ASEN 5022 - Spring 2005

Dynamics of Aerospace Structures

Lecture 11: 22 February

Vibration of Continuum Beams

Summary of Elementary Beam Theory

Strain-displacement: $\epsilon_{xx} = \frac{\partial u(x,t)}{\partial x} - z\frac{\partial^2 w(x,t)}{\partial x^2} + \frac{1}{2}(\frac{\partial w(x,t)}{\partial x})^2$

Constitutive relation: $\sigma_{xx} = E\epsilon_{xx}$

Bending moment: $M_y = \int_A z\sigma_{xx} dA = -EI_z \frac{\partial^2 w(x,t)}{\partial x^2}$ (1)

Kinetic energy:

$$T = \frac{1}{2}\int_0^L m(x)\ (w_t)^2 dx,\ \ w_t = \frac{\partial w(x,t)}{\partial t} \tag{2}$$

Potential energy of the beam:

$$
\begin{aligned}
V &= \tfrac{1}{2}\int_0^L [\int_A \sigma_{xx}\epsilon_{xx} dA] dx \\
&= \tfrac{1}{2}\int_0^L [\int_A E(\epsilon_{xx})^2 dA] dx \\
&= \tfrac{1}{2}\int_0^L [\int_A E\{(z^2 w_{xx}^2 + u_x w_x^2) \\
&+ (u_x^2 + 0.25 w_x^4 - 2z u_x w_{xx} - z w_{xx} w_x^2)\} dA] dx \qquad (3) \\
&\approx \tfrac{1}{2}\int_0^L [\int_A E(z^2 w_{xx}^2 + u_x w_x^2) dx \\
&= \tfrac{1}{2}\int_0^L \{\, EI(x)\ (w_{xx})^2 \ + \ P(x)\ (w_x)^2 \,\}\, dx
\end{aligned}
$$

$$
P(x) = EA(x)\ u_x, \quad w_x = \frac{\partial w(x,t)}{\partial x},\ etc.
$$

Hamilton's principle:

$$\int_{t_1}^{t_2} [\delta T - \delta V + \delta \bar{W}\,]\, dt = 0 \tag{4}$$

External energy:

$$\delta \bar{W} = \int_0^L \mathbf{f}(x, t)\, \delta w(x, t)\, dx \tag{5}$$

Variation of the Kinetic energy:

$$\int_{t_1}^{t_2} \delta T \, dt = \int_0^L m(x) \left[w(x,t)_t \delta w(x,t) \right]\Big|_{t_1}^{t_2} dx$$
$$- \int_0^L \int_{t_1}^{t_2} m(x) \; w(x,t)_{tt} \delta w(x,t) \; dt \; dx \qquad (6)$$
$$= - \int_0^L \int_{t_1}^{t_2} m(x) \; w(x,t)_{tt} \delta w(x,t) \; dt \; dx$$

since we consider $w(x,t_1)$ **and** $w(x,t_2)$ **as specified or constant at** $t = t_1$ **and** $t = t_2$**.**

Variation of the Potential energy:

$$\begin{aligned}\delta V = &\{\ EI\ w(x,t)_{xx}\delta w(x,t)_x\ \}|_0^L \\ &\{-[EI\ w(x,t)_{xx}]_x \\ &+[P\ w(x,t)_x \delta w(x,t)\ \}|_0^L \\ &+\int_0^L \{\ \rho[EI\ w(x,t)_{xx}]_{xx} \\ &-[P\ w(x,t)_x]_x\ \}\delta w(x,t)\ dx\end{aligned} \tag{7}$$

Substituting (5)-(7) **into Hamilton's principle** (4)**, we obtain the desired variational equation.**

$$
\begin{aligned}
&\int_{t_1}^{t_2} \langle \; \{EI \; w(x,t)_{xx} \delta w(x,t)_x \; \}|_0^L \\
&\{-([EI \; w(x,t)_{xx}]_x + P \; w(x,t)_x)\delta w(x,t) \; \} \; |_0^L \\
&+ \int_0^L \{ \; m(x) \; w(x,t)_{tt} + [EI \; w(x,t)_{xx}]_{xx} \\
&- [P \; w(x,t)_x]_x \; - \; f(x,t)\}\delta w(x,t) \; dx \; \rangle \; dt = 0
\end{aligned}
\tag{8}
$$

As in the case of string equations, the first two expressions yield the boundary conditions. It is the third expression in the above equation that provides the governing equation of motion for beam.

