

## ASEN 5022 - Final Project: Component Mode Synthesis

Assigned: 27 April 2004

Due: 5:00 p.m., Tuesday, 04 May 2004

Consider the same plate you have analyzed in Homework #6, which is partitioned into two substructures. In that reduced-order modeling, you have selected interior modes consecutively, viz., from the first to the k-th mode. In this term project, you will utilize the following mode selection indicators through which you may selectively retain interior modes, e.g., modes (1,2,5,12) instead of modes (1,2,3,4).

### Individual mode selection indicator:

$$\epsilon_k = 1 - \left| \frac{\mathbf{K}_{\Gamma I} [\Psi_I(:, k) \omega_I^{-2}(k) \Psi_I^T(:, k)] \mathbf{K}_{I\Gamma}}{\mathbf{K}_{\Gamma I} \mathbf{K}_{II}^{-1} \mathbf{K}_{I\Gamma}} \right| \quad (1)$$

### Cummulative mode selection indicator:

$$\epsilon_{1:k} = 1 - \left| \frac{\mathbf{K}_{\Gamma I} [\Psi_I(:, 1:k) \omega_I^{-2}(1:k) \Psi_I^T(:, 1:k)] \mathbf{K}_{I\Gamma}}{\mathbf{K}_{\Gamma I} \mathbf{K}_{II}^{-1} \mathbf{K}_{I\Gamma}} \right| \quad (2)$$

The details on the rationale of the preceding mode selection indicators are provided in Appendix.

In using the above mode selection indicators, note that for each addition of mode k, a mode that yields smaller values of  $\epsilon_k$  and/or  $\epsilon_{1:k}$  is the desired mode to be kept in your reduced-order modeling.

*Tasks to be carried out:*

**Problems to be analyzed:** Three boundary conditions to be examined: free-free-free-free, fixed-free-fixed-free, fixed-fixed-free-free.

1. For each of the three boundary conditions, carry out vibration analysis employing your reduced-order models such that the frequency errors for the six modes compared to the full model will be within 0.5 percent, while modes are selected in a consecutive manner, i.e., from first mode to k-the mode.
2. While doing the above analysis, compute the two mode selection indicators as you increase the number of modes and plot the mode selection indicators vs. the number modes selected as well as individual mode selection indicator.
3. With the insight you have gained from **Tasks 1 and 2**, apply the mode selection indicators judiciously and carry out vibration analysis for the three boundary conditions by *selectively* retaining modes that are identified to offer higher accuracy in the global analysis.
4. For each of the interior mode you have selected, plot their mode shapes (interior) and endeavor to come up with an explanation why their inclusion make sense or no sense at all.
5. Summarize your learning experience through this term project and give your verdict whether the mode selection indicators have been useful to you or not. Explain why they have been useful or they have not been useful. Overall, which of the two indicators do you find more suited for your problem?
6. **Extra credit:** Propose an improved mode selection indicator with numerical evidence!

Have fun!

## Appendix: Derivation of Residual Flexibility Error

The eigenvalue problem for a substructure is given by

$$\begin{aligned} \mathbf{K}\mathbf{u} &= \omega^2 \mathbf{M}\mathbf{u} \\ \mathbf{K} &= \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{\Gamma I} & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \\ \mathbf{M} &= \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{I\Gamma} \\ \mathbf{M}_{\Gamma I} & \mathbf{M}_{\Gamma\Gamma} \end{bmatrix} \end{aligned} \quad (3)$$

As the Craig-Bampton method reduces only the interior modes from

$$\mathbf{K}_{II}\Psi = \mathbf{M}_{II}\Psi\Lambda \quad (4)$$

we partition further the above eigenproblem in the form of

$$\mathbf{K}_{II}[\Psi_I \quad \Psi_R] = \mathbf{M}_{II}[\Psi_I \quad \Psi_R] \begin{bmatrix} \Lambda_I & 0 \\ 0 & \Lambda_R \end{bmatrix} \quad (5)$$

where  $\Psi_I$  and  $\Psi_R$  are mode shapes to be used and to be discarded, respectively.

