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Classical and FEM Solutions of Plate Vibration

§18.1 INTRODUCTION

Classical solutions of plate vibrations have been a subject of many eminent scientists and engineers in the past. A compendium by Leissa [1] is perhaps the most comprehensive source to date. As the simply supported plates (SS-SS-SS-SS) are the simplest to address, we recall from the previous chapter their solution:

$$\begin{aligned}
\text{Vibration mode shapes: } & W(x,y) = \sin\alpha x \sin\gamma y, \\
& \alpha_m a = \pi m, \quad m = 1,2,\ldots; \; \gamma_m a = \pi n, \quad n = 1,2,\ldots \\
\text{Frequency equation: } & \beta_{mn}^4 = \frac{\omega_{mn}^2 \rho}{D}, \; \beta_{mn}^2 = \pi^2[(\frac{m}{a})^2 + (\frac{n}{b})^2] \\
& D = \frac{Eh^3}{12(1-\nu^2)}
\end{aligned} \tag{18.1}$$

where (m, n) denote the number of harmonics along the x and y-coordinate directions.

In conformity with literature, we will modify the frequency equation as

$$\lambda_{mn} = \omega_{mn}\, a^2 \sqrt{\frac{\rho}{D}}, \;\; \lambda_{mn} = (\beta_{mn}\, a)^2 \tag{18.2}$$

where a is the x-directional plate length. Hence, for a rectangular plate $(a = b)$ we have the following frequency equation:

$$\lambda_{mn} = \pi^2\, (m^2 + n^2) \tag{18.3}$$

We now summarize classical solutions of rectangular plate vibration.

§18.2 ASSUMED VIBRATION MODE SHAPES FOR SIX BOUNDARY CONDITIONS

The classical Rayleigh method assumes the plate deflections as the product of beam functions

$$W(x,y) = Y(y)\; X(x) \tag{18.4}$$

each of which can be chosen depending on the boundary conditions.

§18.2.1 Simply supported at $x = 0$ and $x = a$ (SS-SS):

$X(x)$ that satisfies the boundary conditions

$$W(y,0) = W(y,a) = \frac{\partial^2 W(y,0)}{\partial x^2} = \frac{\partial^2 W(y,a)}{\partial x^2} = 0 \tag{18.5}$$

can be expressed as

$$X(x) = \sin\frac{(m-1)\pi x}{a}, \quad m = 2,3,4,\ldots \tag{18.6}$$

It should be noted that if SS boundary conditions are imposed along $y = 0$ and $y = b$, then (x, a, m) in the above expression is simply replaced by (y, b, n).

§18.2.2 Clamped at $x = 0$ and $x = a$ (C-C):

$$\begin{aligned}
\text{For even harmonics: } X(x) &= \cos\gamma_1(\frac{x}{a}-\frac{1}{2}) + \frac{\sin(\gamma_1/2)}{\sinh(\gamma_1/2)}\cosh\gamma_1(\frac{x}{a}-\frac{1}{2}) \\
&\tan(\gamma_1/2) + \tanh(\gamma_1/2) = 0, \quad m = 2, 4, 6, \dots \\
\text{For odd harmonics: } X(x) &= \sin\gamma_2(\frac{x}{a}-\frac{1}{2}) - \frac{\sin(\gamma_2/2)}{\sinh(\gamma_2/2)}\sinh\gamma_2(\frac{x}{a}-\frac{1}{2}) \\
&\tan(\gamma_2/2) - \tanh(\gamma_2/2) = 0, \quad m = 3, 5, 7, \dots
\end{aligned} \tag{18.7}$$

§18.2.3 Free at $x = 0$ and $x = a$ (F-F):

$$\begin{aligned}
\text{For m=0: } X(x) &= 1 \\
\text{For m=1: } X(x) &= 1 - \frac{2x}{a} \\
\text{For even harmonics: } X(x) &= \cos\gamma_1(\frac{x}{a}-\frac{1}{2}) - \frac{\sin(\gamma_1/2)}{\sinh(\gamma_1/2)}\cosh\gamma_1(\frac{x}{a}-\frac{1}{2}) \\
&\tan(\gamma_1/2) + \tanh(\gamma_1/2) = 0, \quad m = 2, 4, 6, \dots \\
\text{For odd harmonics: } X(x) &= \sin\gamma_1(\frac{x}{a}-\frac{1}{2}) + \frac{\sin(\gamma_2/2)}{\sinh(\gamma_2/2)}\sinh\gamma_2(\frac{x}{a}-\frac{1}{2}) \\
&\tan(\gamma_2/2) - \tanh(\gamma_2/2) = 0, \quad m = 3, 5, 7, \dots
\end{aligned} \tag{18.8}$$

The above assumed deflection satisfies only approximately the shear condition, i.e., the zero shear force along the free edges.

