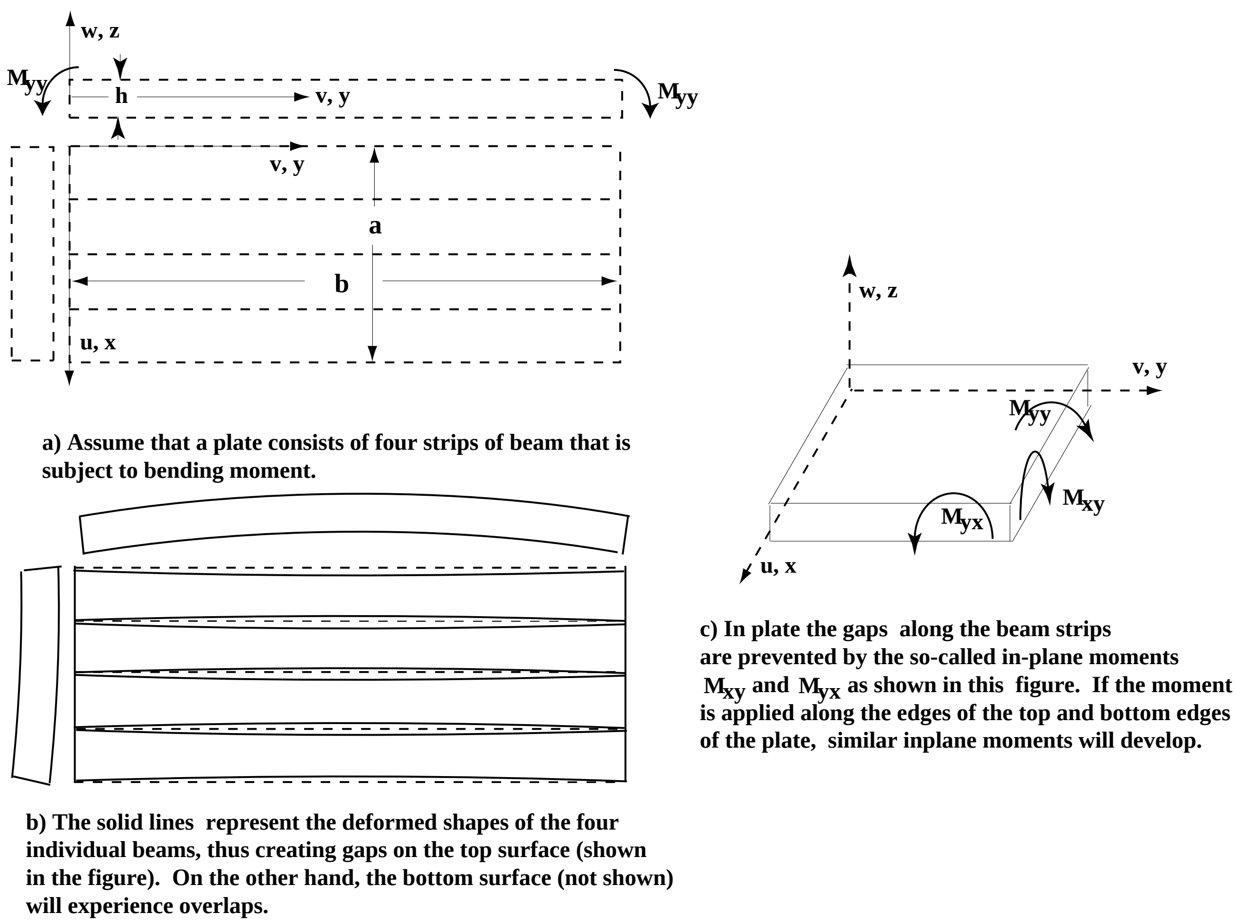
17

MODELING FOR PLATE VIBRATIONS

§17.1 INTRODUCTION

Plate vibrations are distinct from those of beams in that each vibration mode shape consists of two perpendicular motions. In a rectangular plate each mode shape is a product of two functions, with one defined along x coordinate and the other along y-coordinate. For a circular plate each mode shape consists of a product of a radial and circumferential function. In other words, a vibration mode typically induces motions in two directions. This can be understood by examining plate motions as follows.



