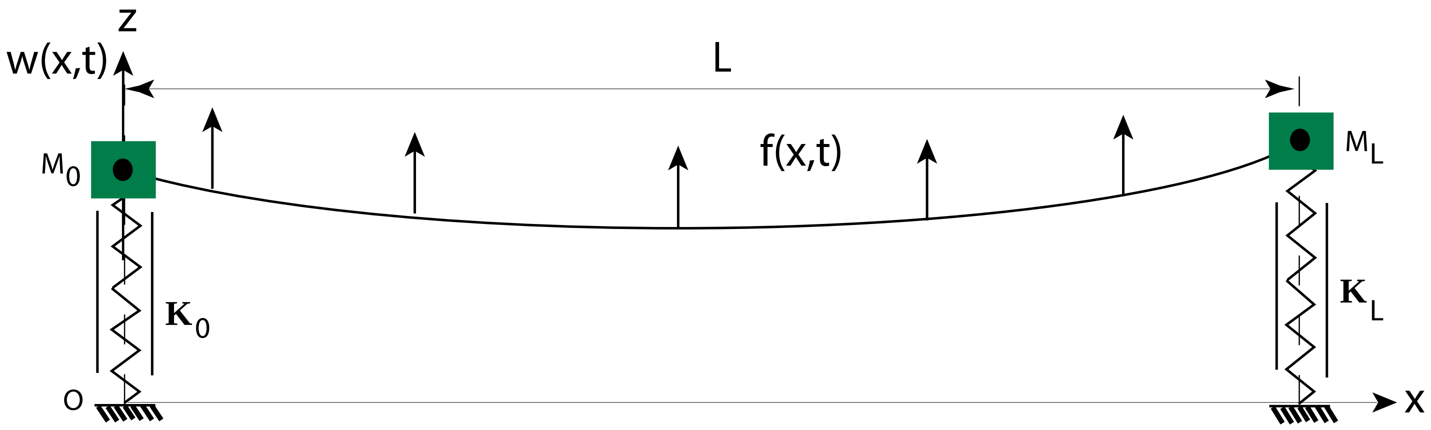
ASEN 5022 - Spring 2005

Dynamics of Aerospace Structures

Lecture 10: 17 February

Cables with End Masses and End Springs



Cables with End Masses and End Springs

Formulation via Hamilton’s principle:

$$\int_{t_1}^{t_2} [\delta L + \delta \bar{W}_{noncons}] dt = 0, \quad L = T - V$$

$$\begin{aligned} T &= \int_0^L \tfrac{1}{2}\rho(x)\dot{w}^2(x,t)\,dx \\ &\quad + \tfrac{1}{2}M_0\,\dot{w}^2(0,t) + \tfrac{1}{2}M_L\,\dot{w}^2(L,t) \end{aligned} \tag{1}$$

$$\begin{aligned} V &= \int_0^L \tfrac{1}{2}T(x)\,w_x^2(x,t)\,dx \\ &\quad + \tfrac{1}{2}K_0\,w^2(0,t) + \tfrac{1}{2}K_L\,w^2(L,t) \end{aligned}$$

Symbol Definitions:

- **Transverse displacement:** $w(x,t)$ (m)
- **Time derivative:** $\dot{w} = \frac{\partial w}{dt}$ (m/s)
- **Spatial derivative:** $w_x = \frac{\partial w}{dx}$ (m/m)
- **String tension:** $T(x)$ (N)
- **Mass per unit cable length:** $\rho(x)$ (kg/m)
- **End springs:** (K_0, K_L) (N/m)
- **End masses:** (M_0, M_L) (N/m)

Evaluation of $\int_{t_1}^{t_2} \delta T\, dt$

$$\boxed{\begin{aligned} &\int_{t_1}^{t_2} \delta T dt = \\ &-\int_{t_1}^{t_2}\{[\int_0^L \rho(x)\ddot{w}(x,t)\,\delta w(x,t)dx] \\ &\quad + M_0\ddot{w}(0,t)\,\delta w(0,t) \\ &\quad + M_L\ddot{w}(L,t)\,\delta w(L,t)\}\,dt \end{aligned}} \tag{2}$$