§18.2.4 Clamped at $x = 0$ and Free at $x = a$ (C-F):

$$\begin{aligned}
X(x) &= (\cos\frac{\gamma_3 x}{a} - \cosh\frac{\gamma_3 x}{a}) + (\frac{\sin\gamma_3 - \sinh\gamma_3}{\cos\gamma_3 - \cosh\gamma_3})(\sin\frac{\gamma_3 x}{a} - \sinh\frac{\gamma_3 x}{a}) \\
&\cos\gamma_3\cosh\gamma_3 + 1 = 0, \quad m = 1, 2, 3, \dots
\end{aligned} \tag{18.9}$$

§18.2.5 Clamped at $x = 0$ and Simply Supported at $x = a$ (C - SS):

$$\begin{aligned}
X(x) &= \sin\gamma_2(\frac{x}{2a}-\frac{1}{2}) - \frac{\sin(\gamma_2/2)}{\sinh(\gamma_2/2)}\sinh\gamma_2(\frac{x}{2a}-\frac{1}{2}) \\
&\tan(\gamma_2/2) - \tanh(\gamma_2/2) = 0, \quad m = 2, 3, 4, \dots
\end{aligned} \tag{18.10}$$

§18.2.6 Free at $x = 0$ and Simply Supported at $x = a$ (F-SS):

$$\begin{aligned}
\text{For m=1: } X(x) &= 1 - \frac{x}{a} \\
X(x) &= \sin\gamma_2(\frac{x}{2a}-\frac{1}{2}) + \frac{\sin(\gamma_2/2)}{\sinh(\gamma_2/2)}\sinh\gamma_2(\frac{x}{2a}-\frac{1}{2}) \\
&\tan(\gamma_2/2) - \tanh(\gamma_2/2) = 0, \quad m = 2, 3, 4, \dots
\end{aligned} \tag{18.11}$$

Once again (m, n) denote the number of vibration nodal lines lying in the x and y-directions including the boundaries as the nodal lines except when the boundary is free. We now list some of the frequency parameters λ_{mn} from Leissa [1] for various boundary conditions for a square plate.

Table 1 Frequency Parameter $\lambda_{mn} = \omega_{mn}\, a^2\, \sqrt{\rho/D}$ for SS-C-SS-C Square Plate

λ_{11}	λ_{21}	λ_{12}	λ_{22}	λ_{31}	λ_{13}
28.946	54.743	69.320	94.584	102.213	129.086

Table 2 Frequency Parameter $\lambda_{mn} = \omega_{mn}\, a^2\, \sqrt{\rho/D}$ for SS-C-SS-SS Square Plate

λ_{11}	λ_{21}	λ_{12}	λ_{22}	λ_{31}	λ_{13}
23.646	51.674	58.641	86.126	100.259	113.217

Table 3 Frequency Parameter $\lambda_{mn} = \omega_{mn}\, a^2\, \sqrt{\rho/D}$ for SS-C-SS-F Square Plate ($\nu = 0.3$)

λ_{11}	λ_{21}	λ_{12}	λ_{22}	λ_{31}	λ_{13}
12.69	33.06	41.70	63.01	72.40	90.61

Table 4 Frequency Parameter $\lambda_{mn} = \omega_{mn}\, a^2\, \sqrt{\rho/D}$ for SS-SS-SS-F Square Plate ($\nu = 0.3$)

λ_{11}	λ_{21}	λ_{12}	λ_{22}	λ_{31}	λ_{13}
11.68	27.76	41.20	59.07	61.86	90.29

Table 5 Frequency Parameter $\lambda_{mn} = \omega_{mn}\, a^2\, \sqrt{\rho/D}$ **for SS-F-SS-F Square Plate (**$\nu = 0.3$**)**

λ_{11}	λ_{12}	λ_{13}	λ_{21}	λ_{22}	λ_{23}
9.8696	16.13	36.72	39.48	46.74	70.75

Table 6 Frequency Parameter $\lambda_{mn} = \omega_{mn}\, a^2\, \sqrt{\rho/D}$ **for C-C-C-C Square Plate (**$\nu = 0.3$**)**

λ_{11}	λ_{21}	λ_{22}	λ_{31}	λ_{32}	λ_{41}
35.10	72.90	107.47	131.63	164.39	210.35

Table 7 Frequency Parameter $\lambda_{mn} = \omega_{mn}\, a^2\, \sqrt{\rho/D}$ **for F-F-F-F Square Plate (**$\nu = 0.3$**)**

First Mode	Second Mode	Third Mode	Fourth Mode	Fifth Mode	Sixth Mode
13.4728	19.5961	24.2702	35.1565	63.6870	77.5896

Frequency parameters for other boundary conditions can be found in Leissa [1].