a) Assume that a plate consists of four strips of beam that is subject to bending moment.

b) The solid lines represent the deformed shapes of the four individual beams, thus creating gaps on the top surface (shown in the figure). On the other hand, the bottom surface (not shown) will experience overlaps.

c) In plate the gaps along the beam strips are prevented by the so-called in-plane moments M_{xy} and M_{yx} as shown in this figure. If the moment is applied along the edges of the top and bottom edges of the plate, similar inplane moments will develop.

Fig. 17.1 Plate bending phenomenon as distinct from beam bending

Consider a thought experiment as illustrated in Fig. 17.1, which shows a plate consisting of four beam strips. If the beam strips were individually subject to bending M_{yy} along $x = (0, a)$ as indicated in Fig. 17.1a while constraining the cross sections of the beam ends, the top surfaces of the beam $z = h/2$ will have shrunken as a result of Poisson's ratio. The bottom surfaces $z = -h/2$, on the other hand, will want to expand, thus creating overlaps. Since the beam strips are part of the plate, the crevices created by the individual beam bending cannot occur. In terms of equilibrium considerations, this is only possible by the resisting shear stresses, which in turn create moments M_{xy} and M_{yx} as indicated in Fig. 17.1c. Note that if two edge moments M_{xx} are applied along $y = (0, b)$, similar twisting moments M_{xy} and M_{yx} would have developed.

§17.2 STRAIN-DISPLACEMENT AND CONSTITUTIVE RELATIONS FOR A THIN PLATE

The thin plate theory assumes

$$
\begin{aligned}
u &= u_0(x, y) - z\,\frac{\partial w}{\partial x} \\
v &= v_0(x, y) - z\,\frac{\partial w}{\partial y} \\
w &= w(x, y)
\end{aligned}
\tag{17.1}
$$

where u, v and w are the displacements along the x, y and z coordinates; $u_0(x, y)$ and $v_0(x, y)$ represent the stretching the plate neutral surface; z is measured from the neutral surface of the plate, respectively.

The strain-displacement relations are obtained as

$$
\begin{aligned}
\epsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2} \\
\epsilon_{yy} &= \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial x} - z\,\frac{\partial^2 w}{\partial y^2} \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} - 2z\,\frac{\partial^2 w}{\partial x \partial y} \\
\gamma_{xz} &= \gamma_{yz} = 0
\end{aligned}
\tag{17.2}
$$

For an isotropic material, the constitutive relations are modeled by

$$
\sigma_{xx} = E(\epsilon_{xx} - \nu\epsilon_{yy}), \qquad \sigma_{yy} = E(\epsilon_{yy} - \nu\epsilon_{xx}), \qquad \tau_{xy} = G\gamma_{xy}
$$

where σ_{xx}, σ_{xx} and τ_{xy} are stress components; E and G are Young's modulus and shear modulus; and, ν is Poisson's ratio.

§17.3 VARIATIONAL FORMULATION OF PLATE VIBRATIONS

The bending strain energy of a plate can be expressed as

$$
\delta V_b = \int_V \{\, \delta\epsilon_{xx}\; E\; (\epsilon_{xx} - \nu\epsilon_{yy}) + \delta\epsilon_{yy}\; E\; (\epsilon_{yy} - \nu\epsilon_{xx}) + \delta\gamma_{xy}\; G\; \epsilon_{xy}\} dV \tag{17.3}
$$

When the plate is sufficiently thin and the applied loads do no cause the neutral surface to stretch, the above expression reduces, with $u_0 = v_0 = 0$, to

$$
\begin{aligned}
\delta V_b = \int_A D\, \{\, [\, (\frac{\partial^2 w}{\partial x^2} &+ \frac{\partial^2 w}{\partial y^2}) - (1-\nu)\frac{\partial^2 w}{\partial y^2}] \cdot\; \delta\frac{\partial^2 w}{\partial x^2} \\
&+ [\, (\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}) - (1-\nu)\frac{\partial^2 w}{\partial x^2}] \cdot\; \delta\frac{\partial^2 w}{\partial y^2} \\
&+ 2(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \cdot\; \delta\frac{\partial^2 w}{\partial x \partial y}\; \} dA \\
D = &\frac{E h^3}{12(1-\nu^2)}
\end{aligned}
\tag{17.4}
$$

where D is called plate bending stiffness, h is the plate thickness , and A is the plate area.

