix} a_3 & a_4\\ a_4 & a_3\end{bmatrix}, \tag{F.14}$$

in which $a_1 = 3+\sqrt{3}$, $a_2 = 3-\sqrt{3}$, $a_3 = 1+\sqrt{3}$, and $a_4 = 1+\sqrt{3}$. **Source: LL2, Fitted: LL1**

$$\mathbf{B} = \frac{1}{2}\,[\,1 \quad 1\,],\quad \mathbf{G} = \frac{1}{2}\,[\,1 \quad 1\,],\quad \mathbf{F} = \begin{bmatrix} 1\\ 1\end{bmatrix}. \tag{F.15}$$

Source: LL3, Fitted: LL3G

$$\mathbf{B} = \frac{1}{36}\begin{bmatrix} b_1 & b_1 & 4\\ 0 & 0 & 16\\ b_1 & b_1 & 4\end{bmatrix},\quad \mathbf{G} = \frac{1}{10}\begin{bmatrix} b_1 & b_1 & 4\\ 0 & 0 & 10\\ b_1 & b_1 & 4\end{bmatrix},\quad \mathbf{F} = \frac{1}{6}\begin{bmatrix} b_3 & -4 & b_4\\ b_4 & -4 & b_3\\ 0 & 6 & 0\end{bmatrix}, \tag{F.16}$$

in which $b_1 = 3 + \sqrt{15}$, $b_2 = 3 - \sqrt{15}$, $b_3 = 5 + \sqrt{15}$, and $b_4 = 5 - \sqrt{15}$.

Source: LL3, Fitted: LL2

$$\mathbf{B} = \frac{1}{6}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}. \tag{F.17}$$

Source: LL3, Fitted: LL2G

$$\mathbf{B} = \frac{1}{12}\begin{bmatrix} a_3 & a_4 & 4 \\ a_4 & a_3 & 4 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{6}\begin{bmatrix} a_3 & a_4 & 4 \\ a_4 & a_3 & 4 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{2}\begin{bmatrix} a_3 & a_4 \\ a_4 & a_3 \\ 1 & 1 \end{bmatrix}, \tag{F.18}$$

in which $a_1 = 3 + \sqrt{3}$, $a_2 = 3 - \sqrt{3}$, $a_3 = 1 + \sqrt{3}$, and $a_4 = 1 + \sqrt{3}$.

Source: LL3, Fitted: LL1

$$\mathbf{B} = \frac{1}{6}\,[\,1 \quad 1 \quad 4\,], \quad \mathbf{G} = \frac{1}{6}\,[\,1 \quad 1 \quad 4\,], \quad \mathbf{F} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{F.19}$$

Source: LL4, Fitted: LL3

$$\mathbf{B} = \frac{1}{120}\begin{bmatrix} 11 & 0 & 18 & -9 \\ 0 & 11 & -9 & 18 \\ 4 & 4 & 36 & 36 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{80}\begin{bmatrix} 62 & 18 & 54 & -54 \\ 18 & 62 & -54 & 54 \\ -5 & -5 & 45 & 45 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 2 & -1 & 8 \\ -1 & 2 & 8 \end{bmatrix}. \tag{F.20}$$

Source: LL4, Fitted: LL3G

$$\mathbf{B} = \frac{1}{720}\begin{bmatrix} h_1 & h_2 & h_3 & h_4 \\ -20 & -20 & 180 & 180 \\ h_2 & h_1 & h_4 & h_3 \end{bmatrix}, \mathbf{G} = \frac{1}{400}\begin{bmatrix} h_5 & h_6 & h_7 & h_8 \\ -25 & -25 & 225 & 225 \\ h_6 & h_5 & h_8 & h_7 \end{bmatrix}, \mathbf{F} = \frac{1}{54}\begin{bmatrix} h_9 & -36 & h_{10} \\ h_{10} & -36 & h_9 \\ h_{11} & 44 & h_{12} \\ h_{12} & 44 & h_{11} \end{bmatrix}. \tag{F.21}$$

in which $h_1 = 55 + 11\sqrt{15}$, $h_2 = 55 - 11\sqrt{15}$, $h_3 = 45 + 27\sqrt{15}$, $h_4 = 45 - 27\sqrt{15}$, $h_5 = 110 + 22\sqrt{15}$, $h_6 = 110 - 22\sqrt{15}$, $h_7 = 90 + 54\sqrt{15}$, $h_8 = 90 - 54\sqrt{15}$, $h_9 = 45 + 9\sqrt{15}$, $h_{10} = 45 - 9\sqrt{15}$, $h_{11} = 5 + 3\sqrt{15}$, and $h_{12} = 5 - 3\sqrt{15}$.

