Solution of ASEN 5022 - Second Mid-Term Quiz - 18 March 2004

Problem 1: Understanding cable vibrations *(30 points)*

1.1 *Consider a cable with a concentrated mass of M as shown in Figure 1.1 below. Rank their fundamental frequencies from the highest to the lowest and explain your answer. (15 points) (Hint: Assume W(x) =q(t)* $\sin(\pi x/L)$ *and compute the total kinetic energy of the system.)*

$$T_{cable\ only} = \tfrac{1}{2}\,[\frac{\rho L}{2}]\,\dot{q}^2(t) \qquad \{T_{mass}\}_{x=a} = \tfrac{1}{2}M\{\dot{W}(a)\}^2 \tag{1}$$

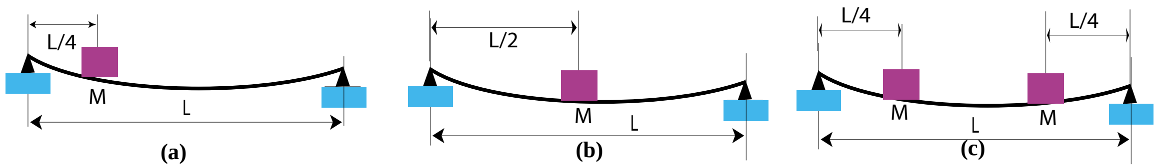


Figure for Problems 1.1

Solution: Suppose the strain energy can be expressed as

$$V_{beam} = \tfrac{1}{2}Kq^2(t), \quad K > 0 \tag{2}$$

Evaluating T_{mass} at $x = \{L/4, L/2, 3L/\}$, one obtains

$$\begin{aligned} T_{mass}(x = L/4) &= \tfrac{1}{2}\{\tfrac{1}{2}M\}\dot{q}^2(t) \\ T_{mass}(x = L/2) &= \tfrac{1}{2}\{M\}\dot{q}^2(t) \\ T_{mass}(x = 3L/4) &= \tfrac{1}{2}\{\tfrac{1}{2}M\}\dot{q}^2(t) \end{aligned} \tag{3}$$

The free vibration equations for the above three cases can be obtained from Euler-Lagrange or Hamilton's principle as follows:

$$\begin{aligned} &\text{Problem 1.1(a):} \quad [\tfrac{1}{2}\rho L + \tfrac{1}{2}M]\ddot{q}(t) + Kq(t) = 0, \quad \omega_a = \sqrt{\frac{K}{\tfrac{1}{2}\rho L + \tfrac{1}{2}M}} \\ &\text{Problem 1.1(b):} \quad [\tfrac{1}{2}\rho L + M]\ddot{q}(t) + Kq(t) = 0, \quad \omega_b = \sqrt{\frac{K}{\tfrac{1}{2}\rho L + M}} \\ &\text{Problem 1.1(c):} \quad [\tfrac{1}{2}\rho L + M]\ddot{q}(t) + Kq(t) = 0, \quad \omega_c = \sqrt{\frac{K}{\tfrac{1}{2}\rho L + M}} \end{aligned} \tag{4}$$

Hence, the frequency ranking is given by

$$\boxed{\omega_a > \omega_b = \omega_c}$$

1.2 *Consider a cable with discrete springs as shown in Figure 1.2 below. Rank their fundamental frequencies from the highest to the lowest and explain your answer. (15 points) (Hint: Assume W(x) =q(t)* $\sin(\pi x/L)$ *and compute the total potential energy of the system.)*

$$V_{cable\ only} = \tfrac{1}{2}\,K\,q^2(t), \quad K = [\frac{\pi^2 T}{2L}] \qquad \{K_{spring}\}_{x=a} = \tfrac{1}{2}k_1\{W(a)\}^2 \tag{5}$$

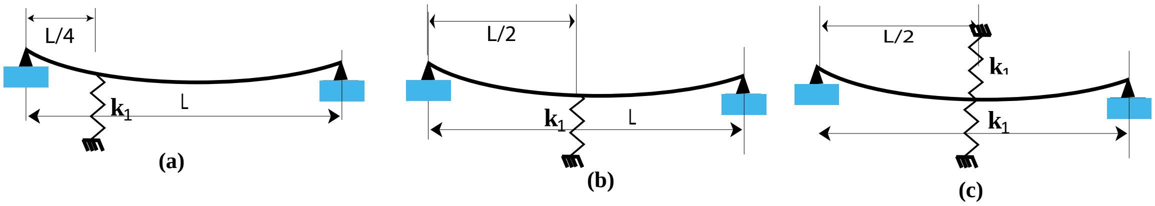


Figure for Problem 1.2

Solution:

