21

Assembling Reduced-Order Substructural Models

§21.1 INTRODUCTION

The previous chapter has devoted to the reduced-order modeling of a single vibrating substructure. Once all of the substructures in a total system are approximated by their corresponding reduced-order models, the next task is to assemble the reduced-order substructural models. Third, the assembled reduced-order total system is either used for performance evaluation and/or design improvements.

In practice, the size of the assembled total structural model that consists of reduced-order substructural models is often considdred too large. For such a case, it is customary to carry out additional reduction via total system modal analysis.

Figure 21.1 illustrates the sequence of model development in large-scale vibrating structural systems.

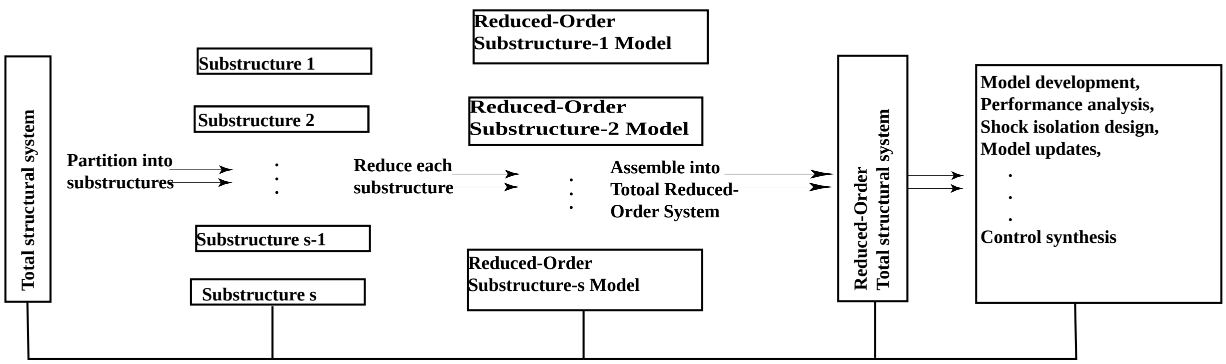

Fig. 21.1 Sequence of Reduced-Order Modeling and Applications

In the following, the variational formulation of partitioned equations of motion will be discussed first. In particular, two treatment of interface constraints will be discussed: classical (or global) λ-method (which reads as *Lagrange multiplier method*), and localized λ-method. The partitioned equations of motion employing the two λ-methods are then derived. we will then focus on one of the most widely used component mode synthesis method, the Craig-Bampton method. Finally, a component mode synthesis technique based on the localized λ-methoid will be described.

§21.2 VARIATIONAL FORMULATION OF PARTITIONED STRUCTURAL SYSTEMS

Consider a structure that consists of two substructures as shown in Fig. 21.2. When the structure is partitioned into two structures, $\Omega^{(1)}$ and $\Omega^{(2)}$, interactions forces, $\boldsymbol{\lambda}^{(1)}$ and $\boldsymbol{\lambda}^{(2)}$ (or $\boldsymbol{\lambda}^{(12)}$), are developed along the interface boundaries of $\Gamma^{(1)}$ and $\Gamma^{(1)}$. In addition, the displacement for substructure 1 consists of the interior ones $\mathbf{u}_I^{(1)}$ and along the partition boundary $\mathbf{u}_\Gamma^{(1)}$. Similarly, for substructure 2 we have $\mathbf{u}_I^{(2)}$ and $\mathbf{u}_\Gamma^{(2)}$. These can be expressed as

$$\mathbf{u}^{(1)} = \begin{bmatrix} \mathbf{u}_I^{(1)} \\ \mathbf{u}_\Gamma^{(1)} \end{bmatrix}, \qquad \mathbf{u}^{(2)} = \begin{bmatrix} \mathbf{u}_I^{(2)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix} \tag{21.1}$$

Using these notations, the energy functionals for substructures 1 and 2 may be written as

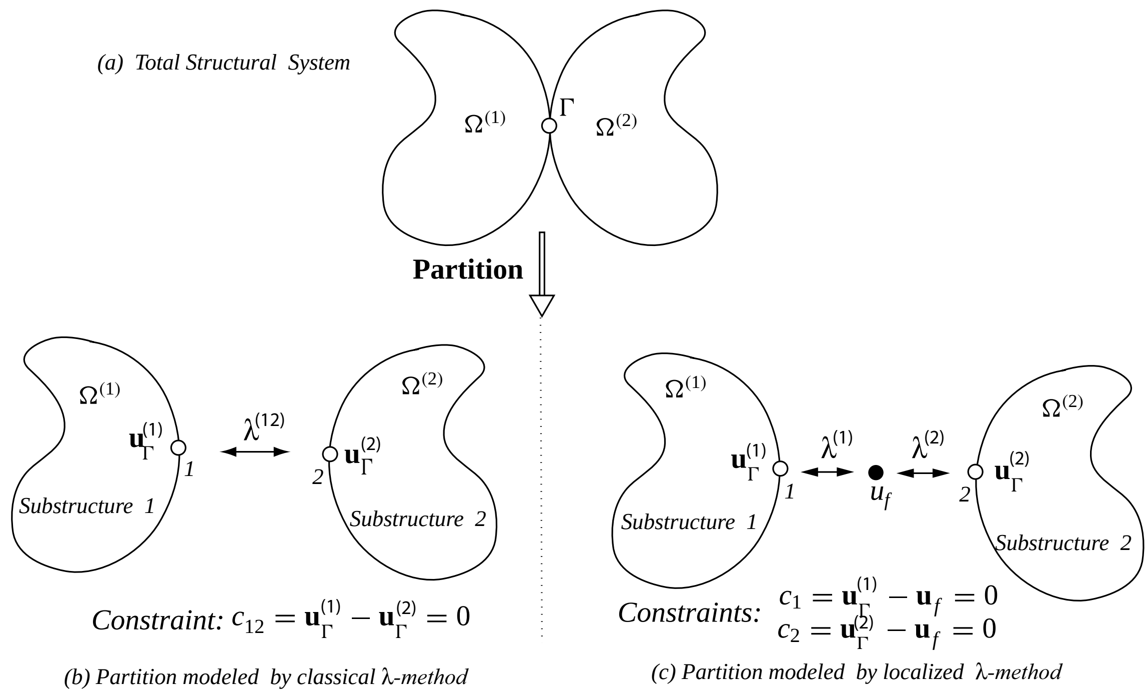

Fig. 21.2 Partitioning of a Structure into Two Substructures

Substructure 1:

$$\delta\Pi^{(1)} = (\delta\mathbf{u}^{(1)})^T\{\mathbf{K}^{(1)}\,\mathbf{u}^{(1)} - (\mathbf{f}^{(1)} - \mathbf{M}^{(1)}\,\ddot{\mathbf{u}}^{(1)})\}$$

Substructure 2:

$$\delta\Pi^{(2)} = (\delta\mathbf{u}^{(2)})^T\{\mathbf{K}^{(2)}\,\mathbf{u}^{(2)} - (\mathbf{f}^{(2)} - \mathbf{M}^{(2)}\,\ddot{\mathbf{u}}^{(2)})\} \tag{21.2}$$

where **M** and **K** are mass and stiffness matrix, resprectively, for a substructure, and the superscripts, $(1, 2)$, denote substructure.