Source: LL4, Fitted: LL2

$$\mathbf{B} = \frac{1}{120}\begin{bmatrix} 13 & 2 & 36 & 9 \\ 2 & 13 & 9 & 36 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{20}\begin{bmatrix} 8 & -3 & 21 & -6 \\ -3 & 8 & -6 & 21 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{F.22}$$

Source: LL4, Fitted: LL2G

$$\mathbf{B} = \frac{1}{240}\begin{bmatrix} g_1 & g_2 & g_3 & g_4 \\ g_2 & g_1 & g_4 & g_3 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{120}\begin{bmatrix} g_1 & g_2 & g_3 & g_4 \\ g_2 & g_1 & g_4 & g_3 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{2}\begin{bmatrix} g_5 & g_6 \\ g_6 & g_5 \\ g_5 & g_6 \\ g_6 & g_5 \end{bmatrix} \tag{F.23}$$

in which $g_1 = 15 + 11\sqrt{3}$, $g_2 = 15 - 11\sqrt{3}$, $g_3 = 45 + 27\sqrt{3}$, $g_4 = 45 - 27\sqrt{3}$, $g_5 = 1 + \sqrt{3}$, and $g_6 = 1 - \sqrt{3}$.

Source: LL4, Fitted: LL1

$$\mathbf{B} = \frac{1}{8}\,[\,1 \quad 1 \quad 3 \quad 3\,], \quad \mathbf{G} = \frac{1}{8}\,[\,1 \quad 1 \quad 3 \quad 3\,], \quad \mathbf{F} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{F.24}$$

Source: HL4, Fitted: LL4

$$\mathbf{B} = \frac{1}{840}\begin{bmatrix} 96 & 6 & 9 & -1 \\ 9 & 1 & 96 & -6 \\ 279 & 45 & 36 & -18 \\ 36 & 18 & 279 & -45 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{27}\begin{bmatrix} 27 & 0 & 0 & 0 \\ 0 & 0 & 27 & 0 \\ 20 & 4 & 7 & -2 \\ 7 & 2 & 20 & -4 \end{bmatrix},$$
$$\mathbf{F} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ -11 & 2 & 18 & -9 \\ 0 & 2 & 0 & 0 \\ -2 & 11 & 9 & -18 \end{bmatrix}. \tag{F.25}$$

Source: HL4, Fitted: LL3

$$\mathbf{B} = \frac{1}{60}\begin{bmatrix} 11 & 1 & -1 & 0 \\ -1 & 0 & 11 & -1 \\ 20 & 4 & 20 & -4 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{40}\begin{bmatrix} 44 & 2 & -4 & 2 \\ -4 & -2 & 44 & -2 \\ 20 & 5 & 20 & -5 \end{bmatrix},$$
$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & -1 & 4 \\ 0 & 1 & 0 \\ 1 & 3 & -4 \end{bmatrix}. \tag{F.26}$$

Source: HL4, Fitted: LL3G

$$\mathbf{B} = \frac{1}{360}\begin{bmatrix} p_1 & p_2 & p_3 & -p_2 \\ 80 & 20 & 80 & -20 \\ p_3 & p_4 & p_1 & -p_4 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{200}\begin{bmatrix} 4\,p_5 & 2\,p_2 & 4\,p_6 & -2\,p_2 \\ 100 & 25 & 100 & -25 \\ 4\,p_6 & 2\,p_2 & 4\,p_5 & -2\,p_2 \end{bmatrix},$$
$$\mathbf{F} = \frac{1}{6}\begin{bmatrix} p_2 & -4 & p_4 \\ -2\,p_7 & 40 & -2\,p_8 \\ p_4 & -4 & p_2 \\ 2\,p_8 & -40 & 2\,p_7 \end{bmatrix}. \tag{F.27}$$

in which $p_1 = 50+12\sqrt{15}$, $p_2 = 5+\sqrt{15}$, $p_3 = 50-12\sqrt{15}$, $p_4 = 5-\sqrt{15}$, $p_5 = 25+6\sqrt{15}$, $p_6 = 25-6\sqrt{15}$, $p_7 = 10+\sqrt{15}$, and $p_8 = 10-\sqrt{15}$.

Source: HL4, Fitted: LL2

$$\mathbf{B} = \frac{1}{60}\begin{bmatrix} 21 & 3 & 9 & -2 \\ 9 & 2 & 21 & -3 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{30}\begin{bmatrix} 33 & 4 & -3 & -1 \\ -3 & 1 & 33 & -4 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \tag{F.28}$$

Source: HL4, Fitted: LL2G

$$\mathbf{B} = \frac{1}{360}\begin{bmatrix} p_1 & p_2 & p_3 & -p_2 \\ 80 & 20 & 80 & -20 \\ p_3 & p_4 & p_1 & -p_4 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{200}\begin{bmatrix} 4\,p_5 & 2\,p_2 & 4\,p_6 & -2\,p_2 \\ 100 & 25 & 100 & -25 \\ 4\,p_6 & 2\,p_2 & 4\,p_5 & -2\,p_2 \end{bmatrix},$$
$$\mathbf{F} = \frac{1}{6}\begin{bmatrix} p_2 & -4 & p_4 \\ -2\,p_7 & 40 & -2\,p_8 \\ p_4 & -4 & p_2 \\ 2\,p_8 & -40 & 2\,p_7 \end{bmatrix}. \tag{F.29}$$