Evaluation of δV

$$
\begin{aligned}
\delta V = & \\
& -\int_0^L T(x)\, w_{xx}(x,t)\delta w(x,t)\, dx \\
& +\left[T(x)\, w_x(x,t)\delta w(x,t)\right]_{x=L} \\
& -\left[T(x)\, w_x(x,t)\delta w(x,t)\right]_{x=0} \\
& \\
& +\left[K_L\, w(x,t)\delta w(x,t)\right]_{x=L} \\
& +\left[K_0\, w(x,t)\delta w(x,t)\right]_{x=0}
\end{aligned}
\tag{3}
$$

Virtual work due to nonconservative force $f(x,t)$

$$\delta \bar{W}_{noncons} = \int_0^L f(x,t)\delta w(x,t)dx \tag{4}$$

Substituting (4), (3) **and** (2) **into** (1), **one obtains Hamilton's principle:**

$$\int_{t_1}^{t_2} [\delta(T - V) + \delta \bar{W}_{noncons}]dt = 0 \tag{1}$$

$$\Downarrow \qquad \Downarrow$$

Hamilton's principle for cable with end masses and end springs

$$
\begin{aligned}
&\int_{t_1}^{t_2} \langle \int_0^L \{[-\rho(x)\ddot{w}(x,t) \\
&+ T(x)\, w_{xx}(x,t) + f(x,t)]\, \delta w(x,t)\} dx \\
&\\
&- \{[(T(x)\, w_x(x,t) + K_L w(x,t) \\
&+ M_L \ddot{w}(x,t)]\delta w(x,t)\}_{x=L} \\
&\\
&- \{[-T(x)\, w_x(x,t) + K_L w(x,t) \\
&+ M_0 \ddot{w}(x,t)]\delta w(x,t)\}_{x=0}\rangle\, dt = 0
\end{aligned}
\tag{5}
$$

The governing equation of motion:

$$\boxed{\rho(x)\ddot{w}(x,t) = T(x)\, w_{xx}(x,t) + f(x,t)} \tag{6}$$

The boundary conditions:

$$\boxed{\begin{aligned} &\{[(T(x)\, w_x(x,t) + K_L w(x,t) \\ &\ + M_L \ddot{w}(x,t)]\delta w(x,t)\}_{x=L} = 0 \end{aligned}} \tag{7}$$

$$\boxed{\begin{aligned} &\{[-T(x)\, w_x(x,t) + K_0 w(x,t) \\ &\ + M_0 \ddot{w}(x,t)]\delta w(x,t)\}_{x=0} = 0 \end{aligned}} \tag{8}$$

Observations:

1. **The governing equation of motion for the case of end masses and end springs is the same as the case without**

any end conditions. This means the governing equation of motion is the same for all possible boundary conditions.

2. **Therefore, the general form of solution**

$$w(x, t) = F(t)\,[C_1 \sin \beta x + C_2 \cos \beta x] \tag{9}$$

is applicable to all the cable vibration problems.

Boundary conditions at $x = L$*:* **From** (7) **we find**

$$T(L)\, w_x(L, t) + K_L w(L, t) + M_L \ddot{w}(L, t) = 0 \tag{10}$$

$$\textbf{or} \quad w(L, t) = 0 \tag{11}$$

Boundary conditions at $x = 0$*:* **From** (8) **we find**

$$-T(0)\, w_x(0, t) + K_0 w(0, t) + M_0 \ddot{w}(0, t) = 0 \tag{12}$$

$$\textbf{or} \quad w(0, t) = 0 \tag{13}$$

From (10)-(13)**, one finds that the two essential boundary conditions given by** (11) **and** (13) **are special cases of the two natural boundary conditions given by** (10) **and** (12)**, since the former two are obtained from the latter two if** $K_L \to \infty$ **and** $K_0 \to \infty$**, respectively.**

Substituting (9) **into** (10) **and** (12)**, together with** $F(t) = \bar{F}e^{j\omega t}$**, yields the following characteristic equation:**

$$\begin{bmatrix} \begin{matrix}\bar{\beta}\cos\bar{\beta}+ \\ (\kappa_L - \mu_L\bar{\beta}^2)\sin\bar{\beta}\end{matrix} & \begin{matrix}-\bar{\beta}\sin\bar{\beta}+ \\ (\kappa_L - \mu_L\bar{\beta}^2)\cos\bar{\beta}\end{matrix} \\ \\ -\bar{\beta} & (\kappa_0 - \mu_0\bar{\beta}^2) \end{bmatrix} \begin{Bmatrix} C_1 \\ \\ \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \\ \\ 0 \end{Bmatrix} \tag{14}$$

$$\begin{gathered} \beta = \omega\sqrt{\frac{\rho}{T}}, \quad \bar{\beta} = \beta L \\ \kappa_L = \frac{K_L L}{T}, \quad \kappa_0 = \frac{K_0 L}{T} \\ \mu_L = \frac{M_L}{\rho L}, \quad \mu_0 = \frac{M_0}{\rho L} \end{gathered} \tag{15}$$

