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Reduced-Order Modeling of Vibrating Structures: Introduction

§20.1 ASSEMBLING SUBSTRUCTURES TO MODEL A TOTAL STRUCTURE

Consider a toy airplane model shown in Figure 20.1. Suppose that your job is to develop a structural model by utilizing the substructural models. Specifically, manufacturers of the substructures have provided only the dominant modes and their mode shapes, and maybe their modal damping properties.

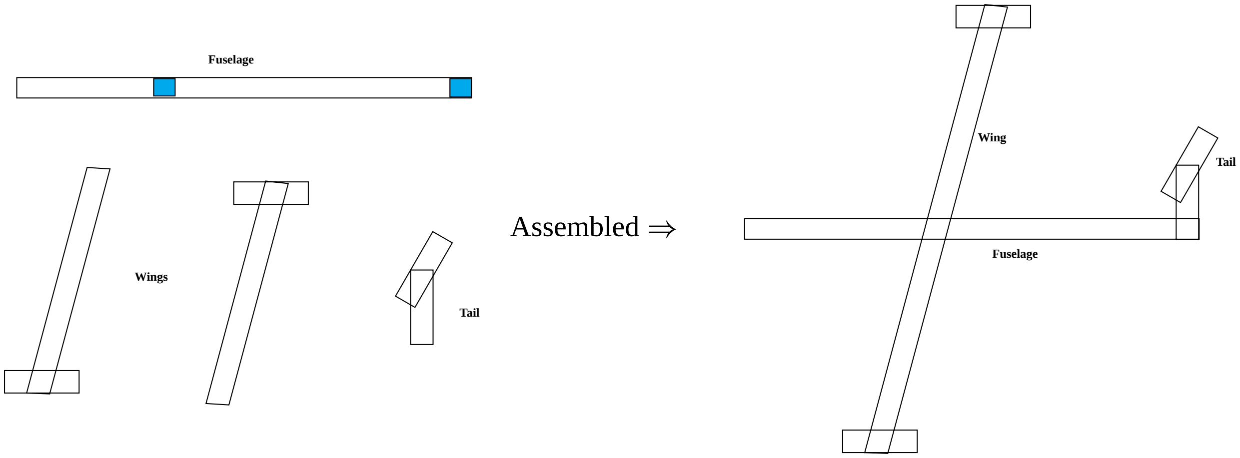

Figure 20.1 A toy strucuture

Mathematically, each of the reduced-order substructural models provided, $(\phi(:, 1:m), \{\omega_k^2, k = 1, 2, ...m\})$, is an approximation of the large-order model given by

$$\mathbf{K}\,\mathbf{\Psi} = \mathbf{M}\,\mathbf{\Psi}\mathbf{\Lambda} \tag{20.1}$$

for each of the substructures.

In order for us to utilize the reduced-order substructural models for the development of the total system model, it is critical to understand the nature of the reduced-order models. This is addressed below.

§20.2 TYPES OF REDUCED-ORDER MODELS

The reduced-order substructural models may be categorize according to how the interface degrees of freedom are treated. The modes ϕ are called:

fixed-interface normal modes if all of the interface degrees of freedom $\mathbf{u}_\Gamma$ are restrained,

free-interface normal modes if none of $\mathbf{u}_\Gamma$ is restrained, and

hybrid-interface normal modes if part of $\mathbf{u}_\Gamma$ are restrained

where subscript $\{\Gamma\}$ denotes the substructural boundaries.

Hence, a typical substructure vibration problem may be expressed as(see Figure 20.2) as

$$\begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{I\Gamma}^T & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \Psi_I & \Psi_{I\Gamma} \\ \Psi_{\Gamma I} & \Psi_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{I\Gamma} \\ \mathbf{M}_{I\Gamma}^T & \mathbf{M}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \Psi_I & \Psi_{I\Gamma} \\ \Psi_{\Gamma I} & \Psi_\Gamma \end{bmatrix} \mathbf{\Lambda} \tag{20.2}$$

where subscript $\{I\}$ denotes the interior nodes, and the physical substructural displacement,$\mathbf{u}$, is related to the modal displacement,$\mathbf{q}$, according to

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_I \\ \mathbf{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \Psi_I & \Psi_{I\Gamma} \\ \Psi_{\Gamma I} & \Psi_\Gamma \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{q}_\Gamma \end{bmatrix} \tag{20.3}$$

