ASEN 5022 - Spring 2005

Dynamics of Aerospace Structures

Lecture 12: 24 February

Beams with Uncertain Boundary Conditions

Consider a beam whose two ends are constrained by transverse springs (k_{w1}, k_{w2}) as well as rotational springs ($k_{\theta 1}$, $k_{\theta 2}$) as shown in the figure below.

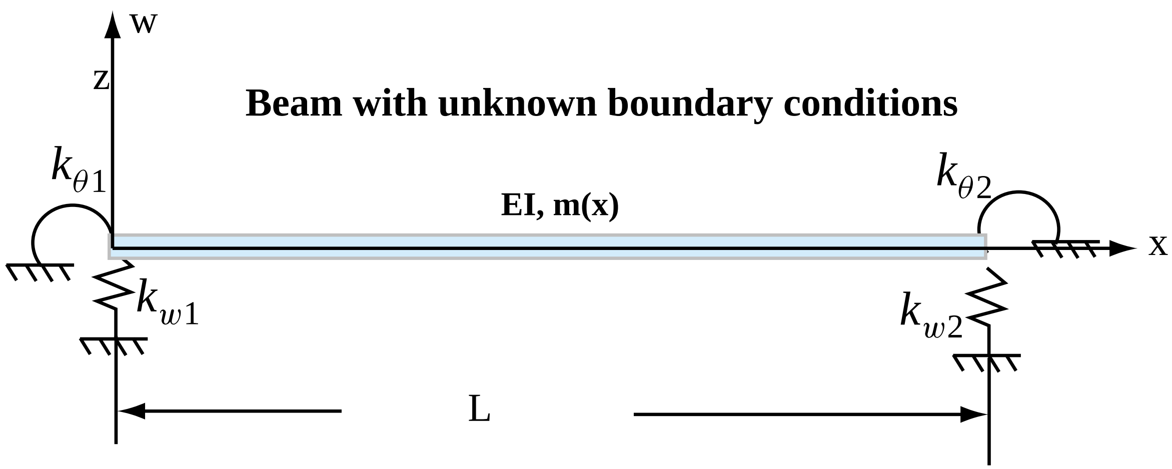


Fig. 1 Modeling of unknown boundary conditions

Before we proceed further, it should be noted that the mathematically known boundary conditions can be realized by taking the limit values of the four spring constants as follows:

$$\begin{aligned}
\textbf{Simply supported}: k_{w1} = k_{w2} \to \infty \quad & \textbf{and} \quad k_{\theta 1} = k_{\theta 2} = 0 \\
\textbf{Cantilever beam } x = 0: k_{w1} = k_{\theta 1} \to \infty \quad & \textbf{and} \quad k_{w2} = k_{\theta 2} = 0 \\
\textbf{Fixed-fixed}: k_{w1} = k_{\theta 1} \to \infty \quad & \textbf{and} \quad k_{w2} = k_{\theta 2} \to \infty \\
\textbf{Free-free}: k_{w1} = k_{\theta 1} = 0 \quad & \textbf{and} \quad k_{w2} = k_{\theta 2} = 0
\end{aligned} \tag{1}$$

For intermediate values of the unknown springs a convenient way of identifying the boundary conditions is to invoke a variational formulation. This is because the appropriate boundary conditions, both natural and essential, are determined as part of variational process.