Evaluating V_{spring} at $x = \{L/4, L/2\}$, one obtains

$$\begin{aligned} V_{spring}(x = L/4) &= \tfrac{1}{2}\{\tfrac{1}{2}k_1\}q^2(t) \\ V_{spring}(x = L/2) &= \tfrac{1}{2}\{k_1\}q^2(t) \end{aligned} \tag{6}$$

The free vibration equations for the above three cases can be obtained from Euler-Lagrange or Hamilton's principle as follows:

$$\begin{aligned} &\text{Problem 1.2(a):} \quad \tfrac{1}{2}\rho L\ddot{q}(t) + [K + \tfrac{1}{2}k_1]q(t) = 0, \quad \omega_a = \sqrt{\frac{K + \frac{1}{2}k_1}{\frac{1}{2}\rho L}} \\ &\text{Problem 1.2(b):} \quad \tfrac{1}{2}\rho L\ddot{q}(t) + [K + k_1]q(t) = 0, \quad \omega_b = \sqrt{\frac{K + k_1}{\frac{1}{2}\rho L}} \\ &\text{Problem 1.2(c):} \quad \tfrac{1}{2}\rho L\ddot{q}(t) + [K + 2k_1]q(t) = 0, \quad \omega_c = \sqrt{\frac{K + 2k_1}{\frac{1}{2}\rho L}} \end{aligned} \tag{7}$$

Hence, the frequency ranking is given by

$$\boxed{\omega_a < \omega_b < \omega_c}$$

Problem 2: *(30 points)*

We have studied that Hamilton's principle for a beam shown below can be expressed as

$$\begin{aligned}&\int_{t_1}^{t_2}\{[EI\ w_{xxx}(x,t)]\ \delta w(x,t)\}\ |_{x=0}^{x=L}\ dt - \int_{t_1}^{t_2}\{k_{w2}\ w(L,t)\ \delta w(L,t) + k_{w1}\ w(0,t)\ \delta w(0,t)\}\ dt \\ &- \int_{t_1}^{t_2}\{[EI\ w_{xx}(x,t)]\delta w_x(x,t)\}|_{x=0}^{x=L}\ dt - \int_{t_1}^{t_2}\{k_{\theta 2}\ w_x(L,t)\ \delta w_x(L,t) + k_{\theta 1}\ w_x(0,t)\ \delta w_x(0,t)\}\ dt \\ &- \int_{t_1}^{t_2}\int_0^L\{\ m(x)w_{tt}(x,t) + [EI\ w_{xxxx}(x,t)] - f(x,t)\ \}\ \delta w(x,t)\ dxdt = 0\end{aligned} \tag{8}$$

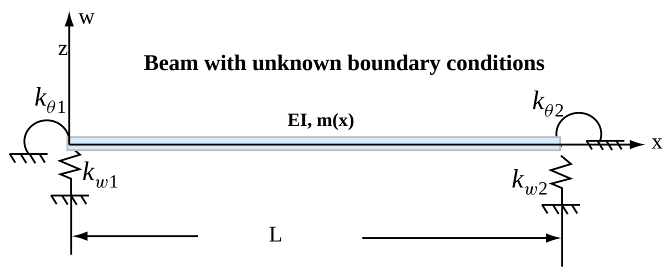


Figure for Problem 2

2.1 Identify the boundary condition(s) when $k_{w1} \neq 0$, $k_{w2} \neq 0$, $k_{\theta 1} \neq 0$, and $k_{\theta 2} \neq 0$. Are the boundary conditions identified natural or essential ones? (5 pts)

Solution:

$$\begin{aligned}&[EI\ w_{xxx}(x,t)]|_{x=L} - k_{w2}\ w_x(L,t)\ = 0 \\ &-[EI\ w_{xx}(x,t)]|_{x=L} - k_{\theta 2}\ w_x(L,t) = 0 \\ &-[EI\ w_{xxx}(x,t)]|_{x=0} - k_{w1}\ w(0,t)\ = 0 \\ &[EI\ w_{xx}(x,t)]|_{x=0}\ - k_{\theta 1}\ w_x(0,t) = 0\end{aligned} \tag{9}$$

All of the above four boundary conditions are natural.