In carrying out the indicated variations, we make use of the identities:

$$\begin{aligned}\int_A G(w)\,\delta\frac{\partial w}{\partial x}\,dA &= \int_\Gamma \ell\,G(w)\delta w\,d\Gamma - \int_A \frac{\partial G(w)}{\partial x}\delta w dA\\ \int_A G(w)\,\delta\frac{\partial w}{\partial y}\,dA &= \int_\Gamma m\,G(w)\delta w\,d\Gamma - \int_A \frac{\partial G(w)}{\partial y}\delta w dA\end{aligned} \tag{17.5}$$

where ℓ and m are the directional cosines of the unit normal vector outward from the surface contour Γ. Hence, δV_b can be shown to be

$$\begin{aligned}\delta V_b =& \int_\Gamma [\ell\; D(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) + m\,(1-\nu)\;D\;\frac{\partial^2 w}{\partial x\partial y}]\cdot\;\delta\frac{\partial w}{\partial x}\,d\Gamma\\ &+ \int_\Gamma [m\;D(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}) + \ell\,(1-\nu)\;D\;\frac{\partial^2 w}{\partial x\partial y}]\cdot\;\delta\frac{\partial w}{\partial y}\,d\Gamma\\ &- \int_\Gamma [\ell\;D\frac{\partial}{\partial x}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) + \ell(1-\nu)\;D\;\frac{\partial^2 w}{\partial x\partial y^2}]\cdot\;\delta w\,d\Gamma\\ &- \int_\Gamma [m\;D\frac{\partial}{\partial y}(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}) + m(1-\nu)\;D\;\frac{\partial^2 w}{\partial x^2\partial y}]\cdot\;\delta w\,d\Gamma\\ &+ \int_A D\,\nabla^4 w\cdot\delta w dA,\quad \nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}\end{aligned} \tag{17.6}$$

In order to identify the natural and essential boundary conditions, we introduce the following definitions:

$$\begin{aligned}M_{xx} &= -D(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2})\\ M_{yy} &= -D(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2})\\ M_{xy} &= -D(1-\nu)\;\;\frac{\partial^2 w}{\partial x\partial y^2}\\ Q_x &= \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y}\\ Q_y &= \frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x}\end{aligned} \tag{17.7}$$

so that (17.6) simplifies to

$$\begin{aligned}\delta V_b = -& \int_\Gamma \{\;[\ell\;M_{xx} + m\;M_{xy}]\cdot\;\delta\frac{\partial w}{\partial x}\; + [m\;M_{yy} + \ell\;M_{xy}]\cdot\;\delta\frac{\partial w}{\partial y}\;\}\,d\Gamma\\ &+ \int_\Gamma \{\;[\ell\;\frac{\partial}{\partial x}M_{xx} + \ell\;\frac{\partial}{\partial y}M_{xy}] + [m\;\frac{\partial}{\partial y}M_{yy} + m\;\frac{\partial}{\partial x}M_{xy}]\;\}\;\cdot\;\delta w\,d\Gamma\\ &+ \int_A D\,\nabla^4 w\cdot\delta w dA\end{aligned} \tag{17.8}$$

In order for the above expression to be valid for all possible boundary contour shapes including a circular plate, we introduce the following transformation (see Fig. 17.2)

$$\begin{aligned}\frac{\partial}{\partial x} &= \ell\frac{\partial}{\partial n} - m\frac{\partial}{\partial s}\\ \frac{\partial}{\partial y} &= m\frac{\partial}{\partial n} + \ell\frac{\partial}{\partial s}, \qquad \ell = \cos\phi, \quad m = \sin\phi\end{aligned} \tag{17.9}$$