Hence, the substructural displacement  $\mathbf{u}$  may be expressed as

$$\mathbf{u} = \begin{bmatrix} \Psi_I & \Psi_R & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{q}_R \\ \mathbf{u}_\Gamma \end{Bmatrix} \quad (6)$$

which, when substituted into (3), yields

$$\begin{aligned} \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{\Gamma I} & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \mathbf{u} &= \omega^2 \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{I\Gamma} \\ \mathbf{M}_{\Gamma I} & \mathbf{M}_{\Gamma\Gamma} \end{bmatrix} \mathbf{u} \\ \Downarrow \\ \begin{bmatrix} \Psi_I & \Psi_R & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{\Gamma I} & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \Psi_I & \Psi_R & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{q}_R \\ \mathbf{u}_\Gamma \end{Bmatrix} &= \\ \omega^2 \begin{bmatrix} \Psi_I & \Psi_R & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{I\Gamma} \\ \mathbf{M}_{\Gamma I} & \mathbf{M}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \Psi_I & \Psi_R & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{q}_R \\ \mathbf{u}_\Gamma \end{Bmatrix} \\ \Downarrow \\ \begin{bmatrix} \Psi_I^T \mathbf{K}_{II} \Psi_I & \Psi_I^T \mathbf{K}_{II} \Psi_R & \Psi_I^T \mathbf{K}_{I\Gamma} \\ \Psi_R^T \mathbf{K}_{II} \Psi_I & \Psi_R^T \mathbf{K}_{II} \Psi_R & \Psi_R^T \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{\Gamma I} \Psi_I & \mathbf{K}_{\Gamma I} \Psi_R & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{q}_R \\ \mathbf{u}_\Gamma \end{Bmatrix} &= \omega^2 \begin{bmatrix} \Psi_I^T \mathbf{M}_{II} \Psi_I & \Psi_I^T \mathbf{M}_{II} \Psi_R & \Psi_I^T \mathbf{M}_{I\Gamma} \\ \Psi_R^T \mathbf{M}_{II} \Psi_I & \Psi_R^T \mathbf{M}_{II} \Psi_R & \Psi_R^T \mathbf{M}_{I\Gamma} \\ \mathbf{M}_{\Gamma I} \Psi_I & \mathbf{M}_{\Gamma I} \Psi_R & \mathbf{M}_{\Gamma\Gamma} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{q}_R \\ \mathbf{u}_\Gamma \end{Bmatrix} \\ \Downarrow \\ \begin{bmatrix} \Lambda_I & 0 & \Psi_I^T \mathbf{K}_{I\Gamma} \\ 0 & \Lambda_R & \Psi_R^T \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{\Gamma I} \Psi_I & \mathbf{K}_{\Gamma I} \Psi_R & \mathbf{K}_{\Gamma\Gamma} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{q}_R \\ \mathbf{u}_\Gamma \end{Bmatrix} &= \omega^2 \begin{bmatrix} \mathbf{I}_I & 0 & \Psi_I^T \mathbf{M}_{I\Gamma} \\ 0 & \mathbf{I}_R & \Psi_R^T \mathbf{M}_{I\Gamma} \\ \mathbf{M}_{\Gamma I} \Psi_I & \mathbf{M}_{\Gamma I} \Psi_R & \mathbf{M}_{\Gamma\Gamma} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{q}_R \\ \mathbf{u}_\Gamma \end{Bmatrix} \\ \text{since } \Psi_I^T \mathbf{K}_{II} \Psi_I &= \Lambda_I, \quad \Psi_I^T \mathbf{M}_{II} \Psi_I = \mathbf{I}_I, \quad \Psi_I^T \mathbf{K}_{II} \Psi_R = \Psi_I^T \mathbf{M}_{II} \Psi_R = 0, \quad \text{etc.} \end{aligned} \quad (7)$$