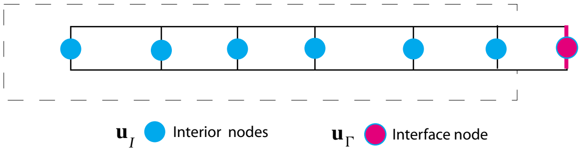

Figure 20.2 Substructural DOF Classification

The preceding distinction is essential in interpreting the results of each substructural vibrations. For some applications, one invokes *loaded-interface normal modes* (pre-stressed concrete columns, cables, prebuckled space trusses), which means one augments the interface stiffness $\mathbf{K}_{\Gamma\Gamma}$ with pre-stress stiffness.

In what follows, we will first discuss various fixed-interface modeling, which is followed by free-interfce modeling approach.

§20.3 REDUCTION OF INTERIOR DEGREES OF FREEDOM AND INTERFACE MODELING

Often, the size of the interior degrees of freedom far exceeds those of the substructural interface degreesof freedom. For this reason, we begin with the reduction of the interior degrees of freedom from the eigenvalue problem:

$$\begin{array}{cccccc} [\,\mathbf{K}_{II}\,] & [\,\Psi_I\,] & = & [\,\mathbf{M}_{II}\,] & [\,\Psi_I\,] & \mathbf{\Lambda}_I \\ (n_I \times n_I) & (n_I \times m_I) & & (n_I \times n_I) & (n_I \times m_I) & (m_I \times m_I) \\ & m_I << n_I & & & & \end{array} \tag{20.4}$$

Referring to (20.3), it is clear that Ψ_I is only one of the four submatric elements that are needed to relate the modal displacement, $\mathbf{q}$, to the physical displacement, $\mathbf{u}$. We present two techniques to construct the remainder submatrices.

It is important to observe that the above process of mode truncation,viz., retaining only m_I-modes from n_I-modes, implies

$$\Psi_{\Gamma I} \approx \mathbf{0} \tag{20.5}$$

§20.3.1 Quasistatic Constraint Interface Modes

We consider a quasistatic equilibrium state in which the reaction force $\boldsymbol{\lambda}_\Gamma$ that will produce unit desplacement at the interface $\mathbf{u}_\Gamma = \mathbf{I}_\Gamma$ is applied. For this case,one has the following equilibrium state:

$$\begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{I\Gamma}^T & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \Psi_{I\Gamma} \\ \mathbf{I}_{\Gamma\Gamma} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{\lambda}_\Gamma \end{bmatrix} \tag{20.6}$$

Note that the unit interface displacement constitute a unitary matrix, $\mathbf{I}_{\Gamma\Gamma}$.

Equation(20.6) enables us to obtain

$$\Psi_{I\Gamma} = -\mathbf{K}_{II}^{-1}\,\mathbf{K}_{I\Gamma}\,\mathbf{I}_{\Gamma\Gamma} = -\mathbf{K}_{II}^{-1}\,\mathbf{K}_{I\Gamma} \tag{20.7}$$

Finally, since we retain the physical boundary displacemen, $\mathbf{u}_\Gamma$, we have

$$\Psi_\Gamma = \mathbf{I}_{\Gamma\Gamma} \tag{20.8}$$

Thus, using (20.4) - (20.8), the substructural displacement **u** can be approximated by

$$\mathbf{u} \approx \begin{bmatrix} \mathbf{u}_I \\ \mathbf{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \Psi_I & -\mathbf{K}_{II}^{-1} K_{I\Gamma} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{bmatrix} \tag{20.9}$$