The characteristic equation of (14) **is found by taking the determinant to be zero:**

$$
\begin{aligned}
&(\kappa_0 - \mu_0 \bar{\beta}^2)(\kappa_L - \mu_L \bar{\beta}^2)\sin\bar{\beta} \\
&+ \bar{\beta}\cos\bar{\beta}[(\kappa_0 - \mu_0 \bar{\beta}^2) + (\kappa_L - \mu_L \bar{\beta}^2)] \\
&- \bar{\beta}^2 \sin\bar{\beta} = 0
\end{aligned}
\tag{16}
$$

We now examine several special cases in the following.

1. Fixed-Fixed Ends $(k_L \to \infty,\ k_0 \to \infty)$

This case corresponds to $(k_L \to \infty,\ k_0 \to \infty)$ **so that, by dividing** (16) **by** $(k_L k_0)$ **yields:**

$$\begin{array}{ll} \sin\bar{\beta} = \sin\beta L = 0 \quad \Rightarrow \quad \beta L = k\pi,\, k = 1, 2, \ldots & \\ \qquad\Downarrow & \\ & (17) \\ \omega_k = \dfrac{k\pi}{L}\sqrt{\dfrac{T}{\rho}}, \quad k = 1, 2, 3, \ldots & \end{array}$$

which was already discussed previously.

2. Fixed-Free Ends ($k_L \to 0,\ k_0 \to \infty, \mu_L = \mu_0 = 0$)

This case leads to the following characteristic equation from (16)**:**

$$\begin{aligned} &\cos\bar{\beta} = \sin\beta L = 0 \\ &\beta L = \frac{(2k-1)}{2}\pi, k = 1, 2, \ldots \\ &\Downarrow \\ &\omega_k = \frac{(2k-1)\pi}{2L}\sqrt{\frac{T}{\rho}}, \quad k = 1, 2, 3, \ldots \end{aligned} \tag{18}$$

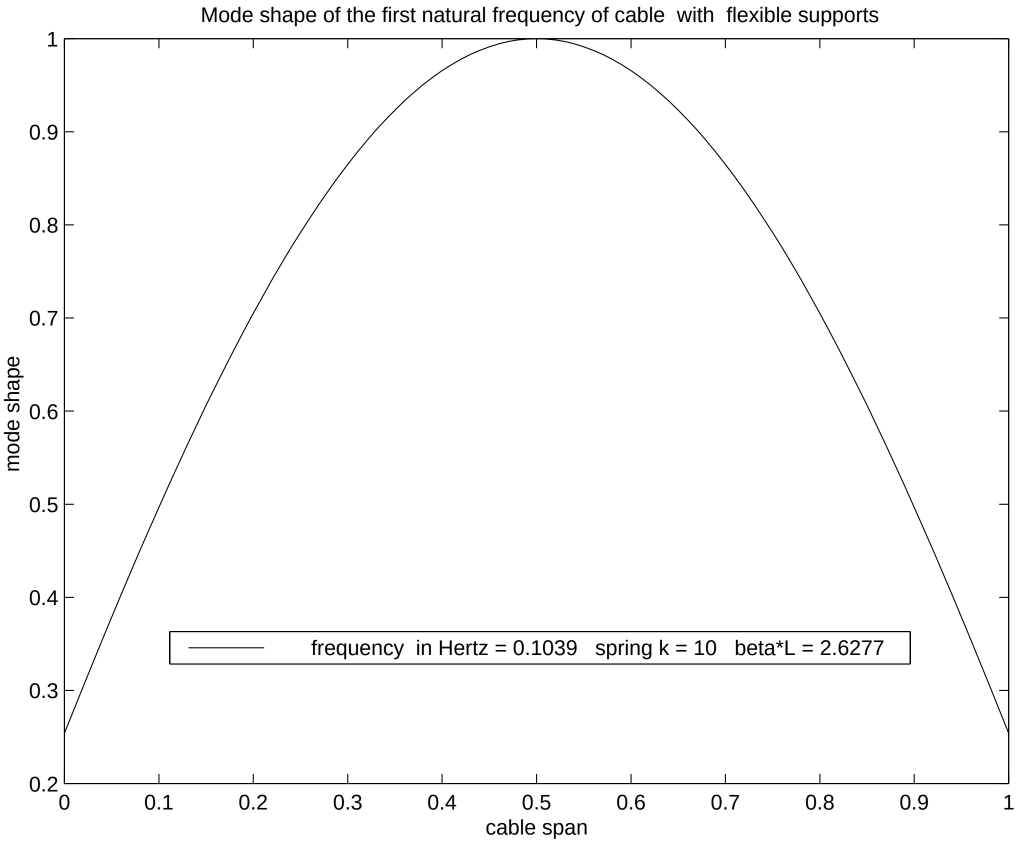
which was also treated before.