Boundary-value problem for beam bending with $P = 0$

Governing equation of motion:

$$\boxed{m(x)\, w(x,t)_{tt} + [EI\ w(x,t)_{xx}]_{xx} = f(x,t)}$$

Moment and/or Slope Boundary Conditions:

$$\boxed{\{M(x,t)\delta w(x,t)_x\}|_0^L = 0, \quad M(x,t) = EI\ w(x,t)_{xx}} \tag{9}$$

Shear and/or Displacement Boundary Condition:

$$\boxed{\{V(x,t)\delta w(x,t)\}\,|_0^L = 0, \ V(x,t) = -[EI\ w(x,t)_{xx}]_x}$$

Boundary-value problem for beam bending - cont'd

Unlike string problems, we need two conditions at each end with $x = c$ being the end coordinate.

Free end at $x = c$:

$$\boxed{M(c,t) = 0 \ \textbf{ and } \ V(c,t) = 0} \tag{10}$$

Fixed end at $x = c$:

$$\boxed{w(c,t) = 0 \ \textbf{ and } \ w(c,t)_x = 0} \tag{11}$$

Simply supported end at $x = c$:

$$\boxed{M(c,t) = 0 \ \textbf{ and } \ w(c,t) = 0} \tag{12}$$

Solution of the homogeneous governing equation assumes the form

$$w(x,t) = F(t)W(x), \quad \textbf{with} F(t) = \bar{F}e^{\pm j\omega t} \tag{13}$$

Remark (a brief tour of hyperbolic partial differential equation theory): In a rigorous treatment, one substitutes (13) **into the governing equation with** $f(x,t) = 0$ **to obtain**

$$m(x)F(t)_{tt}W(x) + EI(x)F(t)W(x)_{xxxx} = 0 \tag{14}$$

Dividing both sides by $F(t)W(x)$ **one obtains**

$$\frac{F(t)_{tt}}{F(t)} = -\frac{EI(x)}{m(x)}\frac{W(x)_{xxxx}}{W(x)} \tag{15}$$

Since the left-hand term is a function of time whereas the right-hand term is a function of only x, both terms must be independent of time, t, and the coordinate, x. Hence, both terms must be a constant if the equality is to hold for all time and for the entire spatial domain range:, viz.,

$$\frac{F(t)_{tt}}{F(t)} = -\frac{EI(x)}{m(x)}\frac{W(x)_{xxxx}}{W(x)} = -\omega^2, \quad \omega \textbf{ being real (why?)} \tag{16}$$

which yields the following two sets of ordinary differential equations:

$$F(t)_{tt} + \omega^2 F(t) = 0 \tag{17}$$

$$\boxed{W(x)_{xxxx} - \beta^4\, W(x) = 0, \quad \beta^4 = \frac{m(x)\omega^2}{EI(x)}} \tag{18}$$

Solution of (17) **can be expressed as**

$$F(t) = \bar{F}e^{st} \quad \Rightarrow \quad s^2 + \omega^2 = 0 \;\Rightarrow\; \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \begin{Bmatrix} +j \\ -j \end{Bmatrix} \omega \tag{19}$$

Therefore, $F(t)$ **has the form of**

$$F(t) = A \sin \omega t + B \cos \omega t \tag{20}$$

which implies that $w(x, t)$ **oscillates with time with the corresponding period** $\tau = 2\pi/\omega$**. Hence,** ω **is its characteristic frequency.**

Solution of (18) **assumes the following form:**

$$W(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \tag{21}$$

1. Free-free beam: **For this the boundary condition** (10) **gives**

$$\left\{ \begin{array}{c} M(x,t) = EI(x)w(x,t)_{xx} = 0 \\ V(x,t) = -EI(x)w(x,t)_{xxx} = 0 \end{array} \right\}$$

$$\Downarrow$$

$$\textbf{at } x = 0 \quad \left\{ \begin{array}{c} \{EI(x)W(x)_{xx}\}|_{x=0} = 0 \\ \{EI(x)W(x)_{xxx}\}|_{x=0} = 0 \end{array} \right\} \tag{22}$$