While the virtual energy is completely contained in the preceding energy expressions, the interface conditions between these two substructures are needed for partitioning as well as assembly. The kinematic interface compatibility conditionmay be described inone of the two possible ways:

Classical (or Global) form:

$$\mathbf{u}_\Gamma^{(1)} - \mathbf{u}_\Gamma^{(2)} = 0$$

Localized form:

$$\begin{bmatrix} \mathbf{u}_\Gamma^{(1)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix} - \begin{bmatrix} \mathbf{L}_f^{(1)} \\ \mathbf{L}_f^{(2)} \end{bmatrix} \mathbf{u}_f = 0 \tag{21.3}$$

which states that the interface displacement along substructure 1, $\mathbf{u}_\Gamma^{(1)}$, must be equal to that of substructure 2, $\mathbf{u}_\Gamma^{(2)}$; and, $\mathbf{L}_f$ is the interface displacement displacement operator.

The constraint functional that incorporates the above constraints canbe expressed as

Classical (or Global) form:

$$\pi_{classical} = (\boldsymbol{\lambda}^{(12)})^T \ (\mathbf{u}_\Gamma^{(1)} - \mathbf{u}_\Gamma^{(2)})$$

Localized form: (21.4)

$$\pi_{localized} = \begin{bmatrix} \boldsymbol{\lambda}^{(1)} \\ \boldsymbol{\lambda}^{(2)} \end{bmatrix}^T \ \{ \begin{bmatrix} \mathbf{u}_\Gamma^{(1)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix} - \begin{bmatrix} \mathbf{L}_f^{(1)} \\ \mathbf{L}_f^{(2)} \end{bmatrix} \mathbf{u}_f \}$$

Finally, the total energy functional is simply the sume of two substructural energy expressions, (21.2), plus one of the the constraint functionals, (21.4):

Classical interface form:

$$\delta\Pi_{system} = \delta\Pi^{(1)} + \delta\Pi^{(2)} + \delta\pi_{classical}$$

Localized interface form: (21.5)

$$\delta\Pi_{system} = \delta\Pi^{(1)} + \delta\Pi^{(2)} + \delta\pi_{localized}$$

We now derive the partitioned equations of motion for the two interface treatment cases.

§21.3 PARTITIONED EQUATIONS OF MOTION EMPLOYING CLASSICAL λ-METHOD

The total energy of the system for this case which is the one given by the first of (21.5), consists of the two substructural energy expressions (21.2) plus the interface constraint functional, viz., the first expression in (21.4), as

$$\begin{aligned}
\delta\Pi^{total} &= (\delta\mathbf{u}^{(1)})^T \{\mathbf{K}^{(1)} \ \mathbf{u}^{(1)} - (\mathbf{f}^{(1)} - \mathbf{M}^{(1)} \ \ddot{\mathbf{u}}^{(1)})\} \\
&+ (\delta\mathbf{u}^{(2)})^T \{\mathbf{K}^{(2)} \ \mathbf{u}^{(2)} - (\mathbf{f}^{(2)} - \mathbf{M}^{(2)} \ \ddot{\mathbf{u}}^{(2)})\} \\
&+ (\delta\boldsymbol{\lambda}^{(12)})^T \ (\mathbf{u}_\Gamma^{(1)} - \mathbf{u}_\Gamma^{(2)}) + \ (\delta\mathbf{u}_\Gamma^{(1)} - \delta\mathbf{u}_\Gamma^{(2)})^T \ \boldsymbol{\lambda}^{(12)} \\
&= (\delta\mathbf{u}^{(1)})^T \{\mathbf{K}^{(1)} \ \mathbf{u}^{(1)} - (\mathbf{f}^{(1)} - \mathbf{M}^{(1)} \ \ddot{\mathbf{u}}^{(1)}) + (\mathbf{B}^{(1)})^T \ \mathbf{u}^{(1)}\} \\
&+ (\delta\mathbf{u}^{(2)})^T \{\mathbf{K}^{(2)} \ \mathbf{u}^{(2)} - (\mathbf{f}^{(2)} - \mathbf{M}^{(2)} \ \ddot{\mathbf{u}}^{(2)}) - (\mathbf{B}^{(2)})^T \ \mathbf{u}^{(2)}\} \\
&+ (\delta\boldsymbol{\lambda}^{(12)})^T \ (\mathbf{B}^{(1)} \ \mathbf{u}^{(1)} - \mathbf{B}^{(2)} \ \mathbf{u}^{(2)})
\end{aligned} \tag{21.6}$$

$$\mathbf{u}_\Gamma^{(1)} = \mathbf{B}^{(1)} \ \mathbf{u}^{(1)}, \quad \mathbf{u}_\Gamma^{(2)} = \mathbf{B}^{(2)} \ \mathbf{u}^{(2)}$$

where $\mathbf{B}^{(k)}$ is the Boolean matrix that extracts the interface degrees of freedom at the interface of substructure k .

The stationarity of the above variational equation, viz., $\delta\Pi^{total} = 0$, yields the following partitioned equations of motion:

$$
\begin{bmatrix} \mathbf{M}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}^{(1)} \\ \ddot{\mathbf{u}}^{(2)} \\ \ddot{\boldsymbol{\lambda}}^{(12)} \end{bmatrix} + \begin{bmatrix} \mathbf{K}^{(1)} & \mathbf{0} & (\mathbf{B}^{(1)})^T \\ \mathbf{0} & \mathbf{K}^{(2)} & -(\mathbf{B}^{(2)})^T \\ \mathbf{B}^{(1)} & -\mathbf{B}^{(2)} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(1)} \\ \mathbf{u}^{(2)} \\ \boldsymbol{\lambda}^{(12)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \\ \mathbf{0} \end{bmatrix}
$$

$$
\Downarrow \quad \Downarrow \tag{21.7}
$$

$$
\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}} \\ \ddot{\boldsymbol{\lambda}}^{(12)} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{B}_{cl}^T \\ \mathbf{B}_{cl} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda}^{(12)} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}
$$