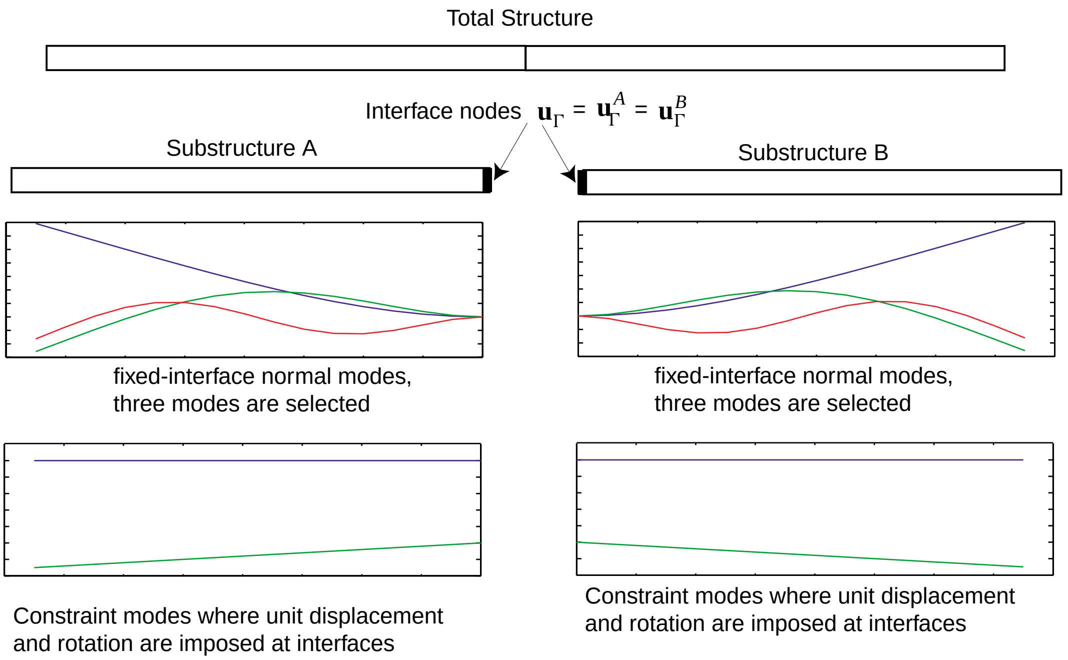

Fig. 20.3 Free-free beam partitioned into two substructures
(Note the fixed-interface modes ϕ_I and the constrained modes ϕ^c)

Figure 20.3 illustrates schematically the fixed-interface normal modes of each substructure and the constraint modes for a free-free beam partitioned into two substructures. Note that the fixed-interface modal amplitudes are zero at the interface while those of the constraint modes are unity, respectively.

§20.3.2 Attachment Interface Modes

Instead of applying a unit displacement on Γ, if a unit force is applied to Γ and no force is applied throughout the rest of the structure, we have teh following static equilibrium condition:

$$\begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{I\Gamma}^T & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \Psi_{I\Gamma} \\ \Psi_{\Gamma\Gamma} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{I}_{\Gamma\Gamma} \end{bmatrix} \tag{20.10}$$

from which one obtains

$$\begin{bmatrix} \Psi_{I\Gamma} \\ \Psi_{\Gamma\Gamma} \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{II}^{-1}\,\mathbf{K}_{I\Gamma} \\ \mathbf{I}_{\Gamma\Gamma} \end{bmatrix} \left(\mathbf{K}_{\Gamma\Gamma} - \mathbf{K}_{I\Gamma}^T\,\mathbf{K}_{II}^{-1}\,\mathbf{K}_{I\Gamma}\right)^{-1} \tag{20.11}$$

The substructural displacement $\mathbf{u}$ can now be approximated by

$$\mathbf{u} \approx \begin{bmatrix} \mathbf{u}_I \\ \mathbf{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \Phi_I & -\mathbf{K}_{II}^{-1}\,K_{I\Gamma}\,\mathbf{K}_{\Gamma\Gamma}^{-S} \\ \mathbf{0} & \mathbf{K}_{\Gamma\Gamma}^{-S} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{bmatrix}, \quad \mathbf{K}_{\Gamma\Gamma}^S = \left(\mathbf{K}_{\Gamma\Gamma} - \mathbf{K}_{I\Gamma}^T\,\mathbf{K}_{II}^{-1}\,\mathbf{K}_{I\Gamma}\right) \tag{20.12}$$

Comparing the constraint mode (20.9) with the above attachment mose (20.12), it is seen that the interface constraint modes are far less expensive to generate than the interface attachment modes.

§20.4 ENERGY CONSIDERATIONS OF REDUCED-ORDER SUBSTRUCTURAL MODEL

Before we launch on reducing the substructures using the constraint interface modes (20.9) or the attachment interface modes (20.12), it would be instructive to examine the resulting substructural energy expressions due to the approximations.