Kinetic energy:

$$T = \tfrac{1}{2} \int_0^L m(x)\ w(x,t)_t^2 dx, \quad w(x,t)_t = \frac{\partial w(x,t)}{\partial t} \tag{2}$$

Potential energy of the beam:

$$\begin{gathered} V_b = \tfrac{1}{2} \int_0^L \{\ EI(x)\ w(x,t)_{xx}^2\ +\ P(x)\ w(x,t)_x^2\ \}\ dx \\ w(x,t)_{xx} = \frac{\partial^2 w(x,t)}{\partial x^2}, \quad w(x,t)_x = \frac{\partial w(x,t)}{\partial x} \end{gathered} \tag{3}$$

where $EI(x)$ **is the bending rigidity and** $P(x)$ **is the pre-stressed axial force.**

Potential energy of four unknown springs:

$$V_s = \frac{1}{2}\{\, k_{w1}\; w(0,t)^2 + k_{\theta 1}\; w(0,t)_x^2 + k_{w2}\; w(L,t)^2 + k_{\theta 2}\; w(L,t)_x^2 \tag{4}$$

External energy due to distributed applied force f(x, t), shear forces $Q_{(1,2)}$ and moments $M_{(1,2)}$ applied at both ends:

$$\begin{aligned}\delta W = \int_0^L \mathbf{f}(x,t)\,\delta w(x,t)\,dx + Q_1\,\delta w(0,t) \\ + M_1\,\delta w(0,t)_x + Q_2\,\delta w(L,t) \\ + M_2\,\delta w(L,t)_x\end{aligned} \tag{5}$$

$$\begin{aligned}Q_1 = Q(0,t), \quad Q_2 = Q(L,t) \\ M_1 = M(0,t), \quad M_2 = M(L,t)\end{aligned} \tag{6}$$

Hamilton's principle

$$\int_{t_1}^{t_2} [\delta T - \delta V_b - \delta V_s + \delta W \,]\, dt = 0 \tag{7}$$

Kinetic energy

$$\int_{t_1}^{t_2} \delta T \; dt = -\int_0^L \int_{t_1}^{t_2} m(x)\; w(x,t)_{tt} \delta w(x,t)\; dt\; dx \tag{8}$$

Variation of the internal energy of the beam V_b

$$\begin{aligned} \delta V_b =& [EI\; w(x,t)_{xx} \delta w(x,t)_x]|_0^L \\ &- \{[EI\; w(x,t)_{xx}]_x \delta w(x,t)\}|_0^L \\ &+ \int_0^L \{\, [EI\; w(x,t)_{xx}]_{xx} - [P\; w(x,t)_x]_x \,\} \delta w(x,t)\; dx \end{aligned} \tag{9}$$

Variation of the potential energy of the unknown boundary forces and moments

$$\delta V_s = \{\, k_{w1}\ w(0,t)\ \delta w(0,t) + k_{\theta 1}\ w(0,t)_x\ \delta w(0,t)_x \\ + k_{w2}\ w(L,t)\ \delta w(L,t) + k_{\theta 2}\ w(L,t)_x\ \delta w(L,t)_x \tag{10}$$

$$\int_{t_1}^{t_2} [\delta T - \delta V_b - \delta V_s + \delta W\ ]\, dt = 0 \tag{11}$$

$$
\begin{aligned}
&\int_{t_1}^{t_2} [\delta T - \delta V_b - \delta V_s + \delta W\,]\, dt = \\
&- \int_{t_1}^{t_2} \{-[EI\ w(L,t)_{xx}]_x + k_{w2}\ w(L,t) - Q_2\,\}\,\delta w(L,t)dt \\
&+ \int_{t_1}^{t_2} \{-[EI\ w(0,t)_{xx}]_x - k_{w1}\ w(0,t) + Q_1\,\}\,\delta w(0,t)dt \\
&- \int_{t_1}^{t_2} \{[EI\ w(L,t)_{xx}] + k_{\theta 2}\ w(L,t)_x\ - M_2\,\}\,\delta w(L,t)_x dt \\
&+ \int_{t_1}^{t_2} \{[EI\ w(0,t)_{xx}] - k_{\theta 1}\ w(0,t)_x\ + M_1\,\}\,\delta w(0,t)_x dt \\
&- \int_{t_1}^{t_2} \int_0^L \{\, m(x) w(x,t)_{tt} + [EI\ w(x,t)_{xx}]_{xx} \\
&\qquad - f(x,t)\,\}\,\delta w(x,t)\, dxdt = 0
\end{aligned}
\tag{12}
$$