In the above equation set, the first row is the equations of motion for substructure 1, the second row for substructure 2, and the third is the interface constraint equation. To illustrate the compositions of the partitioned equation further, we express each of the three equations in terms of the interior degrees of freedom, $\mathbf{u}_I$, and the interafce degrees of freedom, $\mathbf{u}_\Gamma$ as follows:

For substructure 1:

$$
\begin{bmatrix} M_{II} & M_{I\Gamma} \\ M_{\Gamma I} & M_{\Gamma\Gamma} \end{bmatrix}^{(1)} \begin{bmatrix} \ddot{\mathbf{u}}_I \\ \ddot{\mathbf{u}}_\Gamma \end{bmatrix}^{(1)} + \begin{bmatrix} K_{II} & K_{I\Gamma} \\ K_{I\Gamma} & K_{\Gamma\Gamma} \end{bmatrix}^{(1)} \begin{bmatrix} \mathbf{u}_I \\ \mathbf{u}_\Gamma \end{bmatrix}^{(1)} = \begin{bmatrix} f_I^{(1)} \\ f_\Gamma^{(1)} - \boldsymbol{\lambda}^{(12)} \end{bmatrix} \tag{21.8}
$$

For substructure 2:

$$
\begin{bmatrix} M_{II} & M_{I\Gamma} \\ M_{\Gamma I} & M_{\Gamma\Gamma} \end{bmatrix}^{(2)} \begin{bmatrix} \ddot{\mathbf{u}}_I \\ \ddot{\mathbf{u}}_\Gamma \end{bmatrix}^{(2)} + \begin{bmatrix} K_{II} & K_{I\Gamma} \\ K_{I\Gamma} & K_{\Gamma\Gamma} \end{bmatrix}^{(2)} \begin{bmatrix} \mathbf{u}_I \\ \mathbf{u}_\Gamma \end{bmatrix}^{(1)} = \begin{bmatrix} f_I^{(2)} \\ f_\Gamma^{(2)} + \boldsymbol{\lambda}^{(12)} \end{bmatrix} \tag{21.9}
$$

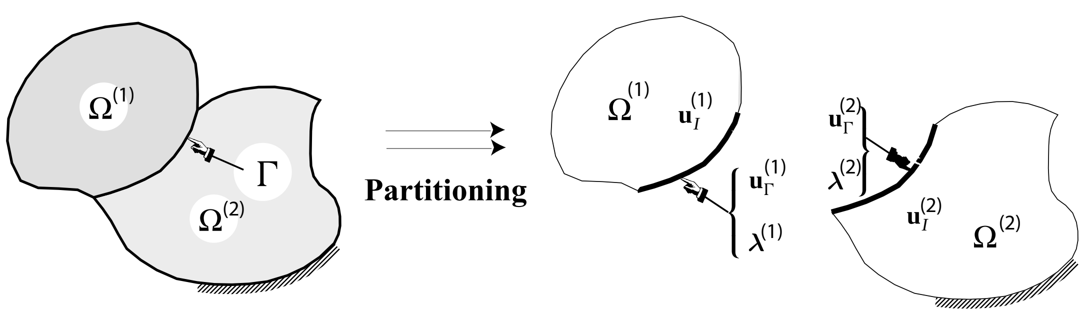

Fig. 21.3 Partitioning of a Structure into Two Substructures

If one is interested in constructing the equations of motion for the entire system, then all one has to do is to assemble the coefficient matrices that are associated with $\mathbf{u}_\Gamma^{(1)}$ and $\mathbf{u}_\Gamma^{(2)}$ into the same rows and the columns. This corresponds to an explicit enforcement of the second of the above constraint condition, e.g., $\mathbf{u}_\Gamma = \mathbf{u}_\Gamma^{(1)} = \mathbf{u}_\Gamma^{(1)}$. The assembled equations of motion therefore becomes

$$
\begin{bmatrix} M_{II}^{(1)} & M_{I\Gamma}^{(1)} & 0 \\ M^{(1)}{}_{\Gamma I} & M_{\Gamma\Gamma}^{(1)} + M_{\Gamma\Gamma}^{(2)} & M_{I\Gamma}^{(2)} \\ 0 & M_{\Gamma I}^{(2)} & M_{II}^{(2)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_I^{(1)} \\ \ddot{\mathbf{u}}_\Gamma \\ \ddot{\mathbf{u}}_I^{(2)} \end{bmatrix} + \begin{bmatrix} K_{II}^{(1)} & K_{I\Gamma}^{(1)} & 0 \\ K^{(1)}{}_{\Gamma I} & K_{\Gamma\Gamma}^{(1)} + K_{\Gamma\Gamma}^{(2)} & K_{I\Gamma}^{(2)} \\ 0 & K_{\Gamma I}^{(2)} & K_{II}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_I^{(1)} \\ \mathbf{u}_\Gamma \\ \mathbf{u}_I^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_I^{(1)} \\ \mathbf{f}_\Gamma \\ \mathbf{f}_I^{(2)} \end{bmatrix} \tag{21.10}
$$

This assembly process is precisely the assembly procedure of a typical finite element software system, except it is repeated several hundreds or thousands times. The modes and mode shapes of the total system can, *in principle*, be extracted from the eigenproblem associated with the above assembled equations of motion.

§21.4 PARTITIONED EQUATIONS OF MOTION EMPLOYING LOCALIZED λ-METHOD

The total energy of the system for this case which is the one given by the second of (21.5), consists of the two substructural energy expressions (21.2) plus the interface constraint functional, viz., the second expression in (21.4), as