§20.4.1 Approximate substructural energy when using the interface constraint modes

The kinetic energy is obtained by

$$\begin{aligned} T &= \tfrac{1}{2}\dot{\mathbf{u}}^T \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{I\Gamma} \\ \mathbf{M}_{I\Gamma}^T & \mathbf{M}_{\Gamma\Gamma} \end{bmatrix} \dot{\mathbf{u}} \\ &\approx \tfrac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}_I \\ \dot{\mathbf{u}}_\Gamma \end{bmatrix}^T \begin{bmatrix} \Psi_I & -\mathbf{K}_{II}^{-1} K_{I\Gamma} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma} \end{bmatrix}^T \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{I\Gamma} \\ \mathbf{M}_{I\Gamma}^T & \mathbf{M}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \Psi_I & -\mathbf{K}_{II}^{-1} K_{I\Gamma} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_I \\ \dot{\mathbf{u}}_\Gamma \end{bmatrix} \\ &\approx \tfrac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}_I \\ \dot{\mathbf{u}}_\Gamma \end{bmatrix}^T \begin{bmatrix} \mathbf{I} & \mathcal{M}_{I\Gamma}^c \\ \mathcal{M}_{I\Gamma}^{T\,c} & \mathcal{M}_{\Gamma\Gamma}^c \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_I \\ \dot{\mathbf{u}}_\Gamma \end{bmatrix} \end{aligned} \tag{20.13}$$

$$\begin{aligned} \mathcal{M}_{I\Gamma}^c &= \mathbf{M}_{I\Gamma} - \Psi_I^T\,\mathbf{M}_{II}\mathbf{K}_{II}^{-1}\,\mathbf{K}_{I\Gamma} \\ \mathcal{M}_{\Gamma\Gamma}^c &= \mathbf{M}_{\Gamma\Gamma} + \mathbf{K}_{I\Gamma}^T\,\mathbf{K}_{II}^{-1}\,\mathbf{M}_{II}\,\mathbf{K}_{II}^{-1}\,\mathbf{K}_{I\Gamma} - \mathbf{K}_{I\Gamma}^T\,\mathbf{K}_{II}^{-1}\,\mathbf{M}_{I\Gamma} - \mathbf{M}_{\Gamma I}\,\mathbf{K}_{II}^{-1}\,\mathbf{K}_{I\Gamma} \end{aligned}$$

where $\dot{\mathbf{u}}$ is approximated by the interface constraint modes given by (20.9).

Similarly, the strain energy is obtained by

$$
\begin{aligned}
U &= \tfrac{1}{2}\mathbf{u}^T \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{I\Gamma}^T & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \mathbf{u} \\
&\approx \tfrac{1}{2} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{bmatrix}^T \begin{bmatrix} \Psi_I & -\mathbf{K}_{II}^{-1} K_{I\Gamma} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{I\Gamma}^T & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \Psi_I & -\mathbf{K}_{II}^{-1} K_{I\Gamma} \\ \mathbf{0} & \mathbf{I}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{bmatrix} \\
&\approx \tfrac{1}{2} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{bmatrix}^T \begin{bmatrix} \boldsymbol{\Lambda}_I & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{\Gamma\Gamma}^S \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{bmatrix}
\end{aligned}
\tag{20.14}
$$

$$
\mathbf{K}_{\Gamma\Gamma}^S = \left(\mathbf{K}_{\Gamma\Gamma} - \mathbf{K}_{I\Gamma}^T\, \mathbf{K}_{II}^{-1}\, \mathbf{K}_{I\Gamma}\right)
$$

Observe that the stiffness matrix in the strain energy given by (20.14) is diagonal corresponding to the interior modes, $\boldsymbol{\lambda}_I$, and block diagonal corresponding to the interfce degrees of freedom. However, the resulting mass matrix corresponding to the interface degrees of freedom is full as can be seen in (20.13). This indicates that, unless the interface degrees of freedom is relatively small, considerable computations are required to generate the approximate reduced-order substructural kinetic energy expression.