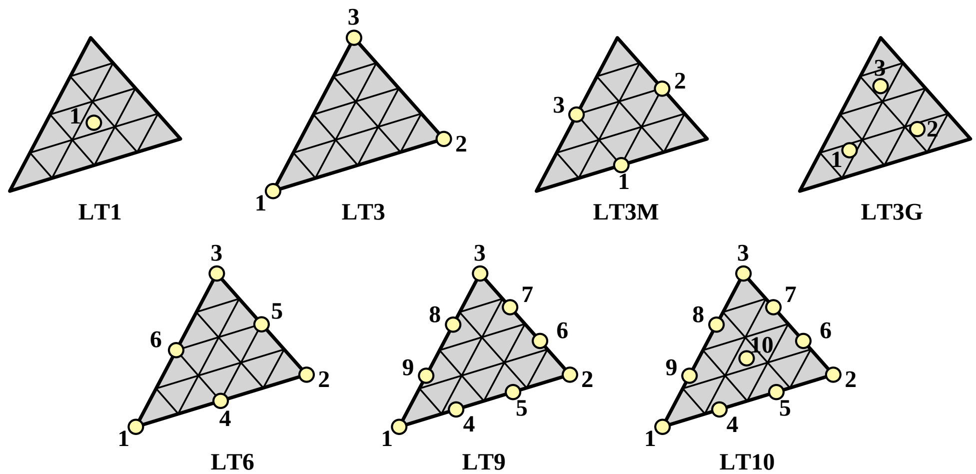

FIGURE F.4. Notation for Lagrangian triangles considered in §F.4. Figures taken from [213].

```
LTShapeFunctions[typ_,tcoor_]:=Module[{ζ1,ζ2,ζ3,Nsf=Null,m=Null,
  ζnc=Null}, {ζ1,ζ2,ζ3}=tcoor;
  If [typ=="LT1",  Nsf={1}; m=0; ζnc={{0,0,0}}];
  If [typ=="LT3",  Nsf={ζ1,ζ2,ζ3}; m=1; ζnc={{1,0,0},{0,1,0},{0,1,0}}];
  If [typ=="LT3M", Nsf={ζ1+ζ2-ζ3,-ζ1+ζ2+ζ3,ζ1-ζ2+ζ3}; m=1;
      ζnc={{1/2,1/2,0},{0,1/2,1/2},{1/2,0,1/2}}];
  If [typ=="LT3G", Nsf={5*ζ1-ζ2-ζ3,-ζ1+5*ζ2-ζ3,-ζ1-ζ2+5*ζ3}/3; m=1;
      ζnc={{2/3,1/6,1/6},{1/6,2/3,1/6},{1/6,1/6,2/3}}];
  If [typ=="LT6",  Nsf={ζ1*(2*ζ1-1),ζ2*(2*ζ2-1),ζ3*(2*ζ3-1),
      4*ζ1*ζ2,4*ζ2*ζ3,4*ζ3*ζ1}; m=2; ζnc={{1,0,0},{0,1,0},{0,0,1},
           {1/2,1/2,0},{0,1/2,1/2},{1/2,0,1/2}}];
  If [typ=="LT9", Nsf={ζ1*(3*ζ1-1)*(3*ζ1-2),ζ2*(3*ζ2-1)*(3*ζ2-2),
      ζ3*(3*ζ3-1)*(3*ζ3-2),9*ζ1*ζ2*(3*ζ1-1),9*ζ1*ζ2*(3*ζ2-1),
      9*ζ2*ζ3*(3*ζ2-1),9*ζ2*ζ3*(3*ζ3-1),9*ζ3*ζ1*(3*ζ3-1),
      9*ζ3*ζ1*(3*ζ1-1)}/2+9*{-2,-2,-2,3,3,3,3,3,3}*ζ1*ζ2*ζ3/4; m=3;
      ζnc={{1,0,0},{0,1,0},{0,0,1},{2/3,1/3,0},{1/3,2/3,0},
           {0,2/3,1/3},{0,1/3,2/3},{2/3,0,1/3},{1/3,0,2/3}}];
  If [typ=="LT10", Nsf={ζ1*(3*ζ1-1)*(3*ζ1-2),ζ2*(3*ζ2-1)*(3*ζ2-2),
      ζ3*(3*ζ3-1)*(3*ζ3-2),9*ζ1*ζ2*(3*ζ1-1),9*ζ1*ζ2*(3*ζ2-1),
      9*ζ2*ζ3*(3*ζ2-1),9*ζ2*ζ3*(3*ζ3-1),9*ζ3*ζ1*(3*ζ3-1),
      9*ζ3*ζ1*(3*ζ1-1),54*ζ1*ζ2*ζ3}/2; m=3;
      ζnc={{1,0,0},{0,1,0},{0,0,1},{2/3,1/3,0},{1/3,2/3,0},{0,2/3,1/3},
           {0,1/3,2/3},{2/3,0,1/3},{1/3,0,2/3},{1/3,1/3,1/3}}];
  Return[{Nsf,m,ζnc}]];
```