3. Flexible Supports $(k_L = k_0 \neq 0, \mu_L = \mu_0 = 0)$

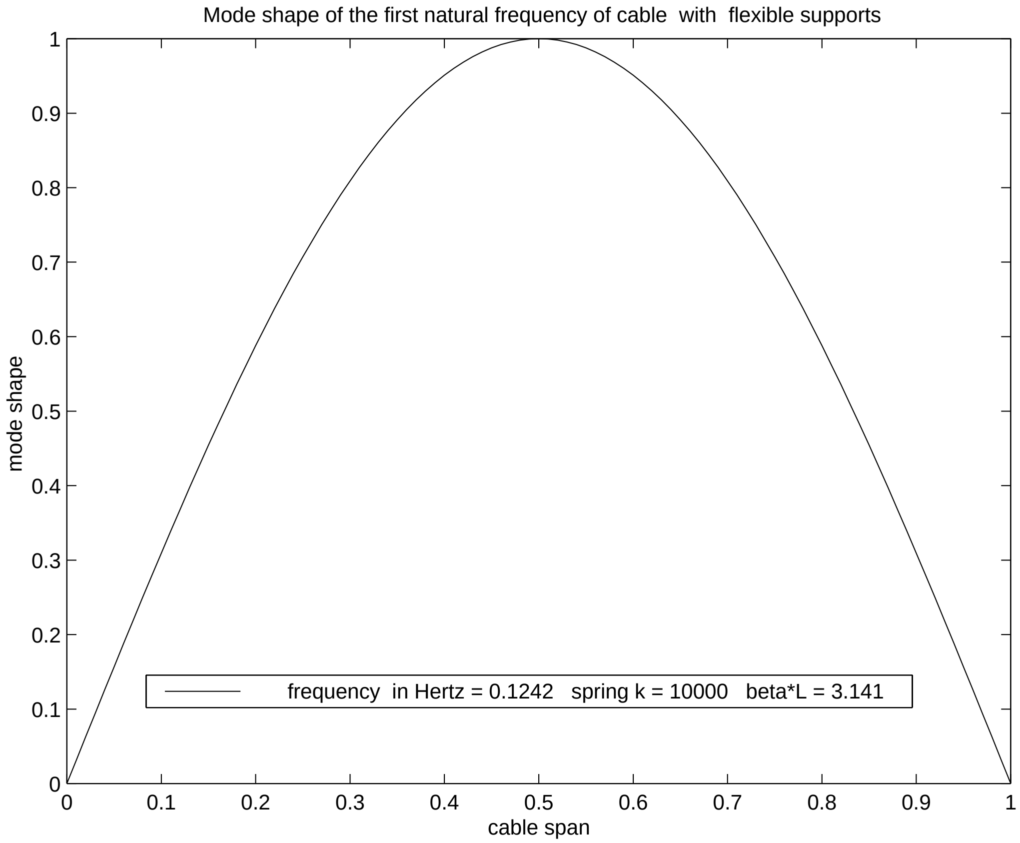
This case leads to the following characteristic equation from (16)**:**

$$(k^2 - \bar{\beta}^2)\tan\bar{\beta} + 2k\bar{\beta} = 0, \quad k = k_L = k_0 \tag{19}$$

Other general cases can be evaluated using (17)**. This is left for exercises**



Mode Shapes of Cable with End Springs



Mode Shapes of Cable with End Springs