$$\begin{aligned}
\delta\Pi^{total} &= (\delta\mathbf{u}^{(1)})^T\{\mathbf{K}^{(1)}\,\mathbf{u}^{(1)} - (\mathbf{f}^{(1)} - \mathbf{M}^{(1)}\,\ddot{\mathbf{u}}^{(1)})\} \\
&+ (\delta\mathbf{u}^{(2)})^T\{\mathbf{K}^{(2)}\,\mathbf{u}^{(2)} - (\mathbf{f}^{(2)} - \mathbf{M}^{(2)}\,\ddot{\mathbf{u}}^{(2)})\} \\
&+ \begin{bmatrix} \delta\boldsymbol{\lambda}^{(1)} \\ \delta\boldsymbol{\lambda}^{(2)} \end{bmatrix}^T \left\{ \begin{bmatrix} \mathbf{u}_\Gamma^{(1)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix} - \begin{bmatrix} \mathbf{L}_f^{(1)} \\ \mathbf{L}_f^{(2)} \end{bmatrix} \mathbf{u}_f \right\} \\
&+ \left\{ \delta \begin{bmatrix} \mathbf{u}_\Gamma^{(1)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix} - \begin{bmatrix} \mathbf{L}_f^{(1)} \\ \mathbf{L}_f^{(2)} \end{bmatrix} \delta\mathbf{u}_f \right\}^T \begin{bmatrix} \boldsymbol{\lambda}^{(1)} \\ \boldsymbol{\lambda}^{(2)} \end{bmatrix} \\
&= (\delta\mathbf{u}^{(1)})^T\{\mathbf{K}^{(1)}\,\mathbf{u}^{(1)} - (\mathbf{f}^{(1)} - \mathbf{M}^{(1)}\,\ddot{\mathbf{u}}^{(1)}) + (\mathbf{B}^{(1)})^T\boldsymbol{\lambda}^{(1)}\} \\
&+ (\delta\mathbf{u}^{(2)})^T\{\mathbf{K}^{(2)}\,\mathbf{u}^{(2)} - (\mathbf{f}^{(2)} - \mathbf{M}^{(2)}\,\ddot{\mathbf{u}}^{(2)}) + (\mathbf{B}^{(2)})^T\boldsymbol{\lambda}^{(2)}\} \\
&+ \begin{bmatrix} \delta\boldsymbol{\lambda}^{(1)} \\ \delta\boldsymbol{\lambda}^{(2)} \end{bmatrix}^T \left\{ \begin{bmatrix} \mathbf{B}^{(1)}\mathbf{u}^{(1)} \\ \mathbf{B}^{(2)}\mathbf{u}^{(2)} \end{bmatrix} - \begin{bmatrix} \mathbf{L}_f^{(1)} \\ \mathbf{L}_f^{(2)} \end{bmatrix} \mathbf{u}_f \right\} \\
&- \delta\mathbf{u}_f^T \begin{bmatrix} \mathbf{L}_f^{(1)} \\ \mathbf{L}_f^{(2)} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{\lambda}^{(1)} \\ \boldsymbol{\lambda}^{(2)} \end{bmatrix} \\
&\quad \mathbf{u}_\Gamma^{(1)} = \mathbf{B}^{(1)}\,\mathbf{u}^{(1)}, \quad \mathbf{u}_\Gamma^{(2)} = \mathbf{B}^{(2)}\,\mathbf{u}^{(2)}
\end{aligned} \tag{21.11}$$

The stationarity of the above variational equation, viz., $\delta\Pi = 0$, yields the following partitioned equations of motion:

$$\begin{bmatrix} \mathbf{M}^{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{(2)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}^{(1)} \\ \ddot{\mathbf{u}}^{(2)} \\ \ddot{\boldsymbol{\lambda}}^{(1)} \\ \ddot{\boldsymbol{\lambda}}^{(2)} \\ \ddot{\mathbf{u}}_f \end{bmatrix} + \begin{bmatrix} \mathbf{K}^{(1)} & \mathbf{0} & (\mathbf{B}^{(1)})^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{(2)} & \mathbf{0} & (\mathbf{B}^{(2)})^T & \mathbf{0} \\ \mathbf{B}^{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{L}_f^{(1)} \\ \mathbf{0} & \mathbf{B}^{(2)} & \mathbf{0} & \mathbf{0} & -\mathbf{L}_f^{(2)} \\ \mathbf{0} & \mathbf{0} & -(\mathbf{L}_f^{(1)})^T & -(\mathbf{L}_f^{(2)})^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(1)} \\ \mathbf{u}^{(2)} \\ \boldsymbol{\lambda}^{(1)} \\ \boldsymbol{\lambda}^{(2)} \\ \mathbf{u}_f \end{bmatrix} = \begin{bmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{21.12}$$

$$\Downarrow \quad \Downarrow$$

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}} \\ \ddot{\boldsymbol{\lambda}}_\ell \\ \ddot{\mathbf{u}}_f \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{B}_\ell^T & \mathbf{0} \\ \mathbf{B}_\ell & \mathbf{0} & -\mathbf{L}_f \\ \mathbf{0} & -\mathbf{L}_f^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda}_\ell \\ \mathbf{u}_f \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

§21.5 EIGENVALUE PROBLEM USING REDUCED-ORDER PARTITIONED EQUATIONS OF MOTION

Model reduction of a substructure has been presented in the previous chapter. In this section we will use the reduction procedure based on the constrained interface modes for assembling the substructures into a total system. To this end, let's express the reduction form for substructure k as

$$\mathbf{u}^{(k)} = \Psi^{(k)}\,\mathbf{q}^{(k)} \tag{21.13}$$

where $\mathbf{u}^{(k)}$ is approximated from the previous chapter as:

$$\mathbf{u}^{(k)} \approx \begin{bmatrix} \mathbf{u}_I \\ \\ \mathbf{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \Psi_I & -\mathbf{K}_{II}^{-1} K_{I\Gamma} \\ \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \\ \mathbf{u}_\Gamma \end{bmatrix} \tag{20.6}$$

Substituting the above reduction formula into (21.7), one obtains

$$\begin{bmatrix} \mathcal{M}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{M}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}^{(1)} \\ \ddot{\mathbf{q}}^{(2)} \\ \ddot{\boldsymbol{\lambda}}^{(12)} \end{bmatrix} + \begin{bmatrix} \mathcal{K}^{(1)} & \mathbf{0} & (\mathcal{B}^{(1)})^T \\ \mathbf{0} & \mathcal{K}^{(2)} & -(\mathcal{B}^{(2)})^T \\ \mathcal{B}^{(1)} & -\mathcal{B}^{(2)} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}^{(1)} \\ \mathbf{q}^{(2)} \\ \boldsymbol{\lambda}^{(12)} \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{(1)} \\ \mathbf{p}^{(2)} \\ \mathbf{0} \end{bmatrix}$$

$$\begin{aligned} &\mathcal{M}^{(1)} = (\Psi^{(1)})^T\,\mathbf{M}^{(1)}\,\Psi^{(1)}, \quad \mathcal{M}^{(2)} = (\Psi^{(2)})^T\,\mathbf{M}^{(2)}\,\Psi^{(1)} \\ &\mathcal{K}^{(1)} = (\Psi^{(1)})^T\,\mathbf{K}^{(1)}\,\Psi^{(1)}, \quad \mathcal{K}^{(2)} = (\Psi^{(2)})^T\,\mathbf{K}^{(2)}\,\Psi^{(1)} \\ &\mathcal{B}^{(1)} = (\Psi^{(1)})^T\,\mathbf{B}^{(1)}, \quad \mathcal{B}^{(2)} = (\Psi^{(2)})^T\,\mathbf{B}^{(2)} \\ &\mathbf{p}^{(1)} = (\Psi^{(1)})^T\,\mathbf{f}^{(1)}, \quad \mathbf{p}^{(2)} = (\Psi^{(2)})^T\,\mathbf{f}^{(2)} \end{aligned} \tag{21.14}$$