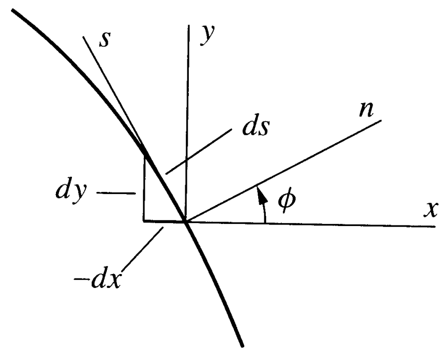


Fig. 17.2 Tangential and normal directions at a plate boundary

Substituting the transformation (17.9) into (17.8) we obtain

$$\begin{aligned}\delta V_b &= \int_s [\,-M_n\,\delta\frac{\partial w}{\partial n} \;-\; M_{ns}\,\delta\frac{\partial w}{\partial s} + Q_n\,\delta w\,]\,ds + \int_A D\,\nabla^4 w\cdot\delta w dA\\ M_n &= \ell^2\,M_{xx} + 2\ell m\,M_{xy} + m^2\,M_{yy}\\ M_{ns} &= \ell m\,(M_{yy} - M_{xx}) + (\ell^2 - m^2)\,M_{xy}\\ Q_n &= \ell\,Q_x + m\,Q_y\\ \nabla^2 &= \frac{\partial^2}{\partial n^2} + \frac{1}{R}\,\frac{\partial}{\partial n} + \frac{\partial^2}{\partial s^2}, \qquad \frac{1}{R} = \frac{\partial\phi}{\partial s}\end{aligned} \tag{17.10}$$

Note that via (17.5) we have

$$\int_s M_{ns}\,\delta\frac{\partial w}{\partial s}\,ds = M_{ns}\,\delta w|_\Gamma - \int_s \frac{\partial M_{ns}}{\partial s}\delta w\,ds \tag{17.11}$$

If the boundary contour Γ is smooth and closed, the virtual work done by the edge twisting moment M_{ns} should vanish. Otherwise, a gap along the contour would exist.

Remark: For rectangular plates, there exist discontinuities at four corners, known as corner conditions, which states that $M_{xy} = 0$. This condition must be satisfied either explicitly or implicitly.

Thus, for a smooth closed contour the variation of the strain energy becomes

$$\delta V_b = \int_s [\,-M_n\,\delta\frac{\partial w}{\partial n} + (Q_n + \frac{\partial M_{ns}}{\partial s})\delta w\,]\,ds + \int_A D\,\nabla^4 w\cdot\delta w dA \tag{17.12}$$

The kinetic energy δT, the energy due to the boundary unknown springs δV_s, and the work performed by the applied forces δW can be expressed in a similar manner as for the case of beam in the form of

$$\int_{t_1}^{t_2} \delta T \; dt = -\int_A \int_{t_1}^{t_2} m(x,y)\; \ddot{w}(x,y,t)\; \delta w(x,t)\; dt\; dA \tag{17.13}$$

The variation of the potential energy of the unknown boundary forces and moments are given by

$$\delta V_s = \int_s [\; k_w \; w \; \delta w + k_\theta \; \frac{\partial w}{\partial n} \cdot \delta \frac{\partial w}{\partial n} \;]\; ds \tag{17.14}$$

The work performed by the applied force

$$\delta W = \int_A f(x,y,t) \delta w(x,y,t) dA \tag{17.15}$$

Hamilton's principle for a continuum plate can be written as

$$\begin{aligned} &\int_{t_1}^{t_2} \{ -\int_A (D\nabla^4 w + m\ddot{w} - \; f)\; \delta w \; dA \\ &+ \int_s [\; -(V_n + k_w \; w\;)\; \delta w + (M_n - k_\theta \; \frac{\partial w}{\partial n})\; \delta \frac{\partial w}{\partial n} \;]\; ds \;\}\; dt = 0 \end{aligned} \tag{17.16}$$

where (V_n, M_n) are given by

$$\begin{aligned} V_n &= -D \frac{\partial}{\partial n} \nabla^2 w - (1-\nu)\; D \; \frac{\partial}{\partial s} \; (\frac{\partial^2 w}{\partial n \partial s} - \frac{1}{R} \frac{\partial w}{\partial s}) \\ M_n &= -D\nabla^2 w + (1-\nu)\; D(\frac{1}{R} \frac{\partial w}{\partial s} + \frac{\partial^2 w}{\partial s^2}) \end{aligned} \tag{17.17}$$

