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Transient Response Analysis of Structural Systems

§21.1 MODAL APPROACH TO TRANSIENT ANALYSIS

Consider the following large-order finite element model equations of motion for linear structures:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{D}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t), \quad \mathbf{D} = (\alpha\mathbf{M} + \beta\mathbf{K}), \ (\alpha, \beta) \text{ are constant.} \tag{21.1}$$

where the size of the displacement vector, n, ranges from several thousands to several millions.

Now suppose we would like to obtain the displacement response, $\mathbf{u}(t)$, for expected applied force, $\mathbf{f}(t)$. There are two approaches: direct time integration and modal superposition. We will differ direct time integration techniques for transient response analysis to the latter part of the course, and concentrate on modal superposition techniques. To this end, we first decompose the displacement vector, $\mathbf{u}(t)$, in terms of its modal components by

$$\mathbf{u}(t) = \mathbf{\Psi}\mathbf{q}(t) \tag{21.2}$$

where $\mathbf{\Psi}$ is the mode shapes of the free-vibration modes, and $\mathbf{q}(t)$ is the generalized modal displacement. The mode shape matrix $\mathbf{\Psi}$ has the property of simultaneously diagonalizing both the mass and stiffness matrices. That is, it is obtained from the following eigenvalue problem:

$$\begin{aligned} \mathbf{K}\,\mathbf{\Psi} &= \mathbf{M}\,\mathbf{\Psi}\,\mathbf{\Lambda} \\ \mathbf{\Lambda} &= \mathrm{diag}(\omega_1^2, \ldots, \omega_N^2) \end{aligned} \tag{21.3}$$

where ω_j is the j-th undamped frequency component of (21.1).

In structural dynamics, one often employs the following special form of mode shapes (eigenvector):

$$\mathbf{\Psi}^T\ \mathbf{K}\ \mathbf{\Psi} = \boldsymbol{\lambda}, \quad \mathbf{\Psi}^T\ \mathbf{M}\ \mathbf{\Psi} = \mathbf{I} = \mathrm{diag}(1, 1, ..., 1) \tag{21.4}$$

Substituting (21.2) into (21.1) and pre-multiplying the resulting equation by $\mathbf{\Psi}^T$ results in the following uncoupled modal equation:

$$\begin{aligned} &\ddot{q}_i(t) + (\alpha + \beta\omega_i^2)\,\dot{q}_i(t) + \omega_i^2\,q_i(t) = p_i(t), \quad i = 1, 2, 3, ..., n. \\ &p_i(t) = \mathbf{\Psi}(i, :)^T\ \mathbf{f}(t) \end{aligned} \tag{21.5}$$

The above equation can be cast into a canonical form

$$\begin{aligned} &\dot{\mathbf{x}}_i(t) = \mathbf{A}_i\ \mathbf{x}_i(t) + \mathbf{b}\ p_i(t), \quad \mathbf{x}_i = [\,q_i(t) \quad \dot{q}_i(t)\,]^T \\ &\mathbf{A}_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -(\alpha + \beta\omega_i^2) \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \tag{21.6}$$

whose solution is given by

$$\mathbf{x}_i(t) = e^{\mathbf{A}_i\ t}\ \mathbf{x}_i(0)\ +\ \int_0^t e^{\mathbf{A}_i\ (t-\tau)}\ \mathbf{b}\ p_i(\tau)\ d\tau \tag{21.7}$$

It should be noted that the above solution provides only for one of the n-vector generalized modal coordinates, $\mathbf{q}(t)$ $(n \times 1)$.

Carrying out for the entire n-modal vector, the physical displacement, $\mathbf{u}(t)$ $(n \times 1)$, can be obtained from (21.2) by

$$\mathbf{u}(t) = \mathbf{\Psi}(1:n, 1:n)\,\mathbf{q}(t)\,(1:n, 1), \quad \mathbf{q} = [\,q_1(t), q_2(t), ..., q_n(t)\,]^T \tag{21.8}$$

While the solution method described in (21.2) - (21.8) appears to be straightfowrad, its practical implementation needs to overcome several computational challenges, which include:

(a) When the size of discrete finite element model increases, the task for obtaining a large number of modes (m), if not all, $m << n$, becomes computationally expensive. In practice, it is customary to truncate only part of the modes and obtain an approximate solution of the form

$$\mathbf{u}(t) \approx \mathbf{\Psi}(1:n, 1:m)\,\mathbf{q}(t)\,(1:m, 1), \quad \mathbf{q} = [\,q_1(t), q_2(t), ..., q_m(t)\,]^T\,, \quad m << n \tag{21.9}$$

It is not uncommon to have $m/n < (1/100 - 1/1000)$.