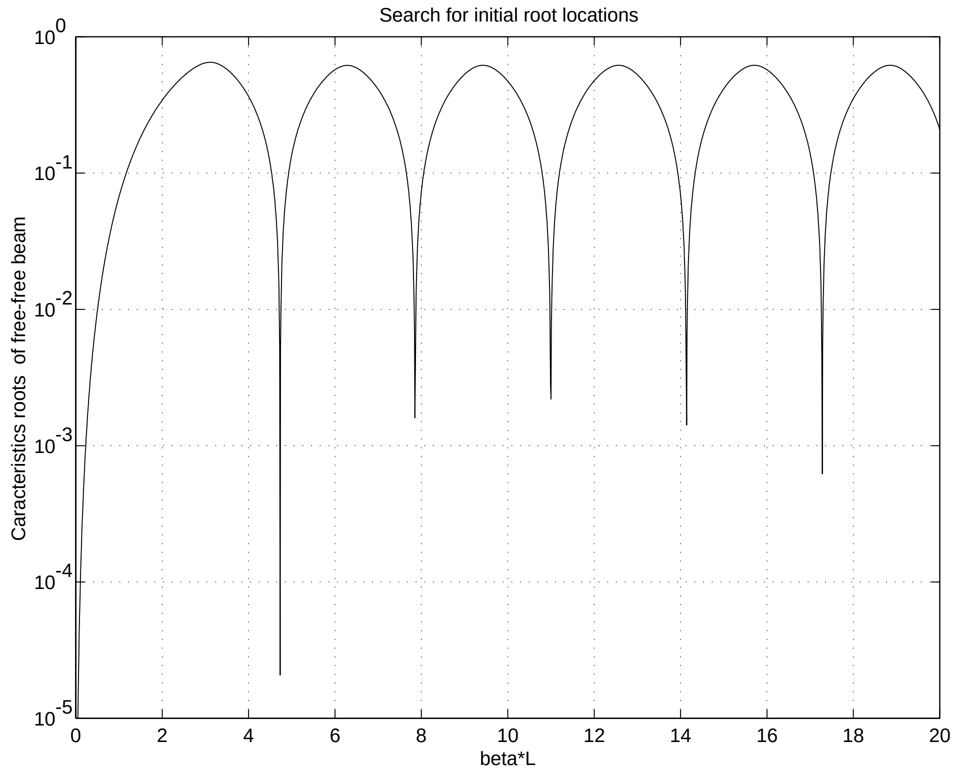
$$\textbf{at } x = L \quad \left\{ \begin{array}{c} \{EI(x)W(x)_{xx}\}|_{x=L} = 0 \\ \{EI(x)W(x)_{xxx}\}|_{x=L} = 0 \end{array} \right\}$$

Substituting $W(x)$ (21) **into the above four boundary conditions, we obtain with** $\bar{\beta} = \beta L$**:**

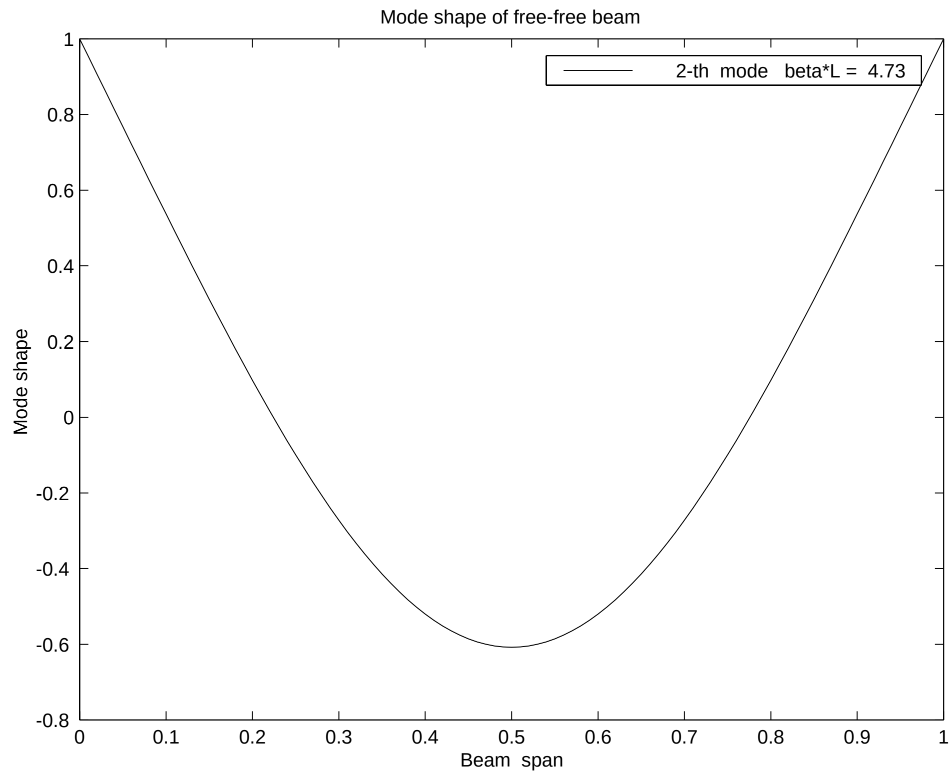
$$
\begin{aligned}
-C_2 + C_4 &= 0 \\
-C_1 + C_3 &= 0 \\
-C_1 \sin\bar{\beta} - C_2 \cos\bar{\beta} + C_3 \sinh\bar{\beta} + C_4 \cosh\bar{\beta} &= 0 \\
-C_1 \cos\bar{\beta} + C_2 \sin\bar{\beta} + C_3 \cosh\bar{\beta} + C_4 \sinh\bar{\beta} &= 0 \\
&\Downarrow \\
\mathbf{AC} &= 0
\end{aligned}
\tag{23}
$$

whose characteristic equation (the determinant of A of the above equation) is given by