2.2 For the case of $\{k_{w1} \neq 0,\ \ k_{w2} \to \infty,\ \ k_{\theta 1} = 0,\ \ k_{\theta 2} \neq 0\}$, sketch the mode shape of the first vibration mode and explain why you believe your prediction makes sense. (Hint: Try to draw for limiting cases of $\{k_{w1} \to \infty, k_{w1} \to 0\}$ and $\{k_{\theta 2} \to \infty, k_{\theta 2} \to 0\}$ then draw for finite cases of $\{k_{w1}, k_{\theta 2}\}$). (15 pts)

2.3 An experimental data of a beam indicates that the fundamental frequency of the beam is far higher than that of the corresponding cantilever beam ($\beta L = 1.875$) but somewhat lower than that of the simply supported beam ($\beta L = \pi$) with a deflection observed at its right end during test. Can you speculate its boundary conditions? (10 points)

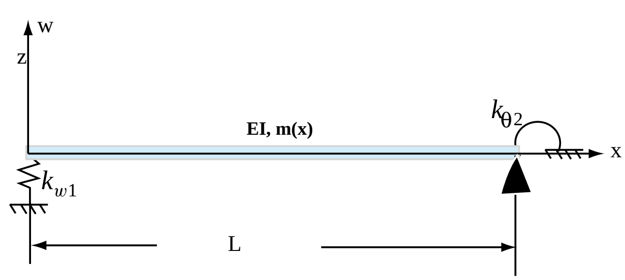


Figure for Problem 2.2

Solution:

Figures (a), (b), (c), and (d) below illustrate the following special cases:

Figure (a): The fundamental mode shape of a cantilever beam, viz., the case of $(k_{w1} = 0, k_{theta1} = 0, k_{w2} \to \infty, \ {}_{theta2} \to \infty)$.

Figure (b): The fundamental mode shape of a beam with simply supported and fixed ends, viz., the case of $(k_{w1} \to \infty, k_{theta1} = 0, k_{w2} \to \infty, \ {}_{theta2} \to \infty)$.

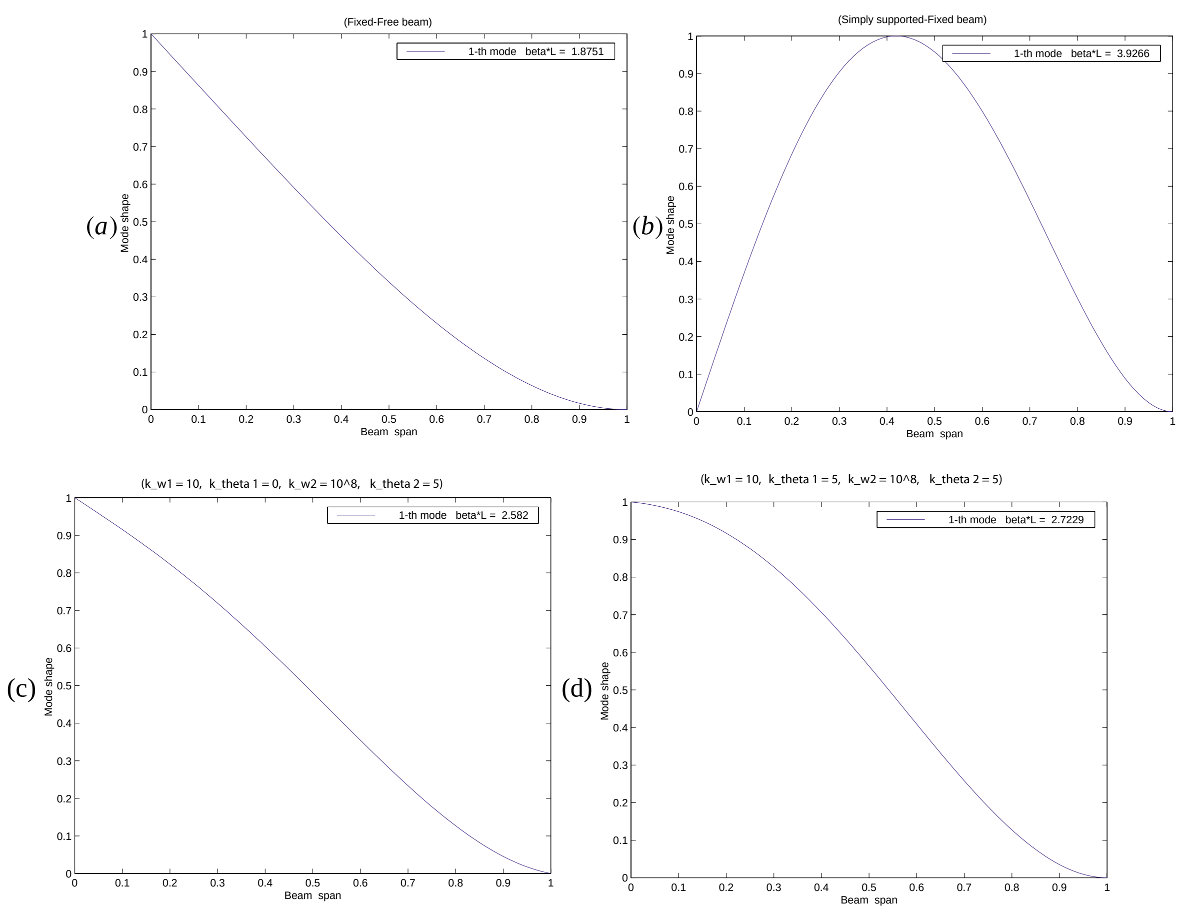
Figure (c): The fundamental mode shape of a beam with simply supported and fixed ends, viz., the case of $(k_{w1} = 10, k_{theta1} = 0, k_{w2} \to \infty, \ {}_{theta2} = 5)$.