FIGURE F.5. Module `LTShapeFunctions` that returns shape function information for the Lagrange triangles of Figure F.4.

in wich the p_i have the same values as in the HL4 → LL3G fitting.

Source: HL4, Fitted: LL1

$$\mathbf{B} = \frac{1}{12}\,[\,6 \quad 1 \quad 6 \quad -1\,], \quad \mathbf{G} = \frac{1}{12}\,[\,6 \quad 1 \quad 6 \quad -1\,], \quad \mathbf{F} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{F.30}$$

§F.4. Lagrangian Triangles

A *Lagrangian triangle* (LT) is one interpolated from node values of the function only. They are primarily used for 2D problems of variational index 1, such as plane stress and heat conduction. This Section tabulates results for the seven triangular elements shown in Figure F.4. It assumes flat triangles with constant metric, in which the area integrations can be performed exactly, without resource to numerical integration.

§F.4.1. Lagrangian Triangle Processing Modules

```
IntegrateOverTriangle[expr_,A_,max_]:=Module[{p,i,j,k,c,s=0},
   p=Expand[expr];
   For [i=0,i<=max,i++, For [j=0,j<=max,j++, For [k=0,k<=max,k++,
       c=Coefficient[Coefficient[Coefficient[p,1,i],2,j],3,k];
       s+=2*c*(i!*j!*k!)/((i+j+k+2)!);
       ]]];
   Return[Simplify[A*s]];
LTFit[ftyp_,gtyp_,tcoor_]:=Module[{Nf={},Ng={},mf,mg,ζf,ζg,
  Am,Bm,Cm,Ainv,Cinv,F,G,GF,FG},
  {Nf,mf,ζf}=LTShapeFunctions[ftyp,tcoor];
  {Ng,mg,ζg}=LTShapeFunctions[gtyp,tcoor];
  If [Nf==Null, Print["Unknown ftyp=",ftyp]; Return[Null]];
  If [Ng==Null, Print["Unknown gtyp=",gtyp]; Return[Null]];
  Am=IntegrateOverTriangle[Transpose[{Nf}].{Nf},1,mf+mf]; Ainv=Inverse[Am];
  Bm=IntegrateOverTriangle[Transpose[{Ng}].{Nf},1,mg+mf]; BmT=Transpose[Bm];
  Cm=IntegrateOverTriangle[Transpose[{Ng}].{Ng},1,mg+mg]; Cinv=Inverse[Cm];
  G=Cinv.Bm; F=Ainv.BmT; FG=F.G; GF=G.F;
  Return[{Am,Bm,Cm,G,F,GF,FG}]];
```

FIGURE F.6. Module `LTFit` that returns the fitting matrices introduced in §F.2 for Lagrangian triangles. `IntegrateOverTriangle` is an auxiliary module.

The shape functions for the LT shown in Figure F.4 are returned by the *Mathematica* module `LTShapeFunctions` listed in Figure F.5. The module is invoked as

`{ Nsf,m,ζnc }=LTShapeFunctions[typ,tcoor]` (F.31)

in which

- `typ` One of the LT identifiers shown in Figure F.4 supplied as a character string (e.g., `"LT3"`)
- `tcoor` A 3-item list of triangular coordinate symbols, usually `{ ζ1,ζ2,ζ3 }`.

The module returns

- `Nfs` A list of shape functions.
- `m` The polynomial order of the shape functions.
- `ζnc` A list of triangular coordinates of the nodes.

If argument `typ` is not implemented, `Nsf` returns `Null`.

The polynomial fit operation is carried out by the *Mathematica* module `LTFit` listed in Figure F.6. The module is invoked as

`{ A,B,C,G,F,FG,GF }=LTFit[ftyp,gtyp,tcoor]` (F.32)

in which

- `ftyp,gtyp` Identifiers of the base and fitted triangle, respectively.
- `tcoor` A 3-item list of triangular coordinate symbols. The module returns a list of seven matrices: $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{G}$, $\mathbf{F}$, $\mathbf{F\,G}$ (which should be the identity), and $\mathbf{GF} = \mathbf{H}$ (which should be a projector). These matrices were introduced in §F.2.

To reduce clutter, the triangle area is set to 1. If the area A were retained as symbolic variable, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ would be scaled by A, but that would cancel out in $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$.