(b) A typical vehicle consists of many substructures whose structural characteristics are distinctly different from one to another. For example, a fuselage has different structural characteristics from wing structures. Likewise, engine blocks are considerably stiffer than the car frame structure. The impact of stiffness differences on the computed modes and mode shapes can lead to accuracy loss, and frequently to an unacceptable level.

(c) In modern manufacturing arrangements, rarely an aerospace company or automobile company designs, manufactures, assembles and tests the entire vehicle system. This means, except for the final performance evaluation, each substructure can be modeled, analyzed and tested before it can be assembled, as a separate and independent structure.

An alternative approach is to numerically integrate the equations of motion (21.1). We will discuss computational procedures of two direction algorithms in the next section. Their algorithmic properties will be examined later in the course.

§21.2 SOLUTION BY DIRECT INTEGRATION METHODS

There are two distinct direct time integration methods: explicit and implicit integration formulas. We summarize their computational sequences below.

§21.2.1 Central Difference Method for Undamped Case ($\mathbf{D} = \mathbf{0}$)

First, we express the acceleration vector $\ddot{\mathbf{u}}$ from(21.1) as

$$\ddot{\mathbf{u}}(t) = \mathbf{M}^{-1} \left(\mathbf{f} - \mathbf{K}\, \mathbf{u}(t)\right) \tag{21.10}$$

Hence, it is clear that if the mass matrix is diagonal, the computation for obtaining the acceleration vector would be greatly simplified. We now describe direction time integration by the central difference method:

Initial step:

Given the initial conditions, $\{\dot{\mathbf{u}}(0), \mathbf{u}(0), \mathbf{f}(t)\}$, obtain the velocity at the half step $\{t = h,\ h = \triangle t\}$ by

$$
\begin{aligned}
\ddot{\mathbf{u}}(0) &= \mathbf{M}^{-1}\ (\mathbf{f}(0) - \mathbf{K}\ \mathbf{u}(0)) \\
\dot{\mathbf{u}}(\tfrac{1}{2}h) &= \dot{\mathbf{u}}(0) + \tfrac{1}{2}h\ \ddot{\mathbf{u}}(0) \\
\mathbf{u}(h) &= \mathbf{u}(0) + h\ \dot{\mathbf{u}}(\tfrac{1}{2}h)
\end{aligned}
\tag{21.11}
$$

Subsequent steps

$$
\begin{aligned}
&T_{total} = hn_{max} \\
&\text{for}\ \ n = 1 : n_{max} \\
&\qquad \begin{aligned}
\ddot{\mathbf{u}}(n) &= \mathbf{M}^{-1}\ (\mathbf{f}(n) - \mathbf{K}\ \mathbf{u}(n)) \\
\dot{\mathbf{u}}(n + \tfrac{1}{2}) &= \dot{\mathbf{u}}(n - \tfrac{1}{2}) + h\ \ddot{\mathbf{u}}(n) \\
\mathbf{u}(n + 1) &= \mathbf{u}(n) + h\ \dot{\mathbf{u}}(n + \tfrac{1}{2})
\end{aligned} \\
&\text{end}
\end{aligned}
\tag{21.12}
$$

§21.2.2 The Trapezoidal Rule for Undamped Case (D = 0)

This method is also referred to Newmark's implicit rule with its free parameter chosen to be ($\alpha = \frac{1}{2}$, $\beta = 1/4$). Among several ways of implementing the trapezoidal integration rule, we will employ a summed form or half-interval rule given as follows.