The governing equation of motion:

$$m(x)\ w(x,t)_{tt} + [EI\ w(x,t)_{xx}]_{xx} - f(x,t) = 0 \tag{13}$$

The boundary conditions:

$$\begin{aligned}
\{-[EI\ w(L,t)_{xx}]_x + k_{w2}\ w(L,t)\ - Q_2\}\ \delta w(L,t) &= 0 \\
\{-[EI\ w(0,t)_{xx}]_x - k_{w1}\ w(0,t)\ + Q_1\}\ \delta w(0,t) &= 0 \\
\{[EI\ w(L,t)_{xx}] + k_{\theta 2}\ w(L,t)_x\ - M_2\}\ \delta w(L,t)_x &= 0 \\
\{[EI\ w(0,t)_{xx}] - k_{\theta 1}\ w(0,t)_x\ + M_1\}\ \delta w(0,t)_x &= 0
\end{aligned} \tag{14}$$

Free vibrations of a beam

The free vibration of beams can be modeled from (13) **and** (14) **by setting**

$$P = Q_1 = Q_2 = M_1 = M_2 = f(x,t) = 0 \tag{15}$$

so that (13) **and** (14) **are simplified to**

$$\begin{aligned} m(x)\ w(x,t)_{tt} + [EI\ w(x,t)_{xx}]_{xx} &= 0 \\ \{-[EI\ w(L,t)_{xx}]_x + k_{w2}\ w(L,t)\ \}\ \delta w(L,t) &= 0 \\ \{-[EI\ w(0,t)_{xx}]_x - k_{w1}\ w(0,t)\ \}\ \delta w(0,t) &= 0 \\ \{[EI\ w(L,t)_{xx}] + k_{\theta 2}\ w(L,t)_x\ \}\ \delta w(L,t)_x &= 0 \\ \{[EI\ w(0,t)_{xx}] - k_{\theta 1}\ w(0,t)_x\ \}\ \delta w(0,t)_x &= 0 \end{aligned} \tag{16}$$

Free Vibration Governing Equation and Four Natural Boundary Conditions:

$$
\begin{aligned}
m(x)\ w(x,t)_{tt} + [EI\ w(x,t)_{xx}]_{xx} &= 0 \\
\{-[EI\ w(L,t)_{xx}]_{x} + k_{w2}\ w(L,t)\} &= 0 \\
\{-[EI\ w(0,t)_{xx}]_{x} - k_{w1}\ w(0,t)\} &= 0 \\
\{[EI\ w(L,t)_{xx}] + k_{\theta 2}\ w(L,t)_{x}\} &= 0 \\
\{[EI\ w(0,t)_{xx}] - k_{\theta 1}\ w(0,t)_{x}\} &= 0
\end{aligned}
\tag{17}
$$

Four natural boundary conditions - cont'd

$$w(x,t) = W(x)e^{j\omega t} \tag{18}$$

which, when substituted into (17)**, yields**

$$\begin{aligned}
-\frac{\omega^2 m}{EI} W(x) + W(x)_{xxxx} = \ 0, \quad 0 \le x \le L \\
-W(0)_{xxx} - \frac{k_{w1}}{EI} W(0) = 0 \\
w(0)_{xx} - \frac{k_{\theta 1}}{EI} W(0)_x = 0 \\
-W(L)_{xxx} + \frac{k_{w2}}{EI} W(L) = 0 \\
w(L)_{xx} + \frac{k_{\theta 2}}{EI} W(L)_x = 0
\end{aligned} \tag{19}$$