Module `IntegrateOverTriangle` evaluates analyticaly the area integral of a polynomial function of the triangular coordinates, as explained in IFEM Chapter 15.

Remark F.1. The shape functions for LT9 and LT10 in triangular coordinates were derived in the author's 1966 thesis [213]. Those for LT3 and LT6, due to Argyris and Fraeijs de Veubeke, respectively, were published one year earlier. (Their expressions in Cartesian coordinates were known earlier). The shape functions of LL3M and LT3G can be trivially derived from those of LT3.

§F.4.2. Lagrangian Triangle Self Kernels

Self kernels are the matrices $\mathbf{A}$ and $\mathbf{C}$ defined in (F.4). Since $\mathbf{C} = \mathbf{A}$ if $\mathbf{N}_f \equiv \mathbf{N}_g$, only $\mathbf{A}$ is listed.

1-Node Triangle with Centroid Node. Labeled LT1 in Figure F.4.

$$\mathbf{A} = [\,1\,]. \tag{F.33}$$

3-Node Triangle With Corner Values. Labeled LT3 in Figure F.4.

$$\mathbf{A} = \frac{1}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}. \tag{F.34}$$

3-Node Triangle With Midpoint Nodes. Labeled LT3M in Figure F.4.

$$\mathbf{A} = \frac{1}{3}\,\mathbf{I}_3. \tag{F.35}$$

3-Node Triangle With Interior Gauss Nodes. Labeled LT3G in Figure F.4.

$$\mathbf{A} = \frac{1}{3}\,\mathbf{I}_3. \tag{F.36}$$

6-Node Triangle. Labeled LT6 in Figure F.4.

$$\mathbf{A} = \frac{1}{180}\begin{bmatrix} 6 & -1 & -1 & 0 & -4 & 0 \\ -1 & 6 & -1 & 0 & 0 & -4 \\ -1 & -1 & 6 & -4 & 0 & 0 \\ 0 & 0 & -4 & 32 & 16 & 16 \\ -4 & 0 & 0 & 16 & 32 & 16 \\ 0 & -4 & 0 & 16 & 16 & 32 \end{bmatrix}, \tag{F.37}$$

9-Node Triangle. Labeled LT9 in Figure F.4.

$$\mathbf{A} = \frac{1}{13440}\begin{bmatrix} 236 & 106 & 106 & -162 & -198 & -144 & -144 & -198 & -162 \\ 106 & 236 & 106 & -198 & -162 & -162 & -198 & -144 & -144 \\ 106 & 106 & 236 & -144 & -144 & -198 & -162 & -162 & -198 \\ -162 & -198 & -144 & 1485 & 27 & 135 & 297 & 135 & 945 \\ -198 & -162 & -144 & 27 & 1485 & 945 & 135 & 297 & 135 \\ -144 & -162 & -198 & 135 & 945 & 1485 & 27 & 135 & 297 \\ -144 & -198 & -162 & 297 & 135 & 27 & 1485 & 945 & 135 \\ -198 & -144 & -162 & 135 & 297 & 135 & 945 & 1485 & 27 \\ -162 & -144 & -198 & 945 & 135 & 297 & 135 & 27 & 1485 \end{bmatrix}. \tag{F.38}$$

10-Node Triangle. Labeled LT10 in Figure F.4.

$$\mathbf{A} = \frac{1}{6720}\begin{bmatrix} 76 & 11 & 11 & 18 & 0 & 27 & 27 & 0 & 18 & 36 \\ 11 & 76 & 11 & 0 & 18 & 18 & 0 & 27 & 27 & 36 \\ 11 & 11 & 76 & 27 & 27 & 0 & 18 & 18 & 0 & 36 \\ 18 & 0 & 27 & 540 & -189 & -135 & -54 & -135 & 270 & 162 \\ 0 & 18 & 27 & -189 & 540 & 270 & -135 & -54 & -135 & 162 \\ 27 & 18 & 0 & -135 & 270 & 540 & -189 & -135 & -54 & 162 \\ 27 & 0 & 18 & -54 & -135 & -189 & 540 & 270 & -135 & 162 \\ 0 & 27 & 18 & -135 & -54 & -135 & 270 & 540 & -189 & 162 \\ 18 & 27 & 0 & 270 & -135 & -54 & -135 & -189 & 540 & 162 \\ 36 & 36 & 36 & 162 & 162 & 162 & 162 & 162 & 162 & 1944 \end{bmatrix}. \tag{F.39}$$

§F.4.3. Lagrangian Triangle Fitting Matrices

Not all combinations are listed.

Base: LT3, Fitted: LT1. Matrices **A** and **C** are given by (F.34) and (F.33), respectively.

$$\mathbf{B} = \mathbf{G} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T. \tag{F.40}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_1$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 1.