§18.3 FINITE ELEMENT ANALYSIS OF PLATE VIBRATION

Before we utilize a typical finite element software for plate vibration analysis, it is instructive to understand how the boundary conditions are treated. Let us begin with the simplest case, that is, the geometric boundary condition. For plate, it is the clamped edge:

$$W(x, y) = 0 \text{ and } \frac{\partial W(x, y)}{\partial n} = 0 \tag{18.12}$$

where n is the normal directional component from the clamped boundary edge.

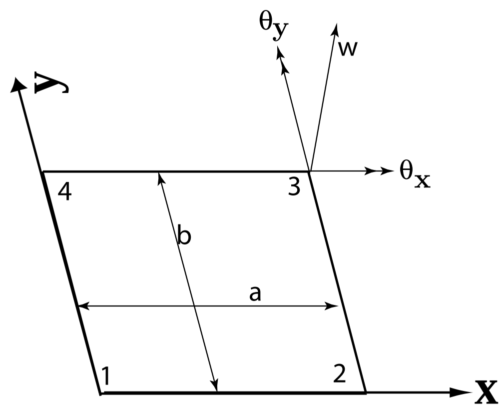

Fig. 18.1 Plate bending element and degrees of freedom per node

Referring to Figure 18.1, there are three discrete degrees of freedom per node, (w, θ_x, θ_y). Since we have, along the edges $(x = 0, x = a)$, $\frac{\partial W(x,y)}{\partial n} = \frac{\partial W(x,y)}{\partial x} = \theta_y$, the clamped edge boundary condition is satisfied if one chooses

$$\begin{aligned} &\text{along } (x = 0, x = a): \quad w = 0 \text{ and } \theta_y = 0 \\ &\text{along } (y = 0, y = b): \quad w = 0 \text{ and } \theta_x = 0 \end{aligned} \tag{18.13}$$

However, simply supported boundary conditions must satisfy

$$W(x, y) = 0 \text{ and } \frac{\partial^2 W(x, y)}{\partial n^2} = 0 \tag{18.14}$$

This boundary conditions are approximately satisfied via

$$\begin{aligned} &\text{along } (x = 0, x = a): \quad w = 0 \text{ and } \theta_y \text{ is free to rotate.} \\ &\text{along } (y = 0, y = b): \quad w = 0 \text{ and } \theta_x \text{ is free to rotate.} \end{aligned} \tag{18.15}$$

It is important to recognize that

$$\begin{aligned} &\text{along } (x = 0, x = a): \ \frac{\partial^2 W(x, y)}{\partial n^2} = \frac{\partial \theta_y}{\partial x} \\ &\text{along } (y = 0, y = b): \ \frac{\partial^2 W(x, y)}{\partial n^2} = \frac{\partial \theta_x}{\partial y} \end{aligned} \tag{18.16}$$

Hence, the second of the simply supported boundary conditions are approximately satisfied as the element size (a, b) becomes smaller and smaller. This can be seen from the finite difference expression of (18.16):

$$\begin{aligned} &\text{along } (x=0, x=a): \ \frac{\partial^2 W(x,y)}{\partial n^2} = \frac{\partial \theta_y}{\partial x} \approx \frac{(\theta_y^{(2)} + \theta_y^{(3)})}{2a} - \frac{(\theta_y^{(1)} + \theta_y^{(4)})}{2a} \\ &\text{along } (y=0, y=b): \ \frac{\partial^2 W(x,y)}{\partial n^2} = \frac{\partial \theta_x}{\partial y} \approx \frac{(\theta_x^{(3)} + \theta_x^{(4)})}{2b} - \frac{(\theta_x^{(1)} + \theta_x^{(2)})}{2b} \end{aligned} \tag{18.17}$$

where the superscript designates the element node as shown in Figure 18.1.