$$
\begin{aligned}
\dot{\mathbf{u}}(n+\tfrac{1}{2}) &= \dot{\mathbf{u}}(n) + \tfrac{1}{2}h\ \ddot{\mathbf{u}}(n+\tfrac{1}{2})\\
\mathbf{u}(n+\tfrac{1}{2}) &= \mathbf{u}(n) + \tfrac{1}{2}h\ \dot{\mathbf{u}}(n+\tfrac{1}{2})\\
\\
\dot{\mathbf{u}}(n+1) &= 2\dot{\mathbf{u}}(n+\tfrac{1}{2}) - \dot{\mathbf{u}}(n)\\
\mathbf{u}(n+1) &= 2\mathbf{u}(n+\tfrac{1}{2}) - \mathbf{u}(n)
\end{aligned}
\tag{21.13}
$$

In using the preceding formula, one multiply the first of (21.13) by **M** to yield

$$
\mathbf{M}\ \dot{\mathbf{u}}(n+\tfrac{1}{2}) = \mathbf{M}\ \dot{\mathbf{u}}(n) + \tfrac{1}{2}h\ \mathbf{M}\ \ddot{\mathbf{u}}(n+\tfrac{1}{2}) \tag{21.14}
$$

The term $\mathbf{M}\ \ddot{\mathbf{u}}(n+\tfrac{1}{2})$ in the above equation is obtained from (21.1) as

$$
\mathbf{M}\ddot{\mathbf{u}}(n+\tfrac{1}{2}) = \mathbf{f}(n+\tfrac{1}{2}) - \mathbf{D}\dot{\mathbf{u}}(n+\tfrac{1}{2}) - \mathbf{K}\ \mathbf{u}(n+\tfrac{1}{2}) \tag{21.15}
$$

which, when substituted into (21.14), results in

$$
\begin{aligned}
&\mathbf{M}\ \dot{\mathbf{u}}(n+\tfrac{1}{2}) = \mathbf{M}\ \dot{\mathbf{u}}(n) + \tfrac{1}{2}h\ \{\mathbf{f}(n+\tfrac{1}{2}) - \mathbf{D}\dot{\mathbf{u}}(n+\tfrac{1}{2}) - \mathbf{K}\ \mathbf{u}(n+\tfrac{1}{2})\}\\
&\Downarrow\\
&[\mathbf{M} + \tfrac{1}{2}h\mathbf{D}]\ \dot{\mathbf{u}}(n+\tfrac{1}{2}) = \mathbf{M}\ \dot{\mathbf{u}}(n) + \tfrac{1}{2}h\ \{\mathbf{f}(n+\tfrac{1}{2}) - \mathbf{K}\ \mathbf{u}(n+\tfrac{1}{2})\}
\end{aligned}
\tag{21.16}
$$

Now multiply the second of (21.13) by $[\mathbf{M} + \frac{1}{2}h\mathbf{D}]$ to obtain

$$[\mathbf{M} + \tfrac{1}{2}h\mathbf{D}]\,\mathbf{u}(n+\tfrac{1}{2}) = [\mathbf{M} + \tfrac{1}{2}h\mathbf{D}]\,\mathbf{u}(n) + \tfrac{1}{2}h\,[\mathbf{M} + \tfrac{1}{2}h\mathbf{D}]\,\dot{\mathbf{u}}(n+\tfrac{1}{2}) \tag{21.17}$$

third, substitute the second term in the righthand side of (21.17) by (21.16), one obtains

$$\begin{gathered}
[\mathbf{M} + \tfrac{1}{2}h\mathbf{D}]\,\mathbf{u}(n+\tfrac{1}{2}) = [\mathbf{M} + \tfrac{1}{2}h\mathbf{D}]\,\mathbf{u}(n) + \tfrac{1}{2}h\,\{\mathbf{M}\,\dot{\mathbf{u}}(n) + \tfrac{1}{2}h\,\{\mathbf{f}(n+\tfrac{1}{2}) - \mathbf{K}\,\mathbf{u}(n+\tfrac{1}{2})\}\\
\Downarrow\\
[\mathbf{M} + \tfrac{1}{2}h\mathbf{D} + (\tfrac{1}{2}h)^2\mathbf{K}]\,\mathbf{u}(n+\tfrac{1}{2}) = \mathbf{M}\,\{\mathbf{u}(n) + \tfrac{1}{2}h\,\dot{\mathbf{u}}(n)\} + \tfrac{1}{2}h\,\mathbf{D}\mathbf{u}(n) + (\tfrac{1}{2}h)^2\,\mathbf{f}(n+\tfrac{1}{2})
\end{gathered} \tag{21.18}$$