Base: LT3, Fitted: LT3M. Matrices **A** and **C** are given by (F.34) and (F.35), respectively.

$$\mathbf{B} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{G} = 3\,\mathbf{B}, \quad \mathbf{F} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}. \tag{F.41}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G} = \mathbf{I}_3$.

Base: LT3, Fitted: LT3G. Matrices **A** and **C** are given by (F.34) and (F.36), respectively.

$$\mathbf{B} = \frac{1}{18}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \quad \mathbf{G} = \frac{1}{6}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{3}\begin{bmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{bmatrix}. \tag{F.42}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G} = \mathbf{I}_3$.

Base: LT6, Fitted: LT3. Matrices **A** and **C** are given by (F.37) and (F.34), respectively.

$$\begin{aligned} \mathbf{B} &= \frac{1}{60}\begin{bmatrix} 2 & -1 & -1 & 8 & 4 & 8 \\ -1 & 2 & -1 & 8 & 8 & 4 \\ -1 & -1 & 2 & 4 & 8 & 8 \end{bmatrix}, \\ \mathbf{G} &= \frac{1}{5}\begin{bmatrix} 2 & -1 & -1 & 3 & -1 & 3 \\ -1 & 2 & -1 & 3 & 3 & -1 \\ -1 & -1 & 2 & -1 & 3 & 3 \end{bmatrix}, \quad \mathbf{F} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{bmatrix}^T \end{aligned} \tag{F.43}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 3.

Base: LT6, Fitted: LT3M. Matrices **A** and **C** are given by (F.37) and (F.35), respectively.

$$\mathbf{B} = \frac{1}{30}\begin{bmatrix} 1 & 1 & -2 & 6 & 2 & 2 \\ -2 & 1 & 1 & 2 & 6 & 2 \\ 1 & -2 & 1 & 2 & 2 & 6 \end{bmatrix}, \quad \mathbf{G} = 3\,\mathbf{B}, \quad \mathbf{F} = \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}^T, \tag{F.44}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 3.

Base: LT6, Fitted: LT3G. Matrices **A** and **C** are given by (F.37) and (F.36), respectively.

$$\mathbf{B} = \frac{1}{90}\begin{bmatrix} 6 & -3 & -3 & 14 & 2 & 14 \\ -3 & 6 & -3 & 14 & 14 & 2 \\ -3 & -3 & 6 & 2 & 14 & 14 \end{bmatrix}, \quad \mathbf{G} = 3\,\mathbf{B}, \quad \mathbf{F} = \frac{1}{3}\begin{bmatrix} 5 & -1 & -1 & 2 & -1 & 2 \\ -1 & 5 & -1 & 2 & 2 & -1 \\ -1 & -1 & 5 & -1 & 2 & 2 \end{bmatrix}^T. \tag{F.45}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 3.

Base: LT6, Fitted: LT1. Matrices **A** and **C** are given by (F.37) and (F.33), respectively.

$$\mathbf{B} = \mathbf{G} = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T. \tag{F.46}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_1$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 1.

Base: LT9, Fitted: LT6. Matrices **A** and **C** are given by (F.38) and (F.37), respectively.

$$
\mathbf{B} = \frac{1}{1680}\begin{bmatrix} 26 & 8 & 8 & 45 & -45 & -21 & -21 & -45 & 45 \\ 8 & 26 & 8 & -45 & 45 & 45 & -45 & -21 & -21 \\ 8 & 8 & 26 & -21 & -21 & -45 & 45 & 45 & -45 \\ -40 & -40 & -32 & 168 & 168 & 120 & 48 & 48 & 120 \\ -32 & -40 & -40 & 48 & 120 & 168 & 168 & 120 & 48 \\ -40 & -32 & -40 & 120 & 48 & 48 & 120 & 168 & 168 \end{bmatrix},
$$

$$
\mathbf{G} = \frac{1}{112}\begin{bmatrix} 64 & 24 & 24 & 72 & -72 & 0 & 0 & -72 & 72 \\ 24 & 64 & 24 & -72 & 72 & 72 & -72 & 0 & 0 \\ 24 & 24 & 64 & 0 & 0 & -72 & 72 & 72 & -72 \\ -10 & -10 & 6 & 63 & 63 & 9 & -9 & -9 & 9 \\ 6 & -10 & -10 & -9 & 9 & 63 & 63 & 9 & -9 \\ -10 & 6 & -10 & 9 & -9 & -9 & 9 & 63 & 63 \end{bmatrix}, \tag{F.47}
$$

$$
\mathbf{F} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 & 2 & -1 & 0 & 0 & -1 & 2 \\ 0 & 9 & 0 & -1 & 2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & -1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 \end{bmatrix}^T
$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_6$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 6.