Vibration analysis of the total system based on the above reduced-order model is carried using the following equation:

$$\hat{\mathbf{K}}^{total}\,\hat{\boldsymbol{\Psi}}^{total} = \hat{\mathbf{M}}^{total}\,\hat{\boldsymbol{\Psi}}^{total}\,\boldsymbol{\Lambda}^{total}$$

$$\begin{aligned} \hat{\mathbf{K}}^{total} &= \begin{bmatrix} \mathcal{M}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{M}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \hat{\mathbf{M}}^{total} &= \begin{bmatrix} \mathcal{K}^{(1)} & \mathbf{0} & (\mathcal{B}^{(1)})^T \\ \mathbf{0} & \mathcal{K}^{(2)} & -(\mathcal{B}^{(2)})^T \\ \mathcal{B}^{(1)} & -\mathcal{B}^{(2)} & \mathbf{0} \end{bmatrix} \end{aligned} \tag{21.15}$$

It is noted that$\hat{\Phi}^{total}$ is not the eigenvectors of the assembled model. The correct eigenvectors (mode shapes) of the assembled model are obtained by

$$\begin{aligned} \begin{bmatrix} \mathbf{u}^{(1)} \\ \mathbf{u}^{(2)} \\ \boldsymbol{\lambda}^{12} \end{bmatrix} &= \begin{bmatrix} \Psi^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Psi^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{q}^{(1)} \\ \mathbf{q}^{(2)} \\ \boldsymbol{\lambda}^{12} \end{bmatrix} \\ &= \begin{bmatrix} \Psi^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Psi^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \hat{\boldsymbol{\Psi}}^{total}\,\hat{\mathbf{q}}^{total} \\ &\Downarrow\Downarrow \\ \boldsymbol{\Psi}^{total} &= \begin{bmatrix} \Psi^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Psi^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \hat{\boldsymbol{\Psi}}^{total} \end{aligned} \tag{21.16}$$

which includes not only the modes that span the substructures but also the interface modes pertaining to the interface force $\boldsymbol{\lambda}^{(12)}$. However, the eigenvalues $\boldsymbol{\Lambda}^{total}$ represent the assembled structural system. In other words, $(\boldsymbol{\Psi}^{total}, \boldsymbol{\Lambda}^{total})$ constitute the mode shapes and modes of the assembled system even though we have obtained them from the partitioned equations of motion.

§21.6 EIGENVALUE PROBLEM USING REDUCED-ORDER ASSEMBLED EQUATIONS OF MOTION

In the preceding section the reduced-order partitioned equations of motion has been directly utilized for the formulation of eigenvalue problem. While computationally equivalent, the resulting eigenvalue problem given by (21.15) involves non-definite matrices, thus requiring a special care. One way to circumvent the non-definite matrices is to assemble the partitioned equations of motion into the assembled form akin to the equation given in (21.10). This can be accomplished in the following way.

First, we note that the constraint condition

$$\mathbf{u}_\Gamma^{(1)} - \mathbf{u}_\Gamma^{(2)} = 0 \tag{21.3}$$

implies that the interface Boolean matrices $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ yield the following relation:

$$\begin{aligned} \mathbf{B}^{(1)}\mathbf{u}^{(1)} &= [\,\mathbf{0} \quad \mathbf{I}_\Gamma^{(1)}\,] \begin{bmatrix} \mathbf{u}_I^{(1)} \\ \mathbf{u}_\Gamma^{(1)} \end{bmatrix} \\ \mathbf{B}^{(2)}\mathbf{u}^{(2)} &= [\,\mathbf{0} \quad \mathbf{I}_\Gamma^{(2)}\,] \begin{bmatrix} \mathbf{u}_I^{(2)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix} \end{aligned} \tag{21.17}$$

This means that the assembled degrees of freedom, $(\mathbf{q}_I^{(1)}, \mathbf{u}_\Gamma, \mathbf{q}_I^{(2)})$, can be related to the partitioned degrees of freedom, $(\mathbf{q}_I^{(1)}, \mathbf{u}_\Gamma, \mathbf{q}_I^{(2)}, \mathbf{u}_\Gamma^{(2)})$,according to

$$\begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{u}_\Gamma \\ \mathbf{q}_I^{(2)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_I^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_\Gamma^{(1)} \\ \mathbf{0} & \mathbf{I}_I^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_\Gamma^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{q}_I^{(2)} \\ \mathbf{u}_\Gamma \end{bmatrix} \quad \Rightarrow \quad \mathbf{q}^{part} = \mathbf{L}^a\ \mathbf{q}^a \tag{21.18}$$

where the superscripts, $(part,\ a)$, denote the partitioned and assembled degrees of freedom, and $\mathbf{L}^a$ is an assembly Boolean matrix. Therefore, teh complete transformation relation can be expressed as

$$\begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{u}_\Gamma \\ \mathbf{q}_I^{(2)} \\ \mathbf{u}_\Gamma^{(2)} \\ \boldsymbol{\lambda}^{(12)} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^{part} \\ \boldsymbol{\lambda}^{(12)} \end{bmatrix} = \mathbf{T}^{total} \begin{bmatrix} \mathbf{q}^a \\ \boldsymbol{\lambda}^{(12)} \end{bmatrix}, \quad \mathbf{T}^{total} = \begin{bmatrix} \mathbf{L}^a & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_\Gamma^{(12)} \end{bmatrix} \tag{21.19}$$

Substituting the above assembly transformation into (21.14) and after some simplifications, one arrives at the following reduced-order assembled equations of motion:

$$\mathcal{M}\,\ddot{\hat{\mathbf{q}}} + \mathcal{K}\,\hat{\mathbf{q}} = \hat{\mathbf{p}}$$

$$\mathcal{M} = (\mathbf{L}^a)^T \begin{bmatrix} \mathcal{M}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathcal{M}^{(2)} \end{bmatrix} \mathbf{L}^a$$

$$\mathcal{K} = (\mathbf{L}^a)^T \begin{bmatrix} \mathcal{K}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathcal{K}^{(2)} \end{bmatrix} \mathbf{L}^a \tag{21.20}$$

$$\hat{\mathbf{p}} = (\mathbf{L}^a)^T \begin{bmatrix} \mathbf{p}^{(1)} \\ \mathbf{p}^{(2)} \end{bmatrix}$$

$$\hat{\mathbf{q}} = (\mathbf{L}^a)^T \begin{bmatrix} \mathbf{q}^{(1)} \\ \mathbf{q}^{(2)} \end{bmatrix}$$

As one can see, the above reduced-order assembled equations of motion is difficult to follow through. We will examine a step-by-step derivation of the above equation below, which is konwn in the literature as the Craig-Bampton component mode synthesis or subtructuring method method.