Implicit integration steps

Assemble: $\mathbf{A} = [\mathbf{M} + \frac{1}{2}h\mathbf{D} + (\frac{1}{2}h)^2\mathbf{K}]$

Factor: $\mathbf{A} = \mathbf{L}\mathbf{U}$

$$
\begin{aligned}
&\text{for } \; n = 0 : n_{max} \\
&\qquad \mathbf{b}(n) = \mathbf{M}\,\{\mathbf{u}(n) + \tfrac{1}{2}h\,\dot{\mathbf{u}}(n)\} + \tfrac{1}{2}h\,\mathbf{D}\mathbf{u}(n) + (\tfrac{1}{2}h)^2\,\mathbf{f}(n+\tfrac{1}{2}) \\
&\qquad \mathbf{u}(n+\tfrac{1}{2}) = \mathbf{A}^{-1}\,\mathbf{b}(n), \quad \text{where } \; \mathbf{A}^{-1} = \mathbf{U}^{-1}\,\mathbf{L}^{-1} \\
&\qquad \dot{\mathbf{u}}(n+\tfrac{1}{2}) = \{\mathbf{u}(n+\tfrac{1}{2}) - \mathbf{u}(n)\}/(\tfrac{1}{2}h) \\
&\qquad \dot{\mathbf{u}}(n+1) = 2\,\dot{\mathbf{u}}(n+\tfrac{1}{2}) - \dot{\mathbf{u}}(n) \\
&\qquad \mathbf{u}(n+1) = 2\,\mathbf{u}(n+\tfrac{1}{2}) - \mathbf{u}(n) \\
&\text{end}
\end{aligned}
\tag{21.19}
$$

§21.3 DISCRETE APPROXIMATION OF MODAL SOLUTION

The modal-form solution of the equations of motion for linear structures given by (21.7) and (21.8) involves the convolution integral of the applied force. For general applied forces an exact evaluation of the convolution integral can involve a considerable effort. To this end, an approximate solution is utilized in practice. To this end, (21.7) is expressed in discrete form at time $t = nh$:

$$
\mathbf{x}_i(nh) = e^{\mathbf{A}_i\; nh}\,\mathbf{x}_i(0) \; + \int_0^{nh} e^{\mathbf{A}_i\;(nh-\tau)}\,\mathbf{b}\; p_i(\tau)\; d\tau \tag{21.20}
$$

Likewise, at time $t = nh + h$, $\mathbf{x}_i(nh+h)$ is given by

$$
\begin{aligned}
\mathbf{x}_i(nh+h) &= e^{\mathbf{A}_i\ nh+h}\ \mathbf{x}_i(0) + \int_0^{nh+h} e^{\mathbf{A}_i\ (nh+h-\tau)}\ \mathbf{b}\ p_i(\tau)\ d\tau \\
&= e^{\mathbf{A}_i h}\ [e^{\mathbf{A}_i\ nh}\ \mathbf{x}_i(0) + \int_0^{nh} e^{\mathbf{A}_i\ (nh-\tau)}\ \mathbf{b}\ p_i(\tau)\ d\tau] \\
&\quad + \int_{nh}^{nh+h} e^{\mathbf{A}_i\ (nh+h-\tau)}\ \mathbf{b}\ p_i(\tau)\ d\tau
\end{aligned}
\tag{21.21}
$$

The bracketed term in the above equation is $\mathbf{x}_i(nh)$ in view of (21.20) and the second term is approximated as

$$
\begin{aligned}
\int_{nh}^{nh+h} e^{\mathbf{A}_i\ (nh+h-\tau)}\ \mathbf{b}\ p_i(\tau)\ d\tau &\approx [\int_{nh}^{nh+h} e^{\mathbf{A}_i\ (nh+h-\tau)}\ d\tau]\ \mathbf{b}\ p_i(nh) \\
&\approx \mathbf{A}_i^{-1}(e^{\mathbf{A}_i h} - \mathbf{I})\ \mathbf{b}\ p_i(nh)
\end{aligned}
\tag{21.22}
$$