Base: LT9, Fitted: LT3. Matrices **A** and **C** are given by (F.38) and (F.34), respectively.

$$
\mathbf{B} = \frac{1}{240}\begin{bmatrix} -2 & -4 & -4 & 27 & 9 & 9 & 9 & 9 & 27 \\ -4 & -2 & -4 & 9 & 27 & 27 & 9 & 9 & 9 \\ -4 & -4 & -2 & 9 & 9 & 9 & 27 & 27 & 9 \end{bmatrix},
$$

$$
\mathbf{G} = \frac{1}{80}\begin{bmatrix} 2 & -6 & -6 & 63 & -9 & -9 & -9 & -9 & 63 \\ -6 & 2 & -6 & -9 & 63 & 63 & -9 & -9 & -9 \\ -6 & -6 & 2 & -9 & -9 & -9 & 63 & 63 & -9 \end{bmatrix}, \tag{F.48}
$$

$$
\mathbf{F} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 2 \\ 0 & 3 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix}^T .
$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 3.

Base: LT9, Fitted: LT3M. Matrices **A** and **C** are given by (F.38) and (F.35), respectively.

$$
\mathbf{B} = \frac{1}{240}\begin{bmatrix} -2 & -2 & -6 & 27 & 27 & 27 & -9 & -9 & 27 \\ -6 & -2 & -2 & -9 & 27 & 27 & 27 & 27 & -9 \\ -2 & -6 & -2 & 27 & -9 & -9 & 27 & 27 & 27 \end{bmatrix}, \quad \mathbf{G} = 3\mathbf{B},
$$

$$
\mathbf{F} = \frac{1}{3}\begin{bmatrix} 3 & 3 & -3 & 3 & 3 & 1 & -1 & -1 & 1 \\ -3 & 3 & 3 & -1 & 1 & 3 & 3 & 1 & -1 \\ 3 & -3 & 3 & 1 & -1 & -1 & 1 & 3 & 3 \end{bmatrix}^T . \tag{F.49}
$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 3.

Base: LT9, Fitted: LT3G. Matrices **A** and **C** are given by (F.38) and (F.36), respectively.

$$\mathbf{B} = \frac{1}{720}\begin{bmatrix} -2 & -14 & -14 & 117 & 9 & 9 & 9 & 9 & 117 \\ -14 & -2 & -14 & 9 & 117 & 117 & 9 & 9 & 9 \\ -14 & -14 & -2 & 9 & 9 & 9 & 117 & 117 & 9 \end{bmatrix}, \quad \mathbf{G} = 3\mathbf{B},$$
$$\mathbf{F} = \frac{1}{3}\begin{bmatrix} 5 & -1 & -1 & 3 & 1 & -1 & -1 & 1 & 3 \\ -1 & 5 & -1 & 1 & 3 & 3 & 1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & 1 & 3 & 3 & 1 \end{bmatrix}^T. \tag{F.50}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 3.

Base: LT9, Fitted: LT1. Matrices **A** and **C** are given by (F.38) and (F.33), respectively.

$$\mathbf{B} = \mathbf{G} = \frac{1}{48}\,[\,-2 \quad -2 \quad -2 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9\,],$$
$$\mathbf{F} = [\,1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\,]^T. \tag{F.51}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_1$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 1.

Base: LT10, Fitted: LT9. Matrices **A** and **C** are given by (F.39) and (F.38), respectively.

$$\mathbf{B} = \frac{1}{13440}\begin{bmatrix} 140 & 10 & 10 & -18 & -54 & 0 & 0 & -54 & -18 & -576 \\ 10 & 140 & 10 & -54 & -18 & -18 & -54 & 0 & 0 & -576 \\ 10 & 10 & 140 & 0 & 0 & -54 & -18 & -18 & -54 & -576 \\ 54 & 18 & 72 & 1161 & -297 & -189 & -27 & -189 & 621 & 1296 \\ 18 & 54 & 72 & -297 & 1161 & 621 & -189 & -27 & -189 & 1296 \\ 72 & 54 & 18 & -189 & 621 & 1161 & -297 & -189 & -27 & 1296 \\ 72 & 18 & 54 & -27 & -189 & -297 & 1161 & 621 & -189 & 1296 \\ 18 & 72 & 54 & -189 & -27 & -189 & 621 & 1161 & -297 & 1296 \\ 54 & 72 & 18 & 621 & -189 & -27 & -189 & -297 & 1161 & 1296 \end{bmatrix},$$
$$\mathbf{G} = \frac{1}{70}\begin{bmatrix} 64 & -6 & -6 & 9 & 9 & 9 & 9 & 9 & 9 & -36 \\ -6 & 64 & -6 & 9 & 9 & 9 & 9 & 9 & 9 & -36 \\ -6 & -6 & 64 & 9 & 9 & 9 & 9 & 9 & 9 & -36 \\ 4 & 4 & 4 & 64 & -6 & -6 & -6 & -6 & -6 & 24 \\ 4 & 4 & 4 & -6 & 64 & -6 & -6 & -6 & -6 & 24 \\ 4 & 4 & 4 & -6 & -6 & 64 & -6 & -6 & -6 & 24 \\ 4 & 4 & 4 & -6 & -6 & -6 & 64 & -6 & -6 & 24 \\ 4 & 4 & 4 & -6 & -6 & -6 & -6 & 64 & -6 & 24 \\ 4 & 4 & 4 & -6 & -6 & -6 & -6 & -6 & 64 & 24 \end{bmatrix}, \tag{F.52}$$
$$\mathbf{F} = \frac{1}{12}\begin{bmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \\ -2 & -2 & -2 & 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}.$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_9$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 9.