Figure (d): The fundamental mode shape of a beam with simply supported and fixed ends, viz., the case of $(k_{w1} = 10, k_{theta1} > 0, k_{w2} \to \infty, \ {}_{theta2} >> 0)$.

Remark 1: The magnitude of the mode shape at $x = 0$ as unity should not be confused as the tip displacement! In real vibration analysis, the physical magnitude of the deflection at $x = 0$ should be very small.

Problem 2.2: The probable mode shapes of the case would fall between Figures (a) and (b), depending on the mode shape slope at $x = 0$ as their fundamental frequency ranges from $\beta L = 1.875$ and $\beta L = \pi$. Two mode shapes resembling Figure (a) are given in figures (c) and (d).

Problem 2.3: Both Figures (c) and (d) will satisfy the observed cased. Furthermore, if the slope conditions are known (not stated in Problem 2.3), then one should evaluate carefully evaluate the rotational spring constants at both ends. The distinction of mode shapes between figure (c) and (d) should be noted.



Problems 2.2 and 2.3: Mode shapes for Cantilever and Simply-supported-and-fixed Beams

Problem 3: Understanding beam vibrations *(40 points)*

3.1 *Consider a simply supported beam with a concentrated mass of M as shown in Figure 3.1 below. Rank their fundamental frequencies from the highest to the lowest. (10 points) (Hint: Assume W(x) =q(t)* $\sin(\pi x/L)$ *and compute the total kinetic energy of the system.)*

$$T_{beam\ only} = \tfrac{1}{2}\,[\frac{\rho L}{2}]\,\dot{q}^2(t)$$
$$\{T_{mass}\}_{x=a} = \tfrac{1}{2}M\{\dot{W}(a)\}^2 \tag{10}$$

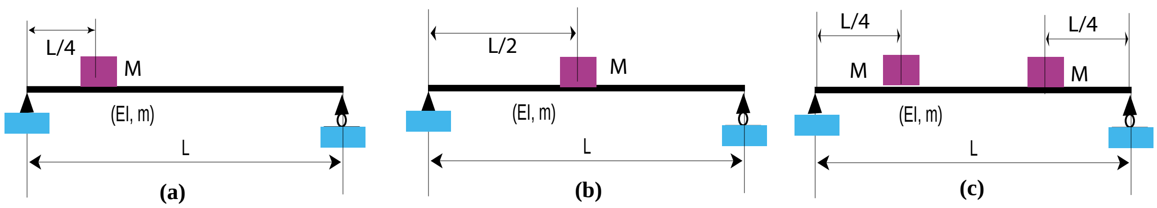


Figure for Problems 3.1

3.2 *Referring to Figure 3.1, will the lowest fundamental frequency (that you have chosen in Problem 3.1) among the three cases be lower or higher than that of the simply supported beam without discrete mass? Explain your answer. (10 points)*