The governing equation of motion for a plate can thus be obtained as

$$D\nabla^4 w + m\ddot{w} = \; f \tag{17.18}$$

with the boundary conditions:

$$\begin{aligned} &(V_n + k_w \; w\;) = 0 \\ &(M_n - k_\theta \; \frac{\partial w}{\partial n}) = 0 \end{aligned} \tag{17.19}$$

§17.4 FREE VIBRATIONS OF A RECTANGULAR PLATE

Free vibrations of a rectangular plate can be studied by specializing the boundary conditions (17.19) according to the convention shown in Fig. 17.3.

First, the solution of the homogeneous equation of motion for the plate with $f = 0$ in (17.18) may assume

$$w(x, y, t) = W(x, y)e^{j\omega t}$$
$$\Downarrow$$
$$(\nabla^4 - \beta^4)W(x, y) = 0, \quad \beta^4 = \frac{\omega^2 m}{D} \tag{17.20}$$
$$\Downarrow$$
$$(\nabla^2 + \beta^2)(\nabla^2 - \beta^2)W(x, y) = 0$$

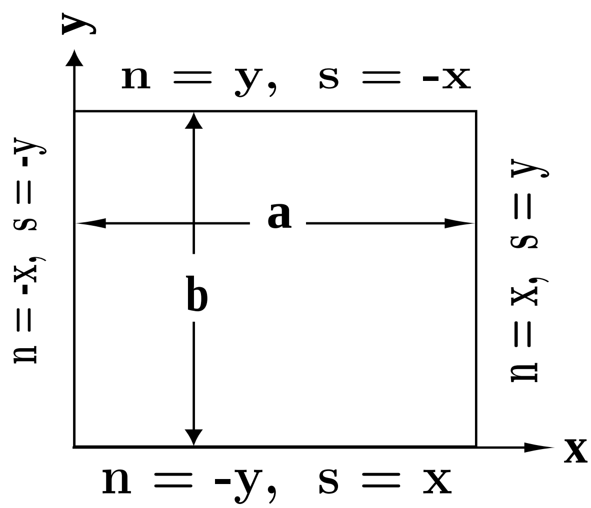


Fig. 17.3 Boundary variable convention for a rectangular plate

Hence, its solution consists of

$$\begin{aligned}
&(\nabla^2 + \beta^2)W_1 = 0 \quad \Rightarrow \quad W_1 = e^{j\alpha x}e^{j\gamma y}, \quad \alpha^2 + \gamma^2 = \beta^2\\
&(\nabla^2 - \beta^2)W_2 = 0 \quad \Rightarrow \quad W_2 = e^{\alpha_1 x}e^{\gamma_1 y}, \quad \alpha_1^2 + \gamma_1^2 = \beta^2\\
&W = W_1 + W_2\\
&\qquad\Downarrow\\
W(x, y) =& A_1 \sin\alpha x \sin\gamma y + A_2 \sin\alpha x \cos\gamma y\\
&+ A_3 \cos\alpha x \sin\gamma y + A_4 \cos\alpha x \cos\gamma y\\
&+ A_5 \sinh\alpha_1 x \sinh\gamma_1 y + A_6 \sinh\alpha_1 x \cosh\gamma_1 y\\
&+ A_7 \cosh\alpha_1 x \sinh\gamma_1 y + A_8 \cosh\alpha_1 x \cosh\gamma_1 y
\end{aligned} \tag{17.21}$$

Specializing the boundary condition(17.19) to a rectangular case requires the conversion of (n, s) in terms of (x, y) along the four edges as shown in Fig. 17.3. The boundary conditions along the four edges are obtained from (17.19) as follows.