The frequency equation of vibrations of a uniform beam can be obtained by assuming the solution in form of

$$
\begin{aligned}
W &= c_1 \sin \beta x + c_2 \cos \beta x + c_3 \sinh \beta x + c_4 \cosh \beta x \\
\beta^4 &= \frac{\omega^2 m}{EI} \\
W_x &= \beta \left(c_1 \cos \beta x - c_2 \sin \beta x + c_3 \cosh \beta x + c_4 \sinh \beta x\right) \\
W_{xx} &= \beta^2 \left(-c_1 \sin \beta x - c_2 \cos \beta x + c_3 \sinh \beta x + c_4 \cosh \beta x\right) \\
W_{xxx} &= \beta^3 \left(-c_1 \cos \beta x + c_2 \sin \beta x + c_3 \cosh \beta x + c_4 \sinh \beta x\right)
\end{aligned} \tag{20}
$$

$$\begin{bmatrix} \bar{\beta}^3 & -\bar{k}_{w1} & -\bar{\beta}^3 & -\bar{k}_{w1} \\ -\bar{k}_{\theta 1} & -\bar{\beta} & -\bar{k}_{\theta 1} & \bar{\beta} \\ \begin{matrix} -\bar{\beta}^3 \cos\bar{\beta} \\ -\bar{k}_{w2} \sin\bar{\beta} \end{matrix} & \begin{matrix} \bar{\beta}^3 \sin\bar{\beta} \\ -\bar{k}_{w2} \cos\bar{\beta} \end{matrix} & \begin{matrix} \bar{\beta}^3 \cosh\bar{\beta} \\ -\bar{k}_{w2} \sinh\bar{\beta} \end{matrix} & \begin{matrix} \bar{\beta}^3 \sinh\bar{\beta} \\ -\bar{k}_{w2} \cosh\bar{\beta} \end{matrix} \\ \begin{matrix} \bar{\beta} \sin\bar{\beta} \\ -\bar{k}_{\theta 2} \cos\bar{\beta} \end{matrix} & \begin{matrix} \bar{\beta} \cos\bar{\beta} \\ +\bar{k}_{\theta 2} \sin\bar{\beta} \end{matrix} & \begin{matrix} -\bar{\beta} \sinh\bar{\beta} \\ -\bar{k}_{\theta 2} \cosh\bar{\beta} \end{matrix} & \begin{matrix} -\bar{\beta} \cosh\bar{\beta} \\ -\bar{k}_{\theta 2} \sinh\bar{\beta} \end{matrix} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \mathbf{0} \tag{21}$$

where

$$\bar{\beta} = \beta L, \quad \bar{k}_{\theta i} = k_{\theta i}/(EI/L), \quad \bar{k}_{wi} = k_{wi}/(EI/L^3) \tag{22}$$

Free-free beam ($k_{w1} = k_{w2} = k_{\theta 1} = k_{\theta 2} = 0$)

$$\begin{bmatrix} \bar{\beta}^3 & 0 & -\bar{\beta}^3 & 0 \\ 0 & -\bar{\beta} & 0 & \bar{\beta} \\ -\bar{\beta}^3 \cos\bar{\beta} & \bar{\beta}^3 \sin\bar{\beta} & \bar{\beta}^3 \cosh\bar{\beta} & \bar{\beta}^3 \sinh\bar{\beta} \\ \bar{\beta}\ \sin\bar{\beta} & \bar{\beta}\ \cos\bar{\beta} & -\bar{\beta}\ \sinh\bar{\beta} & -\bar{\beta}\ \cosh\bar{\beta} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \mathbf{0}$$

$$\Downarrow \tag{23}$$

$$\det \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -\cos\bar{\beta} & \sin\bar{\beta} & \cosh\bar{\beta} & \sinh\bar{\beta} \\ \sin\bar{\beta} & \cos\bar{\beta} & -\sinh\bar{\beta} & -\cosh\bar{\beta} \end{bmatrix} = 0$$