3.3 *Consider a simply supported beam with discrete springs as shown in Figure 3.2 below. Rank their fundamental frequencies from the highest to the lowest. (10 points) (Hint: Assume W(x) =q(t)* $\sin(\pi x/L)$ *and compute the total potential energy of the system.)*

$$V_{beam\ only} = \tfrac{1}{2}\,[\frac{\pi^4 EI}{2L^3}]\,q^2(t)$$
$$\{K_{spring}\}_{x=a} = \tfrac{1}{2}k_1\{W(a)\}^2 \tag{11}$$

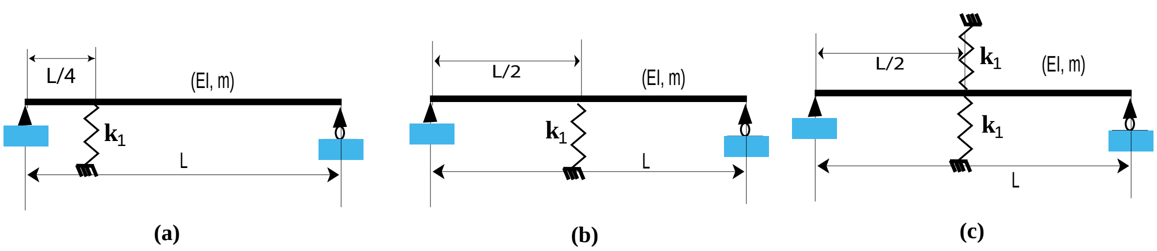


Figure for Problem 3.3 and 3.4

3.4 *Referring to Figure 3.2, will the lowest fundamental frequency (that you have chosen in Problem 2.3) among the three cases be lower or higher than that of the simply supported beam without discrete spring? Explain your answer. (10 points)*

Solution:

Problem 3.1. Their rank is exactly the same as in Problem 1.1, viz.,

$$\boxed{\omega_a > \omega_b = \omega_c}$$

Problem 3.2. Suppose the strain energy can be expressed as

$$V_{beam} = \tfrac{1}{2}Kq^2(t), \quad K > 0 \tag{12}$$

Since

$$T_{mass}(x = L/4) = \frac{1}{2}\{\frac{1}{2}M\}\dot{q}^2(t) \tag{13}$$

we have

$$\begin{aligned} \text{Cable without tip mass:} \quad & \omega_a^{no\ tip\ mass} = \sqrt{\frac{K}{\frac{1}{2}\rho L}} \\ \text{with tip mass:} \quad & \omega_a^{with\ tip\ mass} = \sqrt{\frac{K}{\frac{1}{2}\rho L + \frac{1}{2}M}} \end{aligned} \tag{14}$$

Hence, we conclude

$$\boxed{\omega_a^{with\ tip\ mass} < \omega_a^{no\ tip\ mass}}$$

Problem 3.3 The ranking is exactly the same as in Problem 1.2, viz.,

$$\boxed{\omega_a < \omega_b < \omega_c}$$

Problem 3.4 The case of (a) from the Figures for Problem 3.3. and 3.4 yields the smallest fundamental frequency from Problem 3.3.

Since we have

$$V_{spring}(x = L/4) = \frac{1}{2}\{\frac{1}{2}k_1\}q^2(t) \tag{15}$$

we have

$$\begin{aligned} \text{Cable without discrete spring:} \quad & \omega_a^{no\ spring} = \sqrt{\frac{K}{\frac{1}{2}\rho L}} \\ \text{Cable with spring:} \quad & \omega_a^{with\ spring} = \sqrt{\frac{K + \frac{1}{2}k_1}{\frac{1}{2}\rho L}} \end{aligned} \tag{16}$$

Hence, we conclude

$$\boxed{\omega_a^{no\ spring} < \omega_a^{with\ spring}}$$