Similarly, the free edge boundary conditions

$$\partial^2 W(x,y)\partial n^2 + \nu \frac{\partial^2 W(x,y)}{\partial s^2} = 0 \ \text{ and } \ \frac{\partial^3 W(x,y)}{\partial n^3} + (2-\nu)\frac{\partial^3 W(x,y)}{\partial n \partial s^2} = 0 \tag{18.18}$$

can be only approximately satisfied by leaving (w, θ_x, θ_y) as free variables along the edges. Specifically, the corner conditions given by

$$\frac{\partial^2 W(x,y)}{\partial s \partial n} = 0 \tag{18.19}$$

at the four corners, $\{(x=0, y=0), (x=a, y=0), (x=a, y=b), (x=0, y=a)\}$, are difficult to satisfy and it is the level of violating these corner conditions that is largely responsible for somewhat slower convergence of F-F-F-F plate higher modes.

Figure 18.2 shows the first four modes and mode shapes of a square rectangular plate with $\nu = 0.3$ so that the FEM-based computations can be compared with classical solution tabulated in Table 7. The errors of FEM-based modes are 7.5%, 0.015%, 0.04%, and 2.6% for the first, second, third and fourth modes, respectively. The large errors of the first mode is clearly due to not satisfying the zero shear and moment conditions along the four plate edges. This aspect should be noted when modeling plates by the finite element method.

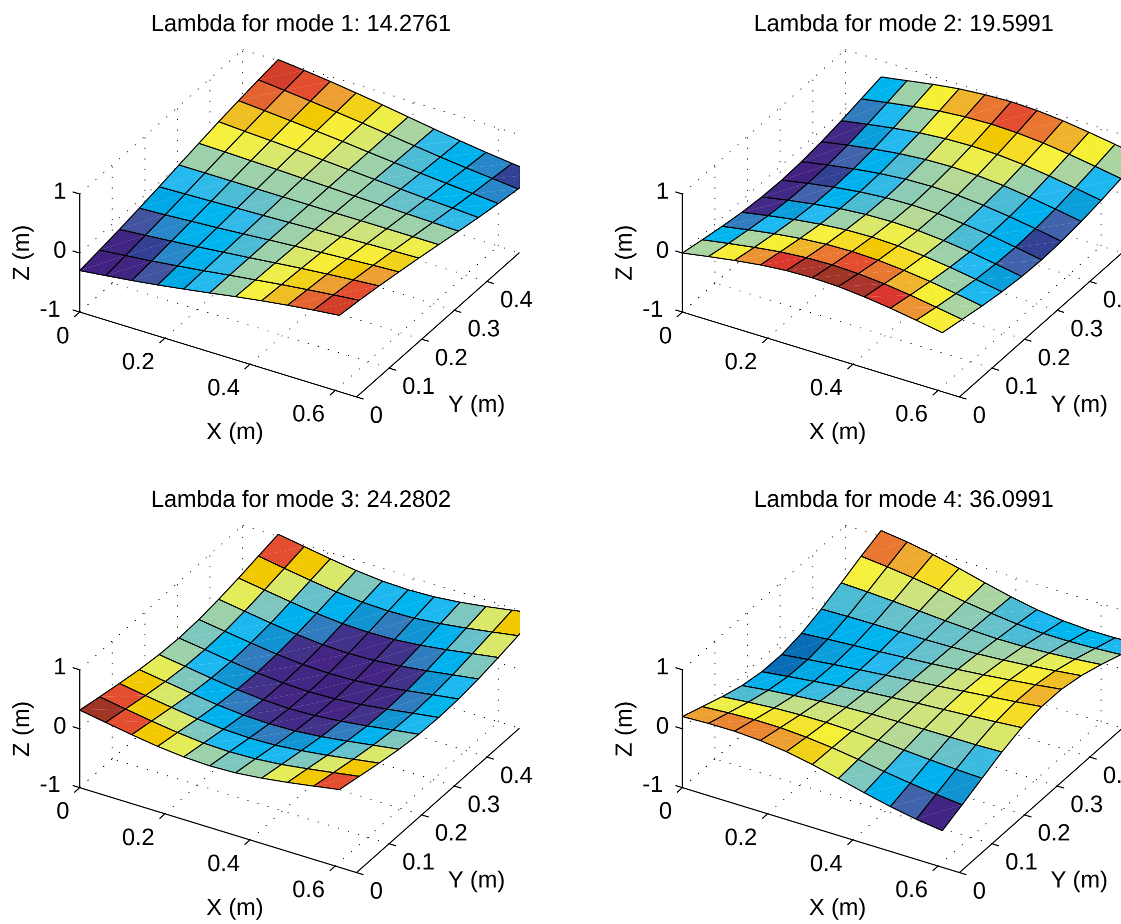

Fig. 18.2 F-F-F-F square plate vibration modes and mode shapes ($\nu = 0.3$)

Reference

1. Leissa, A.W., *Vibration of Plates*, NASA SP-160, 1969,