§21.7 THE CRAIG-BAMPTON METHOD

Equation(21.20) may be considered a generic component mode synthesis as it can accommodate several possible substructural reduction methods. One of its specializations was proposed by R.R Craig M.C.C. Bampton in 1968. As the Craig-Bampton component mode synthesis technique is perhaps the most widely used substructuring method, we present a step-by-step formulation of their method below.

§21.7.1 Step 1: Approximate the substructural displacements

It approximates the displacement of each substructure by a set of fixed-interface normal modes plus a set of constraint modes. Specifically, for substructure 1, $\mathbf{u}^{(1)}$ is approximated by (see Eq. (20.6))

$$\mathbf{u}^{(1)} \approx \begin{bmatrix} \mathbf{u}_I^{(1)} \\ \mathbf{u}_\Gamma^{(1)} \end{bmatrix} = \begin{bmatrix} \phi_I^{(1)} & \psi_{I\Gamma}^{(1)} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{u}_\Gamma^{(1)} \end{bmatrix}, \quad \psi_{I\Gamma}^{(1)} = -(\mathbf{K}_{II}^{(1)})^{-1}\, K_{I\Gamma}^{(1)} \tag{21.21}$$

Similarly, the substructural displacement $\mathbf{u}^{(2)}$ for substructure is approximated by

$$\mathbf{u}^{(2)} \approx \begin{bmatrix} \mathbf{u}_I^{(2)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix} = \begin{bmatrix} \phi_I^{(2)} & \psi_{I\Gamma}^{(2)} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I^{(2)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix}, \quad \psi_{I\Gamma}^{(2)} = -(\mathbf{K}_{II}^{(2)})^{-1}\, K_{I\Gamma}^{(2)} \tag{21.22}$$

§21.7.2 Step 2: Obtain the approximate substructural kinetic and strain energy

The approximate substructural strain energy and kinetic energy derived in the previous chapter (see 20.13) and (20.14)) are restated below.

$$U^{(1)} = \tfrac{1}{2}(\mathbf{u}^{(1)})^T\,\mathbf{K}^{(1)}\,\mathbf{u}^{(1)} \approx \begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{u}_\Gamma^{(1)} \end{bmatrix}^T \begin{bmatrix} \Psi_I^{(1)} & \Psi_{I\Gamma}^{(1)} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma}^{(1)} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_{II}^{(1)} & \mathbf{K}_{I\Gamma}^{(1)} \\ \mathbf{K}_{\Gamma I}^{(1)} & \mathbf{K}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} \Psi_I^{(1)} & \Psi_{I\Gamma}^{(1)} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{u}_\Gamma^{(1)} \end{bmatrix}$$

$$T^{(2)} = \tfrac{1}{2}\dot{\mathbf{u}}^{(2)}\,\mathbf{M}^{(2)}\,\dot{\mathbf{u}}^{(2)} \approx \begin{bmatrix} \dot{\mathbf{q}}_I^{(2)} \\ \dot{\mathbf{u}}_\Gamma^{(2)} \end{bmatrix}^T \begin{bmatrix} \Psi_I^{(2)} & \Psi_{I\Gamma}^{(2)} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma}^{(2)} \end{bmatrix}^T \begin{bmatrix} \mathbf{M}_{II}^{(2)} & \mathbf{M}_{I\Gamma}^{(2)} \\ \mathbf{M}_{\Gamma I}^{(2)} & \mathbf{M}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \Psi_I^{(2)} & \Psi_{I\Gamma}^{(2)} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_I^{(1)} \\ \dot{\mathbf{u}}_\Gamma^{(1)} \end{bmatrix} \tag{21.23}$$

The reduced-order mass for substructure 1, $\mathcal{M}^{(1)}$, is thus given by

$$\mathcal{M}^{(1)} = \begin{bmatrix} \Psi_I^{(1)T} & \mathbf{0} \\ \Psi_{I\Gamma}^{(1)T} & \mathbf{I}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{II}^{(1)} & \mathbf{M}_{I\Gamma}^{(1)} \\ \mathbf{M}_{\Gamma I}^{(1)} & \mathbf{M}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} \Psi_I^{(1)} & \Psi_{I\Gamma}^{(1)} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma}^{(1)} \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{II}^{(1)} & \mathcal{M}_{I\Gamma}^{(1)} \\ \mathcal{M}_{\Gamma I}^{(1)} & \mathcal{M}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \tag{21.24}$$

where

$$\begin{aligned} \mathcal{M}_{II}^{(1)} &= \Psi_I^{(1)T}\,\mathbf{M}_{II}^{(1)}\Phi_I^{(1)} = \mathbf{I}_{II} \quad \text{(due to massnormalization)} \\ \mathcal{M}_{I\Gamma}^{(1)} &= \Psi_I^{(1)T}\,(\mathbf{M}_{I\Gamma}^{(1)} + \mathbf{M}_{II}^{(1)}\,\Psi_{I\Gamma}^{(1)}) \\ \mathcal{M}_{\Gamma\Gamma}^{(1)} &= \Psi_{I\Gamma}^{(1)T}\,(\mathbf{M}_{II}^{(1)}\,\Psi_{I\Gamma}^{(1)} + \mathbf{M}_{I\Gamma}^{(1)}) + \mathbf{M}_{\Gamma I}^{(1)}\,\Psi_{I\Gamma}^{(1)} + \mathbf{M}_{\Gamma\Gamma}^{(1)} \end{aligned}$$