Base: LT10, Fitted: LT6. Matrices **A** and **C** are given by (F.39) and (F.37), respectively.

$$\mathbf{B} = \frac{1}{840}\begin{bmatrix} 10 & 1 & 1 & 27 & -18 & -6 & -6 & -18 & 27 & -18 \\ 1 & 10 & 1 & -18 & 27 & 27 & -18 & -6 & -6 & -18 \\ 1 & 1 & 10 & -6 & -6 & -18 & 27 & 27 & -18 & -18 \\ 4 & 4 & 8 & 48 & 48 & 24 & -12 & -12 & 24 & 144 \\ 8 & 4 & 4 & -12 & 24 & 48 & 48 & 24 & -12 & 144 \\ 4 & 8 & 4 & 24 & -12 & -12 & 24 & 48 & 48 & 144 \end{bmatrix},$$

$$\mathbf{G} = \frac{1}{280}\begin{bmatrix} 136 & 36 & 36 & 216 & -144 & 36 & 36 & -144 & 216 & -144 \\ 36 & 136 & 36 & -144 & 216 & 216 & -144 & 36 & 36 & -144 \\ 36 & 36 & 136 & 36 & 36 & -144 & 216 & 216 & -144 & -144 \\ -4 & -4 & 36 & 126 & 126 & -9 & -54 & -54 & -9 & 126 \\ 36 & -4 & -4 & -54 & -9 & 126 & 126 & -9 & -54 & 126 \\ -4 & 36 & -4 & -9 & -54 & -54 & -9 & 126 & 126 & 126 \end{bmatrix}, \tag{F.53}$$

$$\mathbf{F} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 & 2 & -1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 9 & 0 & -1 & 2 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 9 & 0 & 0 & -1 & 2 & 2 & -1 & -1 \\ 0 & 0 & 0 & 8 & 8 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 4 \end{bmatrix}^T.$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_6$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 6.

Base: LT10, Fitted: LT3. Matrices **A** and **C** are given by (F.39) and (F.34), respectively.

$$\mathbf{B} = \frac{1}{120}\begin{bmatrix} 2 & 1 & 1 & 9 & 0 & 0 & 0 & 0 & 9 & 18 \\ 1 & 2 & 1 & 0 & 9 & 9 & 0 & 0 & 0 & 18 \\ 1 & 1 & 2 & 0 & 0 & 0 & 9 & 9 & 0 & 18 \end{bmatrix},$$

$$\mathbf{G} = \frac{1}{40}\begin{bmatrix} 4 & 0 & 0 & 27 & -9 & -9 & -9 & -9 & 27 & 18 \\ 0 & 4 & 0 & -9 & 27 & 27 & -9 & -9 & -9 & 18 \\ 0 & 0 & 4 & -9 & -9 & -9 & 27 & 27 & -9 & 18 \end{bmatrix}, \tag{F.54}$$

$$\mathbf{F} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 3 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 1 & 2 & 2 & 1 & 1 \end{bmatrix}^T.$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_3$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 3.

Base: LT10, Fitted: LT1. Matrices **A** and **C** are given by (F.39) and (F.33), respectively.

$$\mathbf{B} = \mathbf{G} = \frac{1}{120}\,[\,4 \quad 4 \quad 4 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 54\,], \qquad \mathbf{F} = [\,1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\,]^T. \tag{F.55}$$

$\mathbf{G}\,\mathbf{F} = \mathbf{I}_1$, $\mathbf{H} = \mathbf{F}\,\mathbf{G}$ = projection matrix of rank 1.

§F.5. Kirchhoff Triangles

A Kirchhoff triangle (KT) is one primarily used for modeling thin plates that fit the Kirchhoff model, hence the name. Their characteristic feature is the appearance of function derivatives at corner nodes, and possibly at other nodes. If the function represents the transverse displacement of a thin plate, those derivatives may be often interpreted as infinitesimal rotations.

In contrast to the Lagrangian elements studied in previous sections, Kirchhoff triangles bring up two complications:

(I) They are not easily organized into families.

(II) Fit is more geometry dependent because of the presence of derivatives.

Complication (I) means that identifiers are necessarily more individual, and leads to longer character strings. Complication (II) means that the fitting matrices carry geometry dependent information, while that was avoidable in Lagrangian elements.

The elements treated in this section are shown in Figure **?**.