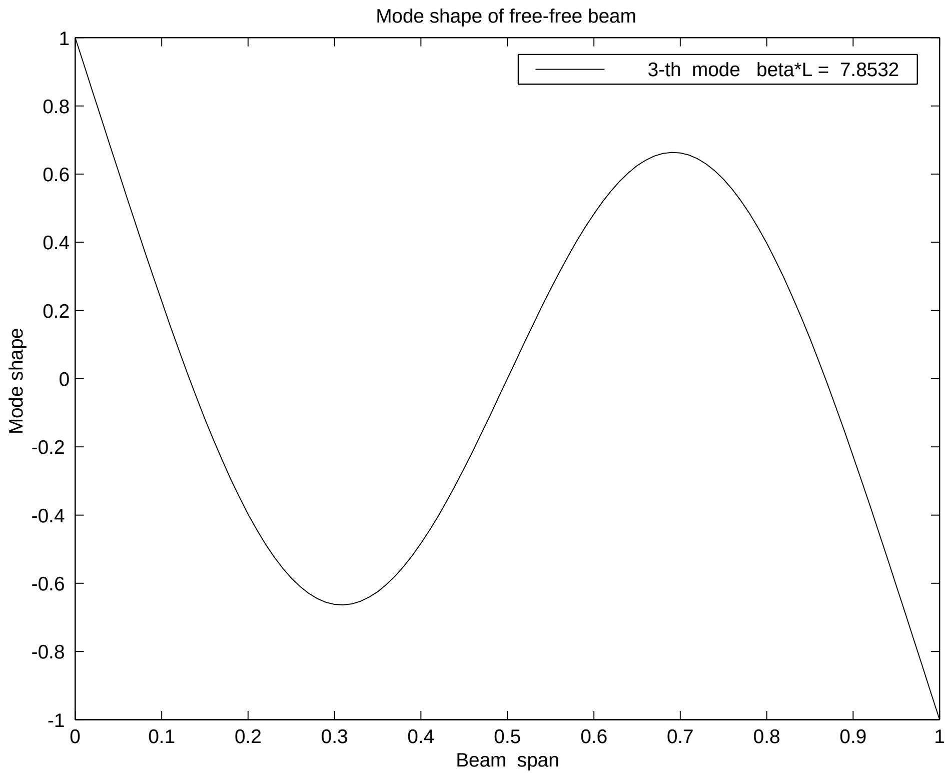
$$
\boxed{1 - \cos\beta L \cdot \cosh\beta L = 0} \tag{24}
$$



Initial root finder for a free-free beam



Second mode of a free-free beam



Third mode of a free-free beam

What are the mode shapes of $\beta L = 0$?

When $\beta L = 0$, one can check the rank of the characteristic matrix A reduces from three to two. This is a degenerative case. Hence, the solution form assumed by (21) is not appropriate. Therefore, one must invoke the governing equation (18), which becomes with $\beta = 0$:

$$\boxed{W(x)_{xxxx} = 0} \tag{25}$$

that is subject to the boundary conditions (22). Integration of the above equation yields

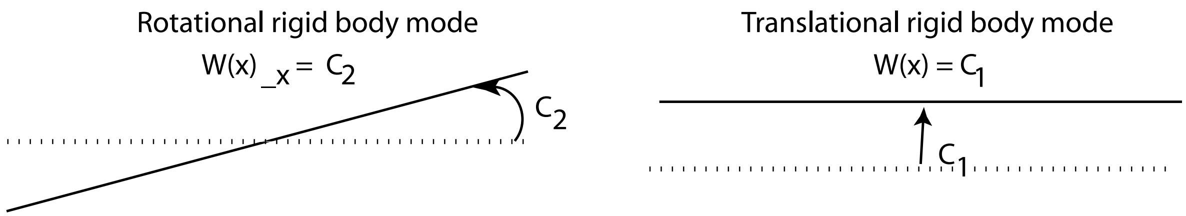
$$W(x) = c_1 = c_2 x + c_3 x^2 + c_4 x^3 \tag{26}$$

which upon satisfying the boundary conditions (22), yields

$$W(x)_x = c_2 \quad \Rightarrow \quad W(x) = c_1 + c_2 x \tag{27}$$

Note that $W(x)_x = c_2$ is the slope of the beam and is a constant. This means the free-free beam with zero frequency ($beta = 0$) rotates as a straight line. This is referred to a rotational rigid-body mode.

There is a second rigid body mode given by $W(x) = c_1$ if $c_2 = 0$ (i.e., no rotational rigid body mode). c_1 represents a uniform displacement across the beam span. This is known as a translational rigid body mode. This is illustrated below.



Two rigid body modes of a free-free beam

Beams with other boundary conditions

There are five additional end conditions that are possible from combinations of the three distinct boundary conditions (10) - (12). *These are left for your practice.*