The reduced-order stiffness for substructure 1, $\mathcal{K}^A$, is thus given by

$$\mathcal{K}^{(1)} = \begin{bmatrix} \mathcal{K}_{II}^{(1)} & \mathcal{K}_{I\Gamma}^{(1)} \\ \mathcal{K}_{\Gamma I}^{(1)} & \mathcal{K}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \tag{21.25}$$

where

$$\begin{aligned} \mathcal{K}_{II}^{(1)} &= \Psi_I^{(1)T}\,\mathbf{K}_{II}^{(1)}\Psi_I^{(1)} = \mathbf{\Lambda}_{II}^{(1)} = \begin{bmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_I^2 \end{bmatrix}^{(1)} \\ \mathcal{K}_{I\Gamma}^{(1)} &= \Psi_I^{(1)T}\,(\mathbf{K}_{I\Gamma}^{(1)} + \mathbf{K}_{II}^{(1)}\,\Psi_{I\Gamma}^{(1)}) = \Psi_I^{(1)T}\,(\mathbf{K}_{I\Gamma}^{(1)} - \mathbf{K}_{II}^{(1)}\,(\mathbf{K}_{II}^{(1)})^{-1}\,\mathbf{K}_{I\Gamma}^{(1)}) = 0 \\ \mathcal{K}_{\Gamma\Gamma}^{(1)} &= \mathbf{K}_{\Gamma\Gamma}^{(1)} + \Psi_{I\Gamma}^{(1)T}\,(\mathbf{K}_{II}^{(1)}\,\Psi_{I\Gamma}^{(1)} + \mathbf{K}_{I\Gamma}^{(1)}) + \mathbf{K}_{\Gamma I}^{(1)}\,\Psi_{I\Gamma}^{(1)} \\ &= \mathbf{K}_{\Gamma\Gamma}^{(1)} - \mathbf{K}_{\Gamma I}^{(1)}\,(\mathbf{K}_{II}^{(1)})^{-1}\,\mathbf{K}_{I\Gamma}^{(1)} \end{aligned}$$

Observe that $\mathcal{K}^{(1)}$ can be written as

$$\mathcal{K}^{(1)} = \begin{bmatrix} \mathbf{\Lambda}_{II}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \tag{21.26}$$

which consists of the diagonal interior substructural modes and the Guyan-reduced matrix $\mathcal{K}_{\Gamma\Gamma}$.

For substructure 2, a similar procedure employed for substructure A can be repeated to yield:

$$\mathcal{M}^{(2)} = \begin{bmatrix} \mathcal{M}_{II}^{(2)} & \mathcal{M}_{I\Gamma}^{(2)} \\ \mathcal{M}_{\Gamma I}^{(2)} & \mathcal{M}_{\Gamma\Gamma}^{(2)} \end{bmatrix}$$
$$\mathcal{K}^{(2)} = \begin{bmatrix} \mathbf{\Lambda}_{II}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \tag{21.27}$$

where it is understood that the substructural displacement $\mathbf{u}^{(2)}$ is approximated by the fixed-interface interior modes plus the constrained modes given by (20.6).

§21.7.3 Step 3: Sum up the substructural kinetic and strain energy expressions

The strain energy and the kinetic energy of the total structure can be approximated by

$$\begin{aligned}
\mathcal{T} &= T^{(1)} + T^{(2)} \\
\mathcal{U} &= U^{(1)} + U^{(2)} \\
U^{(1)} &\approx \tfrac{1}{2} \begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{u}_\Gamma^{(1)} \end{bmatrix}^T \begin{bmatrix} \mathbf{\Lambda}_{II}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{u}_\Gamma^{(1)} \end{bmatrix} \\
T^{(1)} &\approx \tfrac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}_I^{(1)} \\ \dot{\mathbf{u}}_\Gamma^{(1)} \end{bmatrix}^T \begin{bmatrix} \mathcal{M}_{II}^{(1)} & \mathcal{M}_{I\Gamma}^{(1)} \\ \mathcal{M}_{\Gamma I}^{(1)} & \mathcal{M}_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_I^{(1)} \\ \dot{\mathbf{u}}_\Gamma^{(1)} \end{bmatrix} \\
U^{(2)} &\approx \tfrac{1}{2} \begin{bmatrix} \mathbf{q}_I^{(2)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix}^T \begin{bmatrix} \mathbf{\Lambda}_{II}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I^{(2)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix} \\
T^{(2)} &\approx \tfrac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}_I^{(2)} \\ \dot{\mathbf{u}}_\Gamma^{(2)} \end{bmatrix}^T \begin{bmatrix} \mathcal{M}_{II}^{(2)} & \mathcal{M}_{I\Gamma}^{(2)} \\ \mathcal{M}_{\Gamma I}^{(2)} & \mathcal{M}_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_I^{(1)} \\ \dot{\mathbf{u}}_\Gamma^{(2)} \end{bmatrix}
\end{aligned} \tag{21.28}$$

§21.7.4 Step 4: Derive the reduced equations of motion for the total system

The Lagrangian of the total system is given by

$$\mathcal{L} = \mathcal{T} - \mathcal{U} + (\boldsymbol{\lambda}^{(12)})^T \ (\mathbf{u}_\Gamma^{(1)} - \mathbf{u}_\Gamma^{(2)}) \tag{21.29}$$

where the last term involving the Lagrange multiplier $\boldsymbol{\lambda}^{(12)}$ is introduced to enforce the interface displacement compatibility constraint

$$\mathbf{u}_\Gamma^{(1)} - \mathbf{u}_\Gamma^{(2)} = 0 \tag{21.30}$$

The equations of motion for free vibration ($\mathbf{f}^{(1)} = 0, \mathbf{f}^{(2)} = 0$) can be derived from (21.29) as

$$\begin{bmatrix} \mathcal{M}^{(1)} & 0 \\ 0 & \mathcal{M}^{(2)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}^{(1)} \\ \ddot{\mathbf{q}}^{B} \end{bmatrix} + \begin{bmatrix} \mathcal{K}^{(1)} & 0 \\ 0 & \mathcal{K}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{q}^{(1)} \\ \mathbf{q}^{(2)} \end{bmatrix} = \mathbf{C}^T \boldsymbol{\lambda}^{(12)}$$
$$\mathbf{q}^{(1)} = \begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{u}_\Gamma^{(1)} \end{bmatrix}, \qquad \mathbf{q}^{(2)} = \begin{bmatrix} \mathbf{q}_I^{(2)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix} \tag{21.31}$$
$$\mathbf{C}^T = [\,\mathbf{0} \quad \mathbf{I} \quad \mathbf{0} \quad -\mathbf{I}\,]$$

If desired, the interface force $\boldsymbol{\lambda}^{(12)}$ can be eliminated by expressing the interface displacement $\mathbf{u}_\Gamma^{(1)}$ in terms of $\mathbf{u}_\Gamma^{(2)}$ or vice versa. This can be accomplished by the reduction

$$\begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{u}_\Gamma^{(1)} \\ \mathbf{q}_I^{(2)} \\ \mathbf{u}_\Gamma^{(2)} \end{bmatrix} = \mathbf{L}^a \begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{q}_I^{(2)} \\ \mathbf{u}_\Gamma \end{bmatrix}, \quad \mathbf{L}^a = \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{I} \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix}, \quad \mathbf{u}_\Gamma = \mathbf{u}_\Gamma^{(1)} = \mathbf{u}_\Gamma^{(2)} \tag{21.32}$$