A. Along $y = 0$ we have ($s = x, n = -y$):

$$\begin{aligned} V_y = D\frac{\partial}{\partial y}\nabla^2 w + (1-\nu)\ D\ \frac{\partial^3 w}{\partial y \partial x^2} \quad &\Rightarrow \quad [D\frac{\partial}{\partial y}\nabla^2 w + (1-\nu)\ D\ \frac{\partial^3 w}{\partial y \partial x^2} + k_w w\ ]|_{(x,0)} = 0 \\ M_y = -D(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}) \quad &\Rightarrow \quad [\, -D(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}) + k_\theta\ \frac{\partial w}{\partial y}\ ]|_{(x,0)} = 0 \end{aligned} \tag{17.22}$$

B. Along $y = b$ we have ($s = -x, n = y$):

$$\begin{aligned} V_y = -[D\frac{\partial}{\partial y}\nabla^2 w + (1-\nu)\ D\ \frac{\partial^3 w}{\partial y \partial x^2}] \quad &\Rightarrow \quad [D\frac{\partial}{\partial y}\nabla^2 w + (1-\nu)\ D\ \frac{\partial^3 w}{\partial y \partial x^2} - k_w w\ ]|_{(x,b)} = 0 \\ M_y = -D(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}) \quad &\Rightarrow \quad [\, -D(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}) - k_\theta\ \frac{\partial w}{\partial y}\ ]|_{(x,b)} = 0 \end{aligned} \tag{17.23}$$

C. Along $x = 0$ we have ($s = -y, n = -x$):

$$\begin{aligned} V_x = [D\frac{\partial}{\partial x}\nabla^2 w + (1-\nu)\ D\ \frac{\partial^3 w}{\partial x \partial y^2}] \quad &\Rightarrow \quad [D\frac{\partial}{\partial x}\nabla^2 w + (1-\nu)\ D\ \frac{\partial^3 w}{\partial x \partial y^2} + k_w w\ ]|_{(0,y)} = 0 \\ M_x = -D(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) \quad &\Rightarrow \quad [\, -D(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) + k_\theta\ \frac{\partial w}{\partial x}\ ]|_{(0,y)} = 0 \end{aligned} \tag{17.24}$$

D. Along $x = a$ we have ($s = y, n = x$):

$$\begin{aligned} V_x = -[D\frac{\partial}{\partial x}\nabla^2 w + (1-\nu)\ D\ \frac{\partial^3 w}{\partial x \partial y^2}] &\Rightarrow \quad [D\frac{\partial}{\partial x}\nabla^2 w + (1-\nu)\ D\ \frac{\partial^3 w}{\partial x \partial y^2} - k_w w\ ]|_{(a,y)} = 0 \\ M_x = -D(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) &\Rightarrow \quad [\, -D(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}) - k_\theta\ \frac{\partial w}{\partial x}\ ]|_{(a,y)} = 0 \end{aligned} \tag{17.25}$$

where it is understood that the distributed boundary springs (k_w, k_θ) are defined along the four edges.

Note that there are a total of eight boundary conditions for plate vibrations. Hence, the vibration modes $W(x, y)$ must consist of eight unknown coefficients:

§17.4.1 Simply supported along all four edges

The boundary conditions for this case is obtained from (17.22)-(17.25) by setting $k_w \to \infty$ and $k_\theta = 0$:

$$\begin{aligned} &\text{Along} \quad y = 0: \qquad w|_{(x,0)} = 0 \quad \text{and} \quad M_y = -D(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2})|_{(x,0)} = 0 \\ &\text{Along} \quad y = b: \qquad w\ |_{(x,b)} = 0 \quad \text{and} \quad M_y = -D(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2})|_{(x,b)} = 0 \\ &\text{Along} \quad x = 0: \qquad w\ |_{(0,y)} = 0 \quad \text{and} \quad M_x = -D(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2})|_{(0,y)} = 0 \\ &\text{Along} \quad x = a: \qquad w\ |_{(a,y)} = 0 \quad \text{and} \quad M_x = -D(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2})|_{(a,y)} = 0 \end{aligned} \tag{17.26}$$