§20.4.2 Approximate substructural energy when using the interface attachment modes

The kinetic energy is obtained by

$$
\begin{aligned}
T &= \tfrac{1}{2}\dot{\mathbf{u}}^T \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{I\Gamma} \\ \mathbf{M}_{I\Gamma}^T & \mathbf{M}_{\Gamma\Gamma} \end{bmatrix} \dot{\mathbf{u}} \\
&\approx \tfrac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}_I \\ \dot{\mathbf{u}}_\Gamma \end{bmatrix}^T \begin{bmatrix} \Phi_I & -\mathbf{K}_{II}^{-1}\, K_{I\Gamma}\, \mathbf{K}_{\Gamma\Gamma}^{-S} \\ \mathbf{0} & \mathbf{K}_{\Gamma\Gamma}^{-S} \end{bmatrix}^T \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{I\Gamma} \\ \mathbf{M}_{I\Gamma}^T & \mathbf{M}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \Phi_I & -\mathbf{K}_{II}^{-1}\, K_{I\Gamma}\, \mathbf{K}_{\Gamma\Gamma}^{-S} \\ \mathbf{0} & \mathbf{K}_{\Gamma\Gamma}^{-S} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{bmatrix} \\
&\approx \tfrac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}_I \\ \dot{\mathbf{u}}_\Gamma \end{bmatrix}^T \begin{bmatrix} \mathbf{I} & \mathcal{M}_{I\Gamma}^a \\ \mathcal{M}_{I\Gamma}^{T\ a} & \mathcal{M}_{\Gamma\Gamma}^a \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_I \\ \dot{\mathbf{u}}_\Gamma \end{bmatrix}
\end{aligned}
\tag{20.15}
$$

$$
\begin{aligned}
&\mathcal{M}_{I\Gamma}^c = \Psi_I^T\, [-\mathbf{M}_{II}\Psi_{I\Gamma}^a + \mathbf{M}_{I\Gamma}\mathbf{K}_{\Gamma\Gamma}^{-S}] \\
&\mathcal{M}_{\Gamma\Gamma}^c = \Psi_{I\Gamma}^{aT}\mathbf{M}_{\Gamma\Gamma}\Psi_{I\Gamma}^a + \mathbf{K}_{\Gamma\Gamma}^{-S}\, \mathbf{M}_{II}\, \mathbf{K}_{\Gamma\Gamma}^{-S} \ - \Psi_{I\Gamma}^{aT}\, \mathbf{M}_{I\Gamma}\, \mathbf{K}_{\Gamma\Gamma}^{-S} - \mathbf{K}_{\Gamma\Gamma}^{-S}\, \mathbf{M}_{I\Gamma}^T\, \Psi_{I\Gamma}^a \\
&\Psi_{I\Gamma}^a = \mathbf{K}_{II}^{-1}\, K_{I\Gamma}\, \mathbf{K}_{\Gamma\Gamma}^{-S}
\end{aligned}
$$

where $\dot{\mathbf{u}}$ is approximated by the interface constraint modes given by (20.12).

Similarly, the strain energy is obtained by

$$
\begin{aligned}
U &= \tfrac{1}{2}\mathbf{u}^T \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{I\Gamma}^T & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \mathbf{u} \\
&\approx \tfrac{1}{2} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{bmatrix}^T \begin{bmatrix} \Phi_I & -\mathbf{K}_{II}^{-1}\, K_{I\Gamma}\, \mathbf{K}_{\Gamma\Gamma}^{-S} \\ \mathbf{0} & \mathbf{K}_{\Gamma\Gamma}^{-S} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{I\Gamma}^T & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \Phi_I & -\mathbf{K}_{II}^{-1}\, K_{I\Gamma}\, \mathbf{K}_{\Gamma\Gamma}^{-S} \\ \mathbf{0} & \mathbf{K}_{\Gamma\Gamma}^{-S} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{bmatrix} \\
&\approx \tfrac{1}{2} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{bmatrix}^T \begin{bmatrix} \mathbf{\Lambda}_I & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{\Gamma\Gamma}^{-S} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{bmatrix}
\end{aligned}
\tag{20.16}
$$

$$\mathbf{K}_{\Gamma\Gamma}^S = \left(\mathbf{K}_{\Gamma\Gamma} - \mathbf{K}_{I\Gamma}^T\, \mathbf{K}_{II}^{-1}\, \mathbf{K}_{I\Gamma}\right)$$

Observe that, while the form of its mass matrix is the same as in teh case ofthe interface constraint modes, more computations are required as the terms involve the Schur complement, $\mathbf{K}_{\Gamma\Gamma}^S$.