Substituting (21.32) into (21.31) and premultiplying the resulting equation by $(\mathbf{L}^a)^T$ we obtain the following reduced-order free vibration equation:

$$\boxed{\mathcal{M}\,\ddot{\mathbf{q}} + \mathcal{K}\,\mathbf{q} = 0}, \qquad \mathbf{q} = \begin{bmatrix} \mathbf{q}_I^{(1)} \\ \mathbf{q}_I^{(2)} \\ \mathbf{u}_\Gamma \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_{II}^{(1)} & 0 & \mathcal{M}_{I\Gamma}^{(1)} \\ 0 & \mathcal{M}_{II}^{(2)} & \mathcal{M}_{I\Gamma}^{(2)} \\ \mathcal{M}^{(1)}{}_{\Gamma I} & \mathcal{M}_{\Gamma I}^{(2)} & \mathcal{M}_{\Gamma\Gamma} \end{bmatrix}, \quad \mathcal{M}_{\Gamma\Gamma} = \mathcal{M}_{\Gamma\Gamma}^{(1)} + \mathcal{M}_{\Gamma\Gamma}^{(2)} \tag{21.33}$$
$$\mathcal{K} = \begin{bmatrix} \boldsymbol{\Lambda}_{II}^{(1)} & 0 & 0 \\ 0 & \boldsymbol{\Lambda}_{II}^{(2)} & 0 \\ 0 & 0 & \mathcal{K}_{\Gamma\Gamma} \end{bmatrix}, \qquad \mathcal{K}_{\Gamma\Gamma} = \mathcal{K}_{\Gamma\Gamma}^{(1)} + \mathcal{K}_{\Gamma\Gamma}^{(2)}$$

§21.7.5 Step 5: Perform eigenanalysis of the total system

First, we perform an eigenanalysis of (21.33):

$$\mathcal{K}\,\boldsymbol{\Phi} = \mathcal{M}\,\boldsymbol{\Phi}\,\boldsymbol{\Lambda}_g, \qquad \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}_I^{(1)} \\ \boldsymbol{\Phi}_I^{(2)} \\ \boldsymbol{\Phi}_\Gamma \end{bmatrix} \tag{21.34}$$

Second, once $(\boldsymbol{\Phi},\ \boldsymbol{\Lambda}_g)$ are obtained, the *global* eigenvector $\boldsymbol{\Phi}_g$ is obtained by the following expression

$$\mathbf{\Phi}_g = \begin{bmatrix} \phi_I^{(1)} & 0 & \Psi_{I\Gamma}^{(1)} \\ 0 & \Psi_I^{(2)} & \Psi_{I\Gamma}^{(2)} \\ 0 & 0 & \mathbf{I}_\Gamma \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_I^{(1)} \\ \mathbf{\Phi}_I^{(2)} \\ \mathbf{\Phi}_\Gamma \end{bmatrix} \tag{21.35}$$

Observe that the eigenvalues are preserved under a similarity transformation. Thus, the global eigenvalues and eigenvector pairs are given by ($\mathbf{\Phi}_g,\ \boldsymbol{\lambda}_g$).

The component mode synthesis for other techniques due to Benfield and Hruda, Hurty, MacNeal, Rubin, Hintz, Dowell and Klein, and Craig and Chang may be similarly constructed.

Remark 1: Note that from (21.33) equation, in carrying out the component mode synthesis by the Craig-Bampton method, viz.,

$$\omega^2 \boldsymbol{\mathcal{M}}\ \Phi = \mathcal{K}\ \Phi \tag{21.36}$$

the mass matrix $\boldsymbol{\mathcal{M}}$ becomes dense even if the original substructural-level mass matrices are diagonal.

Remakr 2: The stiffness matrix at the interface is given by

$$\begin{aligned} \mathcal{K}_{\Gamma\Gamma} &= \mathcal{K}_{\Gamma\Gamma}^{(1)} + \mathcal{K}_{\Gamma\Gamma}^{(2)} \\ \mathcal{K}_{\Gamma\Gamma}^{(1)} &= \mathbf{K}_{\Gamma\Gamma}^{(1)} - \mathbf{K}_{\Gamma I}^{(1)}\ (\mathbf{K}_{II}^{(1)})^{-1}\ \mathbf{K}_{I\Gamma}^{(1)} \\ \mathcal{K}_{\Gamma\Gamma}^{(2)} &= \mathbf{K}_{\Gamma\Gamma}^{(2)} - \mathbf{K}_{\Gamma I}^{(2)}\ (\mathbf{K}_{II}^{(2)})^{-1}\ \mathbf{K}_{I\Gamma}^{(2)} \end{aligned} \tag{21.37}$$

Note that both $\mathcal{K}_{\Gamma\Gamma}^{(1)}$ and $\mathcal{K}_{\Gamma\Gamma}^{(2)}$ are the Schur complements (or in structural mechanics known as Guyan-reduced matrices) that preserve the strain energy content of each substructure. Hence, no approximation is introduced at the interface strain energy contents.

On the other hand, the same cannot be said regarding the kinetic energy. This can be seen by examing the interface mass traix $\boldsymbol{\mathcal{M}}_{\Gamma\Gamma}$:

$$\begin{aligned} \boldsymbol{\mathcal{M}}_{\Gamma\Gamma} &= \boldsymbol{\mathcal{M}}_{\Gamma\Gamma}^{(1)} + \boldsymbol{\mathcal{M}}_{\Gamma\Gamma}^{(2)} \\ \boldsymbol{\mathcal{M}}_{\Gamma\Gamma}^{(1)} &= \Psi_{I\Gamma}^{(1)^T}\ (\mathbf{M}_{II}^{(1)}\ \Psi_{I\Gamma}^{(1)} + \mathbf{M}_{I\Gamma}^{(1)}) + \mathbf{M}_{\Gamma I}^{(1)}\ \Psi_{I\Gamma}^{(1)} + \mathbf{M}_{\Gamma\Gamma}^{(1)} \\ \boldsymbol{\mathcal{M}}_{\Gamma\Gamma}^{(2)} &= \Psi_{I\Gamma}^{(2)^T}\ (\mathbf{M}_{II}^{(2)}\ \Psi_{I\Gamma}^{(2)} + \mathbf{M}_{I\Gamma}^{(2)}) + \mathbf{M}_{\Gamma I}^{(2)}\ \Psi_{I\Gamma}^{(2)} + \mathbf{M}_{\Gamma\Gamma}^{(2)} \end{aligned} \tag{21.38}$$

In other words, the constraint modes $\Psi_{I\Gamma}$ play the role of augmenting the interface kinetic energy by infusing the interior masses $\mathbf{M}_{II}$ unto the interface nodes.

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