Cantilever beam $(k_{w2} = k_{\theta 2} = 0, k_{w1} \to \infty, k_{\theta 1} \to \infty.)$

$$\mathbf{det} \begin{bmatrix} \bar{\beta}^3 & -\bar{k}_{w1} & -\bar{\beta}^3 & -\bar{k}_{w1} \\ -\bar{k}_{\theta 1} & -\bar{\beta} & -\bar{k}_{\theta 1} & \bar{\beta} \\ -\cos\bar{\beta} & \sin\bar{\beta} & \cosh\bar{\beta} & \sinh\bar{\beta} \\ \sin\bar{\beta} & \cos\bar{\beta} & -\sinh\bar{\beta} & -\cosh\bar{\beta} \end{bmatrix} = 0 \qquad (24)$$

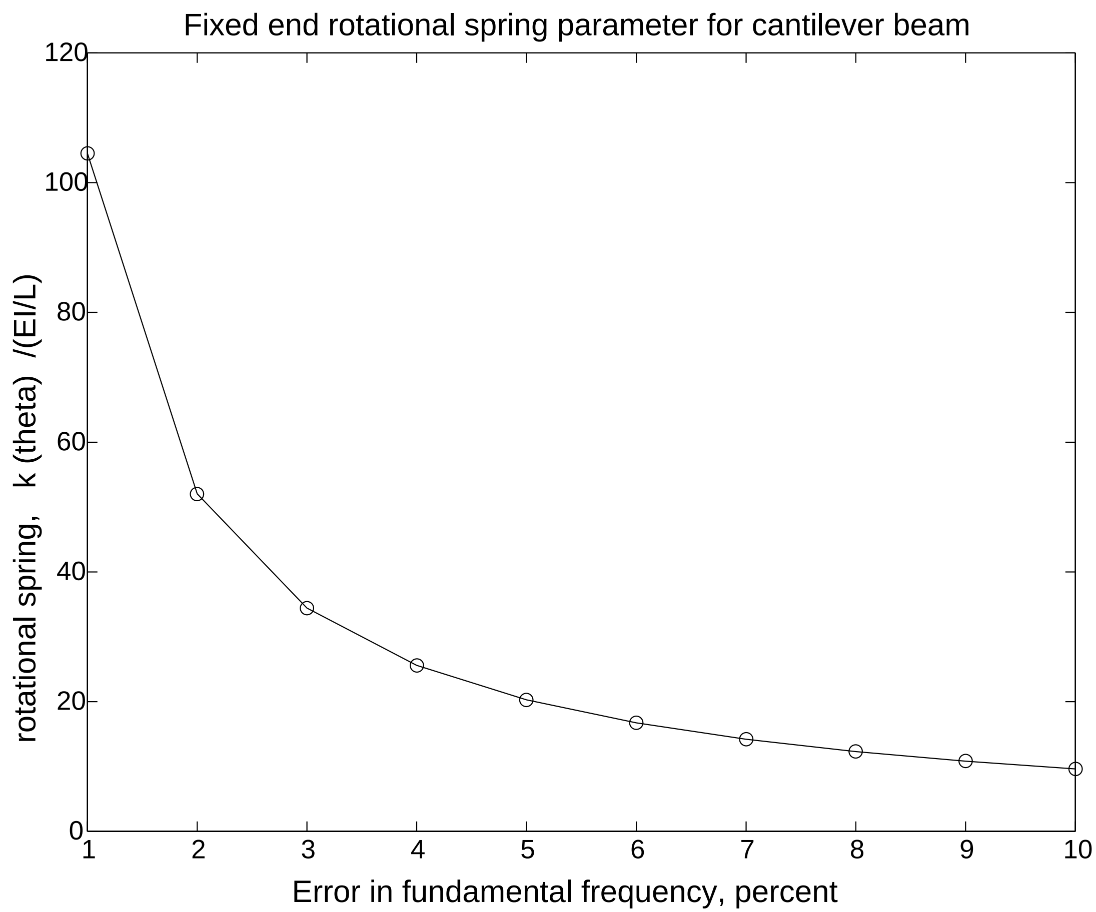
$$
\mathbf{det}\begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ -\cos\bar{\beta} & \sin\bar{\beta} & \cosh\bar{\beta} & \sinh\bar{\beta} \\ \sin\bar{\beta} & \cos\bar{\beta} & -\sinh\bar{\beta} & -\cosh\bar{\beta} \end{bmatrix} = 0
$$