Note that  $\mathbf{q}_R$  can be expressed from the second row of the above equation as

$$\begin{aligned} [\Lambda_R - \omega^2 \mathbf{I}_R] \mathbf{q}_R &= -\Psi_R^T [\mathbf{K}_{I\Gamma} - \omega^2 \mathbf{M}_{I\Gamma}] \mathbf{u}_\Gamma \\ \Downarrow \\ \mathbf{q}_R &= -[\Lambda_R - \omega^2 \mathbf{I}_R]^{-1} \Psi_R^T [\mathbf{K}_{I\Gamma} - \omega^2 \mathbf{M}_{I\Gamma}] \mathbf{u}_\Gamma \end{aligned} \quad (8)$$

Substituting this into the last of (7) gives

$$\begin{aligned} & \begin{bmatrix} \Lambda_I & \Psi_I^T \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{\Gamma I} \Psi_I & \mathbf{K}_{\Gamma\Gamma} - \mathbf{K}_{\Gamma I} \bar{\mathbf{F}}_R \mathbf{K}_{I\Gamma} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{Bmatrix} \\ &= \omega^2 \begin{bmatrix} \mathbf{I}_I & \Psi_I^T \mathbf{M}_{I\Gamma} \\ \mathbf{M}_{\Gamma I} \Psi_I & \mathbf{M}_{\Gamma\Gamma} - \mathbf{K}_{\Gamma I} \bar{\mathbf{F}}_R \mathbf{M}_{I\Gamma} - \mathbf{M}_{\Gamma I} \bar{\mathbf{F}}_R \mathbf{K}_{I\Gamma} + \omega^2 \mathbf{M}_{\Gamma I} \bar{\mathbf{F}}_R \mathbf{M}_{I\Gamma} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_I \\ \mathbf{u}_\Gamma \end{Bmatrix} \\ & \bar{\mathbf{F}} = \Psi_R [\Lambda_R - \omega^2 \mathbf{I}_R]^{-1} \Psi_R^T \approx \Psi_R \Lambda_R^{-1} \Psi_R^T + \omega^2 \Psi_R \Lambda_R^{-2} \Psi_R^T \end{aligned} \quad (9)$$

Using the relation

$$\mathbf{K}_{II} = [\Psi_I \quad \Psi_R] \begin{bmatrix} \Lambda_I & 0 \\ 0 & \Lambda_R \end{bmatrix} [\Psi_I \quad \Psi_R]^T \quad (10)$$

one finds

$$\boxed{\Psi_R \Lambda_R^{-1} \Psi_R^T = \mathbf{K}_{II}^{-1} - \Psi_I \Lambda_I^{-1} \Psi_I^T} \quad (11)$$

The term  $(\Psi_R \Lambda_R^{-1} \Psi_R^T)$  is called a residual flexibility whose magnitude can serve as an indicator on how much approximation is permitted in using the interior modes and mode shape,  $(\Psi_I, \Lambda_I)$  for reduced-order modeling.

Observe that as the number of retained modes  $\{\Lambda_I(1:k)\}$  increases, the contribution of the residual modes  $\{\Lambda_R(k+1:n)\}$  will decrease. Specifically, for each mode addition, say, k-th mode, its contribution may be assessed via

$$\epsilon_k = 1 - \left| \frac{\mathbf{K}_{\Gamma I} [\Psi_I(:, k) \omega_I^{-2}(k) \Psi_I^T(:, k)] \mathbf{K}_{I\Gamma}}{\mathbf{K}_{\Gamma I} \mathbf{K}_{II}^{-1} \mathbf{K}_{I\Gamma}} \right| \quad (12)$$

Alternatively, one may employ a cumulative indicator given by

$$\epsilon_{1:k} = 1 - \left| \frac{\mathbf{K}_{\Gamma I} [\Psi_I(:, 1:k) \omega_I^{-2}(1:k) \Psi_I^T(:, 1:k)] \mathbf{K}_{I\Gamma}}{\mathbf{K}_{\Gamma I} \mathbf{K}_{II}^{-1} \mathbf{K}_{I\Gamma}} \right| \quad (13)$$

Clearly, if both  $\epsilon_k$  and  $\epsilon_{1:k}$  are close to zero, then they are equivalent to taking the entire interior modes. This does not lead to model reduction. It is important to realize that one does not select modes consecutively from the first mode to the k-th modes in order to minimize the number of modes to be selected while capturing the predominant behaviour of the total model.