Substiutting this together the bracketed term by $\mathbf{x}_i(nh)$ into (21.21), $\mathbf{x}_i(nh+h)$ is approximated as

$$
\boxed{\mathbf{x}_i(nh+h) = e^{\mathbf{A}_i h}\ \mathbf{x}_i(nh) + \mathbf{A}_i^{-1}(e^{\mathbf{A}_i h} - \mathbf{I})\ \mathbf{b}\ p_i(nh), \quad \mathbf{x}_i = [\, q_i(nh+h), \dot{q}_i(nh+h)\,]^T}
\tag{21.23}
$$

Once $\mathbf{x}_i(nh+h)$ is computed, the physical displacement $\mathbf{u}(nh+h)$ is obtained via the modal summation expression (21.8).

§21.4 ILLUSTRATIVE PROBLEMS

Consider a beam with boundary constraints as shown in Figure 19.1. For illustrative purposes, two beams will be considered: a fixed-simply supported beam and a beam with boundary constraints with the following specific boundary conditions:

Fixed and simply supported ends:

$$w(0,t) = \frac{\partial w(0,t)}{\partial x} = 0, \quad w(L,t) = 0 \tag{21.24}$$

Beam with boundary springs:

$$w(0,t) = w(L,t) = 0, \qquad k_{\theta 1} = \frac{(1.e+5) * EI}{L}, \quad k_{\theta 2} = \frac{EI}{L} \tag{21.25}$$

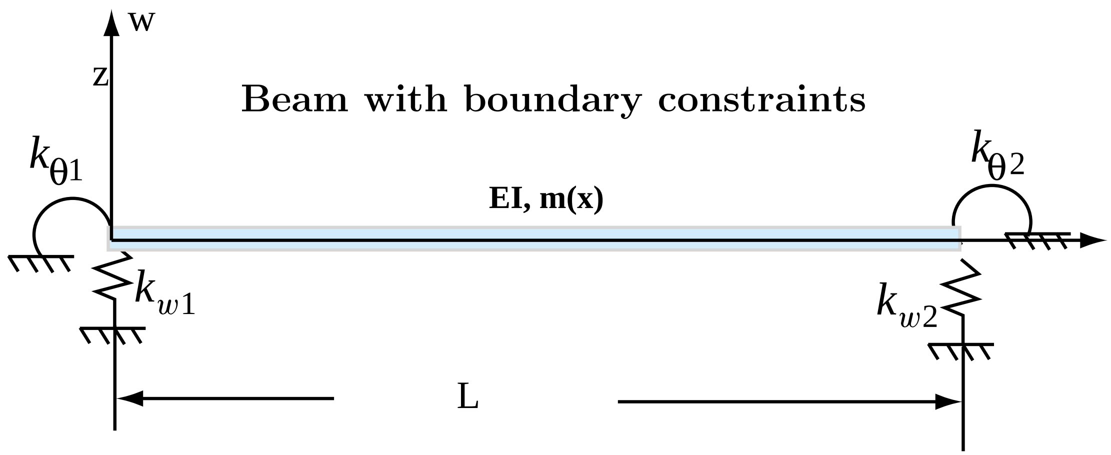

Fig. 19.1 Beam with boundary constraints

It should be noted that the end condition, $(w(0,t) = w(L,t) = 0)$, is equivalent to $(k_{w1} \to \infty, k_{w2} \to \infty)$. However, in computer implementation it is impractical to use $(k_{w1} \to \infty, k_{w2} \to \infty)$ due to limited floating point precision.

The applied force chosen are

$$\begin{aligned} \text{Step load:}\ \ & f(L/2,t) = 1.00, \quad 0 \le t \\ \text{Sinusoidal load:}\ \ & f(L/2,t) = \sin(2\pi f_f\ t) \end{aligned} \tag{21.26}$$

where the forcing frequency is set to $f_f = 1.5(\omega_1/2/\pi)$, with ω_1 being the fundamental frequency of the model problems.

Figures 2 and 3 illustrate time responses of a beam with boundary springs subject to unit mid-span step load. The responses by solid red lines are those obtained by using the central difference method, the ones with $'+'$ are by the trapezoidal rule, and the blue lines by the canonical formula(21.23).