$$
\Downarrow
$$

$$
(1+\cos\bar{\beta}\ \cosh\bar{\beta}) = 0 \tag{25}
$$

which gives

$$
\bar{\beta}_k = \{1.875,\ 4.694,\ 7.855,\ \ldots\} \tag{26}
$$



Effect of fixed end rotational spring on fundamental frequency of a cantilever beam

Simply Supported and Fixed-Fixed Beams

($w(0) = w(L) = 0$ or $k_{w1} = k_{w2} \to \infty$)

$$\det \begin{bmatrix} 0 & -1 & 0 & -1 \\ -\bar{k}_{\theta 1} & -\bar{\beta} & -\bar{k}_{\theta 1} & \bar{\beta} \\ -\sin\bar{\beta} & -\cos\bar{\beta} & -\sinh\bar{\beta} & -\cosh\bar{\beta} \\ \begin{matrix}\bar{\beta}\sin\bar{\beta} \\ -\bar{k}_{\theta 2}\cos\bar{\beta}\end{matrix} & \begin{matrix}\bar{\beta}\ \cos\bar{\beta} \\ +\bar{k}_{\theta 2}\sin\bar{\beta}\end{matrix} & \begin{matrix}-\bar{\beta}\ \sinh\bar{\beta} \\ -\bar{k}_{\theta 2}\cosh\bar{\beta}\end{matrix} & \begin{matrix}-\bar{\beta}\ \cosh\bar{\beta} \\ -\bar{k}_{\theta 2}\sinh\bar{\beta}\end{matrix} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = 0 \tag{27}$$

$$\Downarrow$$

$$\bar{\beta}^2 \sin\bar{\beta}\sinh\bar{\beta} + \bar{k}_{\theta 1}\ \bar{\beta}\ (\sin\bar{\beta}\cosh\bar{\beta} - \cos\bar{\beta}\sinh\bar{\beta}) - \bar{k}_{\theta 2}\ \bar{\beta}\cos\bar{\beta}\sinh\bar{\beta} + \bar{k}_{\theta 1}\ \bar{k}_{\theta 2}\ (1 - \cos\bar{\beta}\cosh\bar{\beta}) = 0$$

Several ideal cases can be obtained from (27):

$$
\begin{aligned}
&\text{Simply supported ends: } \{\bar{k}_{\theta 1} = 0,\ \bar{k}_{\theta 2} = 0\} \quad \Rightarrow \quad \sin\bar{\beta}\sinh\bar{\beta} = 0 \\
&\text{Clamped at } x = 0 \\
&\ \text{and simply supported at } x = L:\ \{\bar{k}_{\theta 1} \to \infty,\ \bar{k}_{\theta 2} = 0\} \\
&\qquad\qquad\qquad\Downarrow \\
&\qquad\qquad \tan\bar{\beta} - \tanh\bar{\beta} = 0 \\
&\text{Clamped at both ends: } \{\bar{k}_{\theta 1} \to \infty,\ \bar{k}_{\theta 2} \to \infty\} \\
&\ \Rightarrow \quad 1 - \cos\bar{\beta}\cosh\bar{\beta} = 0
\end{aligned}
\tag{28}
$$

When the boundary conditions are not ideal, viz.,

$$
\{0 < \bar{k}_{\theta 1} < \infty \quad \text{and} \quad 0 < \bar{k}_{\theta 2} < \infty\} \tag{29}
$$

one needs two or more measured frequencies to determine $\{k_{\theta 1} \quad \text{and} \quad k_{\theta 2}\}$.