Note that along $y = 0$ we have $w|_{(x,0)} = 0$, which implies that w does not vary along the plate edge (meaning with respect to x). Hence we have

$$\text{Along} \quad y = 0: \qquad w|_{(x,0)} = 0 \quad \text{implies} \quad \frac{\partial^2 w}{\partial x^2}|_{(x,0)} = 0 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial y^2}|_{(x,0)} = 0 = 0 \tag{17.27}$$

Applying this observation to the three remaining edges, we arrive at the following boundary conditions for a simply supported plate:

$$\begin{aligned}
&\text{Along} \quad y = 0: \qquad w|_{(x,0)} = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2}|_{(x,0)} = 0 \\
&\text{Along} \quad y = b: \qquad w\,|_{(x,b)} = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2}|_{(x,b)} = 0 \\
&\text{Along} \quad x = 0: \qquad w\,|_{(0,y)} = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2}|_{(0,y)} = 0 \\
&\text{Along} \quad x = a: \qquad w\,|_{(a,y)} = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2}|_{(a,y)} = 0
\end{aligned} \tag{17.28}$$

Clearly, the above boundary conditions are satisfied by retaining A_1-term in (17.21):

$$\begin{aligned}
&W(x, y) = A_1 \sin \alpha x \sin \gamma y, \qquad \alpha^2 + \gamma^2 = \beta^2, \quad \beta^4 = \frac{\omega^2 m(x, y)}{D} \\
&\Downarrow \\
&\sin \alpha a = 0, \quad \sin \gamma b = 0 \\
&\Downarrow \\
&\alpha_m a = \pi m, \quad m = 1, 2, \ldots \\
&\gamma_m a = \pi n, \quad n = 1, 2, \ldots, \\
&\Downarrow \\
&\alpha^2 + \gamma^2 = \beta^2 \quad \Rightarrow \quad \beta_{mn}^2 = \pi^2 [(\frac{m}{a})^2 + (\frac{n}{b})^2]
\end{aligned} \tag{17.29}$$

Therefore, the natural frequencies of a simply supported plate are given by

$$\omega_{mn} = \beta_{mn}^2 \sqrt{\frac{D}{m(x, y)}} = \pi^2 \, [(\frac{m}{a})^2 + (\frac{n}{b})^2] \sqrt{\frac{D}{m(x, y)}} \tag{17.30}$$

Note that the frequencies of a simply supported beam are obtained from the above frequency expression by setting

$$\begin{aligned}
&\{\frac{b}{a} \to 0, \quad \nu = 0\} \quad \Rightarrow \quad \{D = Eh^3/12 = EI, m(x, y) = m(x)\} \\
&\Downarrow \\
&\omega_n = (\frac{m\pi}{a})^2 \sqrt{\frac{EI}{m(x, y)}}
\end{aligned} \tag{17.31}$$

§17.4.2 Completely free along all four edges

The boundary conditions for this case is obtained from (17.22)-(17.25) by setting $k_w = 0$ and $k_\theta = 0$:

$$\text{Along} \quad y = (0, b): \qquad \begin{aligned} \frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial y^2} + (2-\nu)\frac{\partial^2 w}{\partial x^2}\right] = 0 \\ \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0 \end{aligned} \tag{17.32}$$

$$\text{Along} \quad x = (0, a): \qquad \begin{aligned} \frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + (2-\nu)\frac{\partial^2 w}{\partial y^2}\right] = 0 \\ \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) = 0 \end{aligned} \tag{17.33}$$

It should be emphasized that, if an approximation is introduced for $w(x, y)$, it must implicitly satisfy $M_{ns} = M_{xy} = 0$. Otherwise, the following must be explicitly enforced:

$$M_{xy} = -D\frac{\partial^2 w}{\partial x \partial y} = 0 \qquad \text{at the four corners} \tag{17.34}$$

as can be seen from (17.10).