The step increments used for the three methods are

$$h = \begin{cases} 1.8928E - 7, & \text{for the central difference method} \\ 1.5861E - 5, & \text{for the canonical formula} \\ 1.6.3445E - 5, & \text{for the trapezoidal rule} \end{cases} \tag{21.27}$$

It should be noted that the stepsie for the central difference method is dictated by the computational stability whereas taht of the canonical formula and teh trapezoidal rule by accurcy considerations. We hope to revisit this issue later in the course.

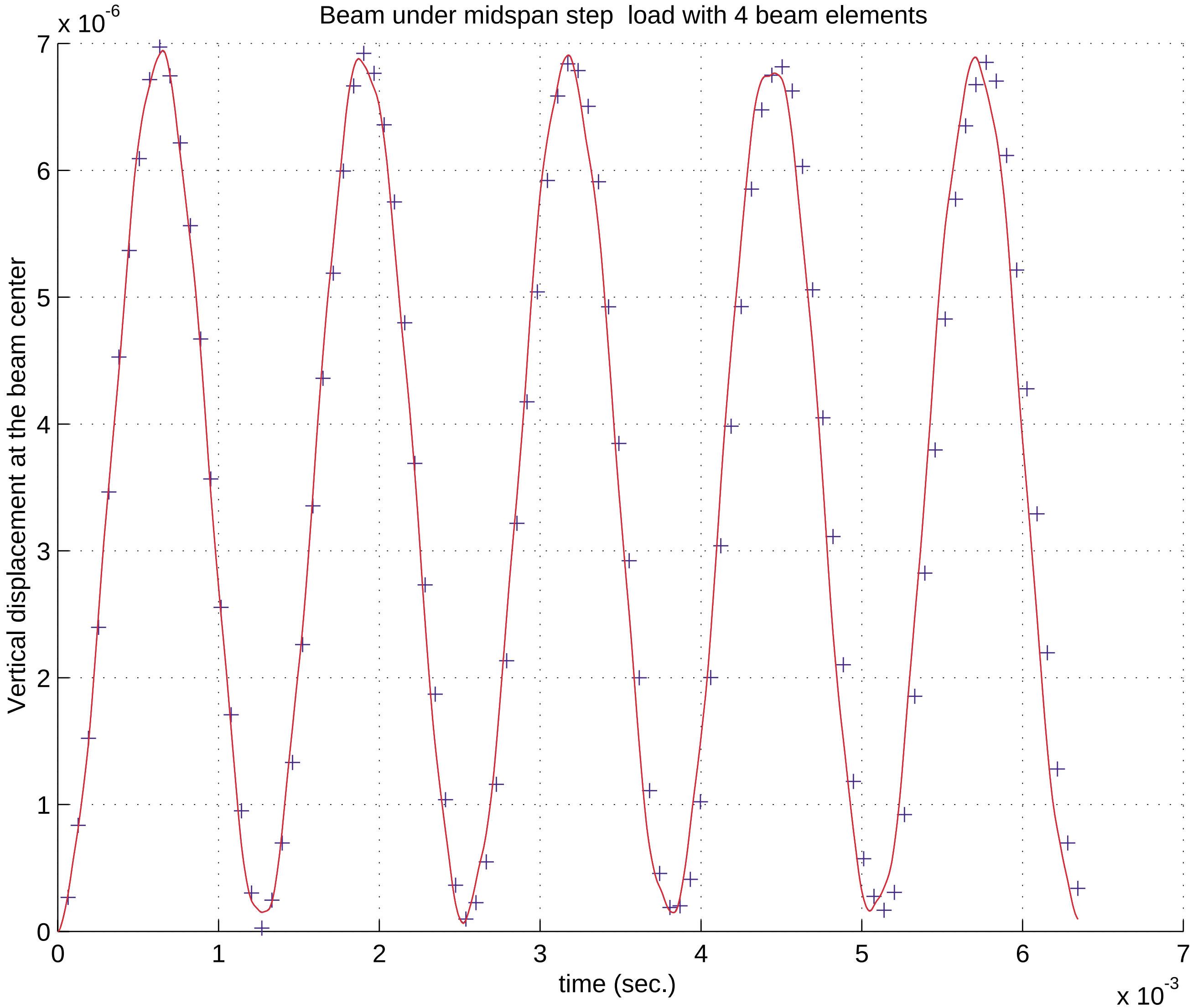

Fig. 19.2 Beam Midspan Vertical Time Response

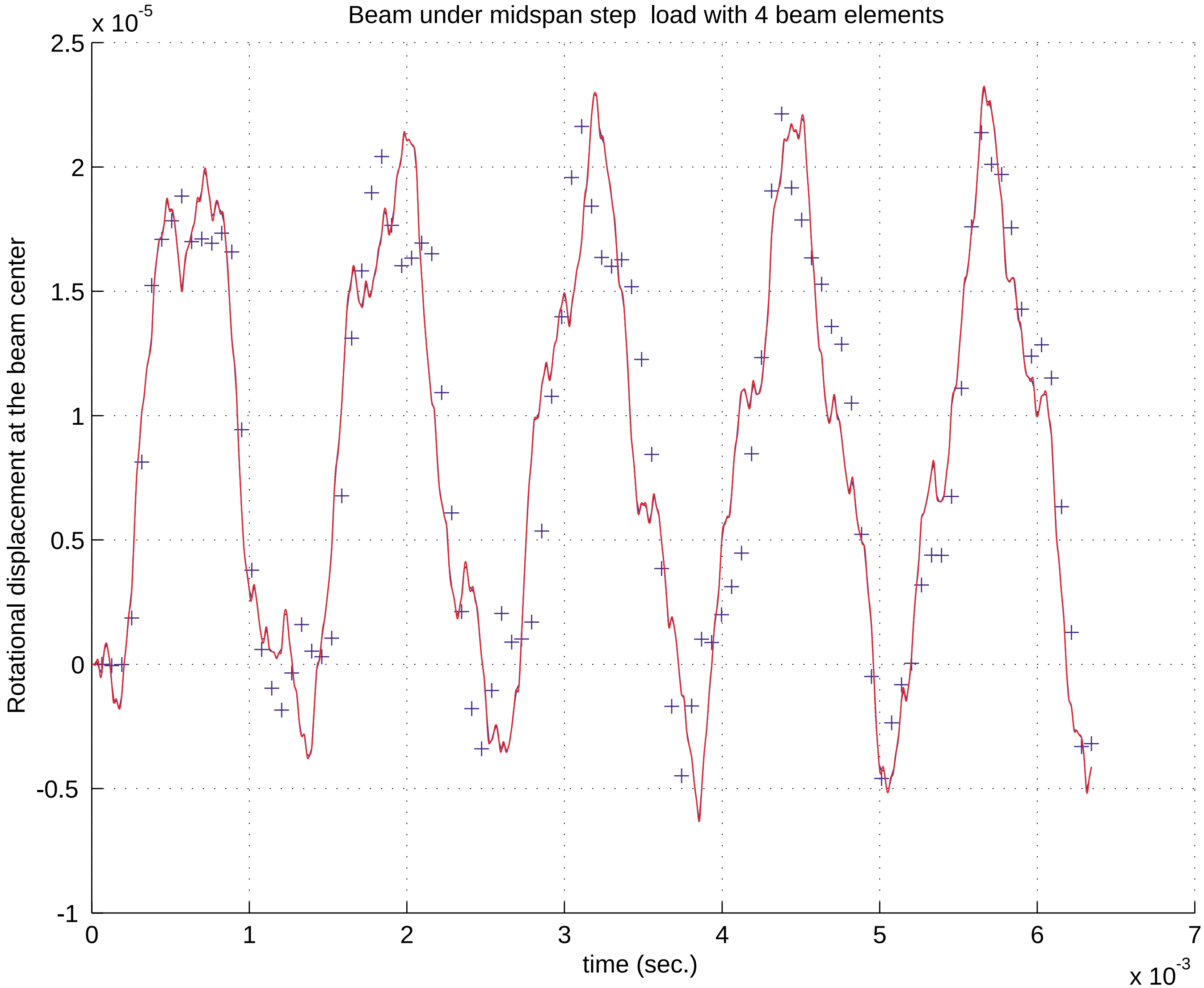

Fig. 19.3 Beam Midspan Rotational Time Response